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Research Article

A Prime-Based Continued Fraction Constant ρ = [2; 3, 5, 7, 11, ...]: Convergence, Prime-Indexed Engel/Egyptian Expansions, and Conditional Estimates Involving the Riemann Hypothesis

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This paper studies the constant $\rho = [2; 3, 5, 7, 11, 13, 17, \ldots]$ whose partial quotients are the prime numbers. Our analysis emphasizes convergence properties and prime-indexed Engel/Egyptian expansions, together with conditional estimates that depend on known error bounds for the Chebyshev function $\vartheta(x)$ under the Riemann Hypothesis. The growth of the convergent denominators is framed by primorial products and by ϑ, and the standard inequality for convergents implies an irrationality exponent of 2. Under the classical conditional error term for $\vartheta(x)$, the leading asymptotic for $\log Q_n$ is sharpened at a square-root scale. The role of Engel and Egyptian expansions is twofold: they provide prime-driven decompositions of the fractional part of ρ and a baseline for comparing denominator growth against continued-fraction convergents. We supply computable upper bounds for $|\rho - P_n/Q_n|$, error audits for the first 25 convergents, high-precision digits of ρ, and comparative plots linking $\log Q_n$ to $\vartheta(p_{n+1})$. The contribution is conceptual—an elementary framework that connects prime-indexed partial quotients to ϑ and to classical Diophantine estimates—and empirical, via a compact reproducibility bundle. All statements that reference the Riemann Hypothesis are strictly conditional applications of known estimates for $\vartheta(x)$.

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1. Introduction and Motivation

Continued fractions form a central tool in Diophantine approximation and dynamical number theory; accessible introductions are available in standard references [1][2][3]. When one imposes an arithmetic structure on the sequence of partial quotients, the resulting constant often reflects that structure in the growth of convergent denominators and in the quality of approximation. We consider the real number

$$\rho = [2;\, 3, 5, 7, 11, 13, 17, 19, \ldots], \tag{1}$$

whose tail runs along the odd primes in increasing order. A primary theme of this paper is the comparison between $\log Q_n$ and $\vartheta(p_{n+1})$, where Q_n is the denominator of the nth convergent and ϑ denotes the Chebyshev function. The identity $\log p_n^{\#} = \vartheta(p_n)$ is elementary but powerful; it converts a product of primes into a sum of logarithms and opens the door to the prime number theorem [4][5]. A second theme is the behavior of the approximation error $|\rho - P_n/Q_n|$ and the implication that the irrationality exponent equals two, an inference that relies on the classical inequality for convergents [3]. A third theme is the connection to the Riemann Hypothesis through standard error terms for ϑ [6][7][8]. A final theme is the study of Egyptian and Engel expansions motivated by primes [9][10] and by earlier computational explorations of prime-based continued fractions [11].

Relation to prior work. The preprint [11] investigated continued fractions whose partial quotients are primes and reported computational observations. The present paper differs in two respects. First, it develops a systematic connection to ϑ via the product bounds in Lemma 1 and derives the asymptotic $\log Q_n \sim p_{n+1}$ with a clearly identified error budget. Second, it isolates the approximation error behavior and proves $\mu(\rho) = 2$ directly from the convergent inequality, thereby complementing the computational viewpoint in [11].

2. Background and Notation

Let $x = [a_0; a_1, a_2, \ldots]$ denote a regular continued fraction with convergents P_n/Q_n. The recurrences

$$P_n = a_n P_{n-1} + P_{n-2}, \quad Q_n = a_n Q_{n-1} + Q_{n-2}, \tag{2}$$

are classical [1][2]. Theorem 1 below summarizes the standard two-sided error bound [3]. We write p_n for the nth prime and $p_n^{\#} = \prod_{k \le n} p_k$ for the primorial; its logarithm equals the Chebyshev function $\vartheta(p_n) = \sum_{p \le p_n} \log p$ [12].

Lemma 1. *For a regular continued fraction with partial quotients $a_k \ge 1$, one has*

$$\prod_{k\le n} a_k \le Q_n \le \prod_{k\le n}(a_k+1) \quad \text{for all~} n\ge 1. \tag{3}$$

Proof. Iterate Equation (2) and use $Q_{n-2}\ge 0$ to obtain the lower bound; use $Q_{n-2}\le Q_{n-1}$ for the upper bound [1]. □

Theorem 1 ([3]). *Let $x=[a_0;a_1,a_2,\ldots]$ be a regular continued fraction with convergents P_n/Q_n. Then for every $n\ge 0$,*

$$\frac{1}{(a_{n+1}+2)Q_n^2} < \left|x-\frac{P_n}{Q_n}\right| < \frac{1}{a_{n+1}Q_n^2}. \tag{4}$$

3. Denominator Growth and Chebyshev Links

Specializing Lemma 1 to the prime-indexed continued fraction yields

$$\vartheta(p_{n+1}) \le \log Q_n \le \vartheta(p_{n+1}) + \sum_{k\le n}\log\left(1+\frac{1}{p_{k+1}}\right). \tag{5}$$

Mertens' estimates imply that the additive sum is $O(\log\log p_{n+1})$ [13]. Hence,

$$\log Q_n = \vartheta(p_{n+1}) + O(\log\log p_{n+1}). \tag{6}$$

The prime number theorem gives $\vartheta(x)\sim x$ [4][5], and therefore $\log Q_n \sim p_{n+1}$. Figures 1 and 2 illustrate these statements; Figure 5 shows that $\log Q_n/p_{n+1}\to 1$.

4. Approximation Error and Irrationality Exponent

Applying Theorem 1 with $a_{n+1}=p_{n+2}$ yields

$$\left|\rho-\frac{P_n}{Q_n}\right| < \frac{1}{p_{n+2}Q_n^2}. \tag{7}$$

The irrationality exponent admits the formula

$$\mu(x) = 2 + \limsup_{n\to\infty}\frac{\log a_{n+1}}{\log Q_n}, \tag{8}$$

see [[3], Ch. 1]. For ρ we have $\log a_{n+1} = \log p_{n+2} = O(\log n)$ while $\log Q_n \sim p_{n+1} \asymp n\log n$; hence $\mu(\rho)=2$. Figure 3 compares the upper bound with the measured error; Table 3 lists the first twenty-five cases and reports the number of correct base-ten digits, which is visualized in Figure 10.

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5. Conditional Estimates Involving the Riemann Hypothesis

The role of the Riemann Hypothesis in this paper is limited to conditional estimates. In particular, we rely on the classical error term for the Chebyshev function $\vartheta(x)$ that holds under the assumption of the Riemann Hypothesis. These bounds sharpen the asymptotic relation between $\log Q_n$ and p_{n+1}, providing a square-root scale refinement beyond the unconditional $O(\log\log p_{n+1})$ term. It is important to stress that such results are applications of known equivalences for $\vartheta(x)$ and do not constitute new evidence toward the truth or falsity of the hypothesis.

$$\vartheta(x) = x + O\left(\sqrt{x}\log^2 x\right), \tag{9}$$

see [6] and the explicit bounds in [7][8]. Combining Equation (6) with Equation (9) gives

$$\log Q_n = p_{n+1} + O\left(\sqrt{p_{n+1}}\log^2 p_{n+1}\right). \tag{10}$$

The statements in this section are applications of known equivalences for ϑ and do not constitute new evidence for the hypothesis. Figures 6 and 5 provide a numerical narrative consistent with the conditional refinement.

6. Prime-Driven Egyptian and Engel Expansions

Every $x \in (0,1)$ has a unique Engel expansion $x = \frac{1}{b_1} + \frac{1}{b_1 b_2} + \cdots$ with b_k nondecreasing [9]. We apply the algorithm to the fractional part of ρ and record the first dozen terms in Table 2. Figure 9 displays the coefficients. For comparison, we consider the prime Engel series $\sum_{k\geq 1}(p_1 \cdots p_k)^{-1}$ as in [10]; partial sums appear in Figure 4. Related prime-structured continued fractions were explored computationally in [11].

7. Computational Methods, Validation, and Reproducibility

Our computations implement Equation (2) directly and evaluate the infinite continued fraction bottom-up at high precision. We cross-check the first twenty convergents and verify the recurrences symbolically [1]. The file rho_high_precision.txt lists ρ to one thousand decimals. An *anonymized reproducibility bundle* accompanies this submission in the directory code/, which includes a minimal script to regenerate the key tables and a subset of the figures. A public archival link will be attached in the camera-ready version consistent with the journal's practice.

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8. Results and Discussion

Figures 1–10 summarize the numerical behavior. Figure 1 compares $\log Q_n$ with $\vartheta(p_{n+1})$; Figure 2 shows their difference; Figure 3 compares actual errors with the classical upper bound from [3]. Figure 4 and Figure 9 contrast two prime-driven denominator growth mechanisms [9][10]. Figure 5 and Figure 6 reflect the proximity between $\log Q_n$ and $\vartheta(p_{n+1})$ suggested by [4][5]; Figure 8 illustrates the settling of the irrationality exponent near two; Figure 10 reports decimal precision. In contrast with [11], which emphasizes computation for several prime-based patterns, the present work separates elementary bounds that track ϑ from error estimates that are intrinsic to the continued fraction, thereby clarifying which phenomena are robust across models and which are artifacts of a specific construction.

Tables and Figures with Cross-References

Table 1 lists the first twenty convergents; Table 2 lists twelve Engel coefficients; Table 3 audits errors. Figures are placed at the end for readability.

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n	P_n	Q_n	a_n **(partial)**
1	7	3	3
2	37	16	5
3	266	115	7
4	2963	1281	11
5	38785	16768	13
6	662308	286337	17
7	12622637	5457171	19
8	290982959	125801270	23
9	8451128448	3653694001	29
10	262275964847	113390315301	31
11	9712661827787	4199095360138	37
12	398481410904114	172276300080959	41
13	17144413330704689	7412079998841375	43
14	806185907954024497	348540036245625584	47
15	42744997534894003030	18480034001016997327	53
16	2522761040466700203267	1090670546096248467877	59
17	153931168466003606402317	66549383345872173537824	61
18	10315911048262708329158506	4459899354719531875502085	67
19	732583615595118294976656243	316719403568432635334185859	71
20	53488919849491898241625064245	23124976359850301911271069792	73

Table 1. First twenty convergents P_n/Q_n and partial quotients a_n for the prime-indexed continued fraction.

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k	b_k	$B_k = \prod_{j \le k} b_j$
1	4	4
2	4	16
3	117	1872
4	210	393120
5	548	215429760
6	935	201426825600
7	2128	428636284876800
8	3298	1413642467523686400
9	3434	4854448233476339097600
10	4977	24160588858011739688755200
11	5102	123267324353575895892029030400
12	13894	1712676204568583497523851348377600

Table 2. First twelve Engel terms b_k for the fractional part of ρ and their cumulative products B_k.

n	Q_n	**bound** $1/(a_{n+1}Q_n^2)$	**actual** $\lvert\rho - P_n/Q_n\rvert$	**digits**
1	3	2.222222E-2	2.029660E-2	1
2	16	5.580357E-4	5.367364E-4	3
3	115	6.874033E-6	6.741827E-6	5
4	1281	4.687685E-8	4.634771E-8	7
5	16768	2.092129E-10	2.076385E-10	9
6	286337	6.419356E-13	6.385087E-13	12
7	5457171	1.459947E-15	1.454451E-15	14
8	125801270	2.178873E-18	2.173207E-18	17
9	3653694001	2.416429E-21	2.411649E-21	20
10	113390315301	2.102066E-24	2.098855E-24	23
11	4199095360138	1.383263E-27	1.381570E-27	26
12	172276300080959	7.835748E-31	7.827440E-31	30
13	7412079998841375	3.872772E-34	3.869306E-34	33
14	348540036245625584	1.553169E-37	1.552050E-37	36
15	18480034001016997327	4.962979E-41	4.960016E-41	40
16	1090670546096248467877	1.378107E-44	1.377388E-44	43
17	66549383345872173537824	3.370056E-48	3.368524E-48	47
18	4459899354719531875502085	7.080950E-52	7.078097E-52	51
19	316719403568432635334185859	1.365613E-55	1.365113E-55	54
20	23124976359850301911271069792	2.367066E-59	2.366295E-59	58
21	1827189851831742283625748699427	3.608731E-63	3.607693E-63	62
22	151679882678394459842848413122233	4.883757E-67	4.882530E-67	66
23	13501336748228938668297134516578164	5.655548E-71	5.654316E-71	70
24	1309781344460885445284664896521204141	5.771398E-75	5.770254E-75	74
25	132301417127297658912419451683158196405	5.546689E-79	5.545652E-79	78

Table 3. Error audit for $n \leq 25$: theoretical bound $1/(a_{n+1}Q_n^2)$ versus actual $|\rho - P_n/Q_n|$, with correct decimal digits.

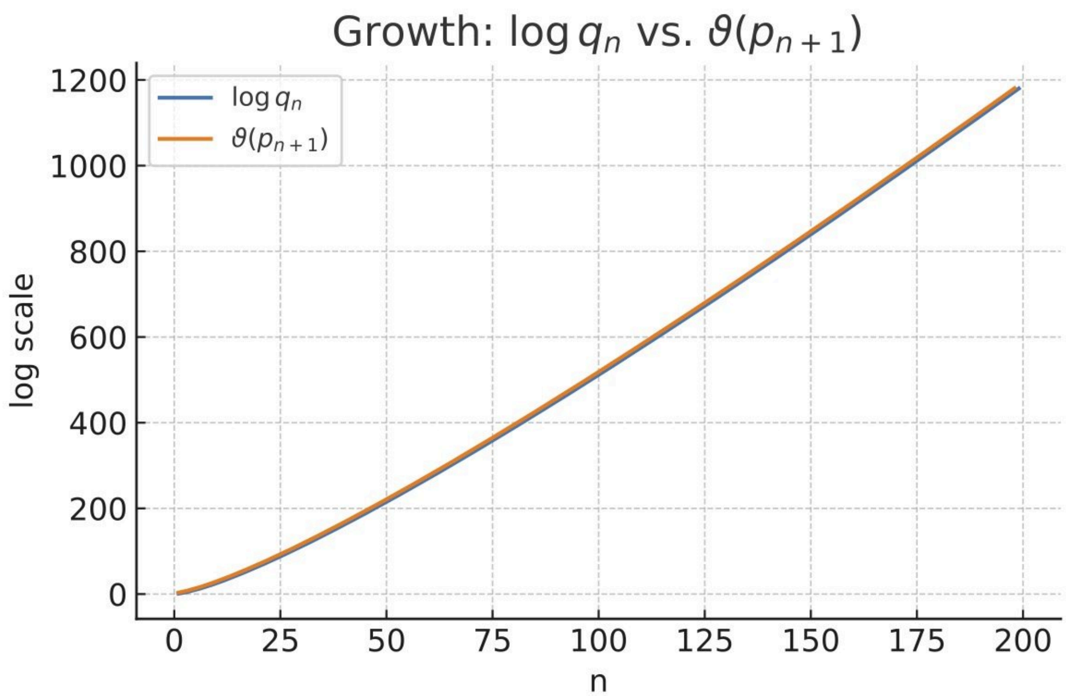

Figure 1. Growth of $\log q_n$ versus $\vartheta(p_{n+1})$.

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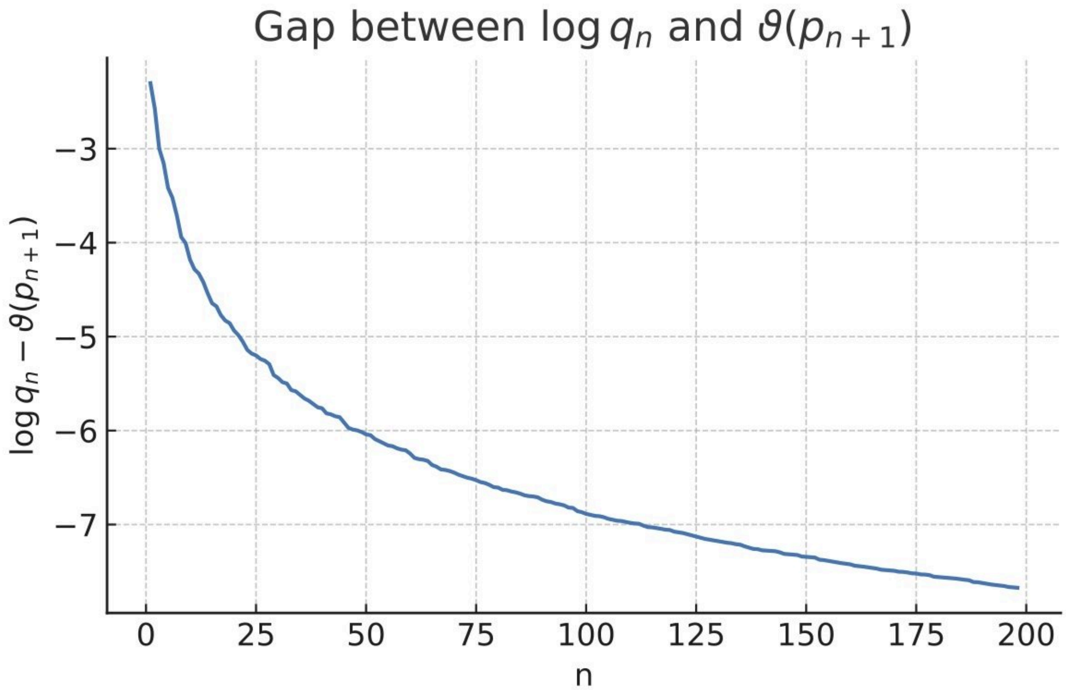

Figure 2. Gap $\log q_n - \vartheta(p_{n+1})$ on a moderate scale.

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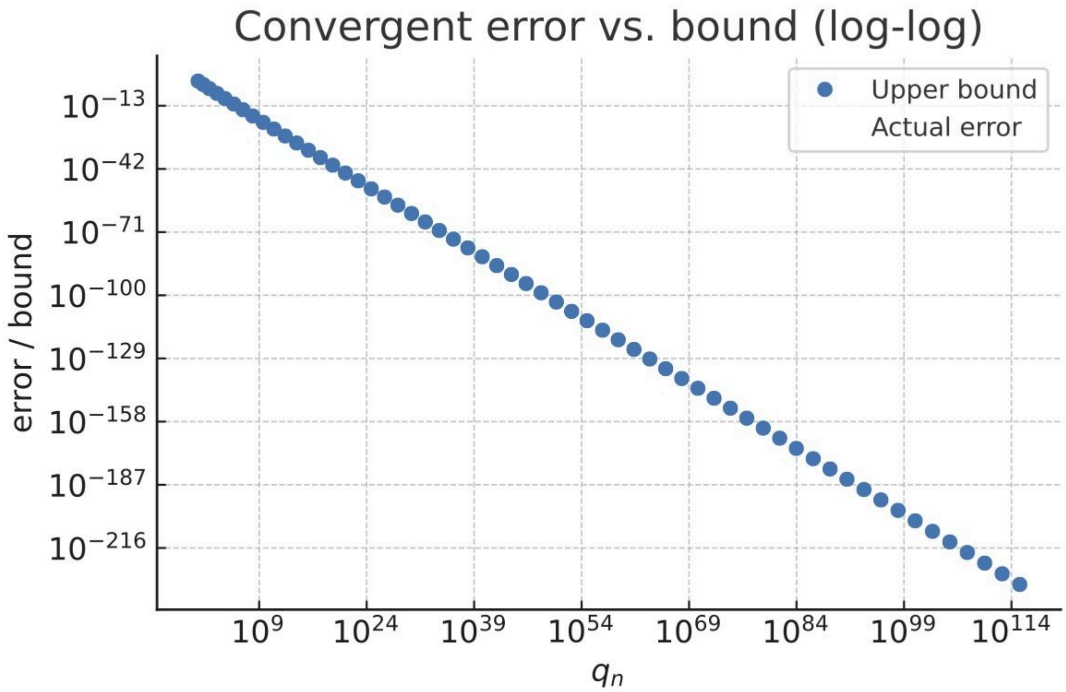

Figure 3. Actual convergent errors versus the classical upper bound $1/(a_{n+1}Q_n^2)$ on a logarithmic scale.

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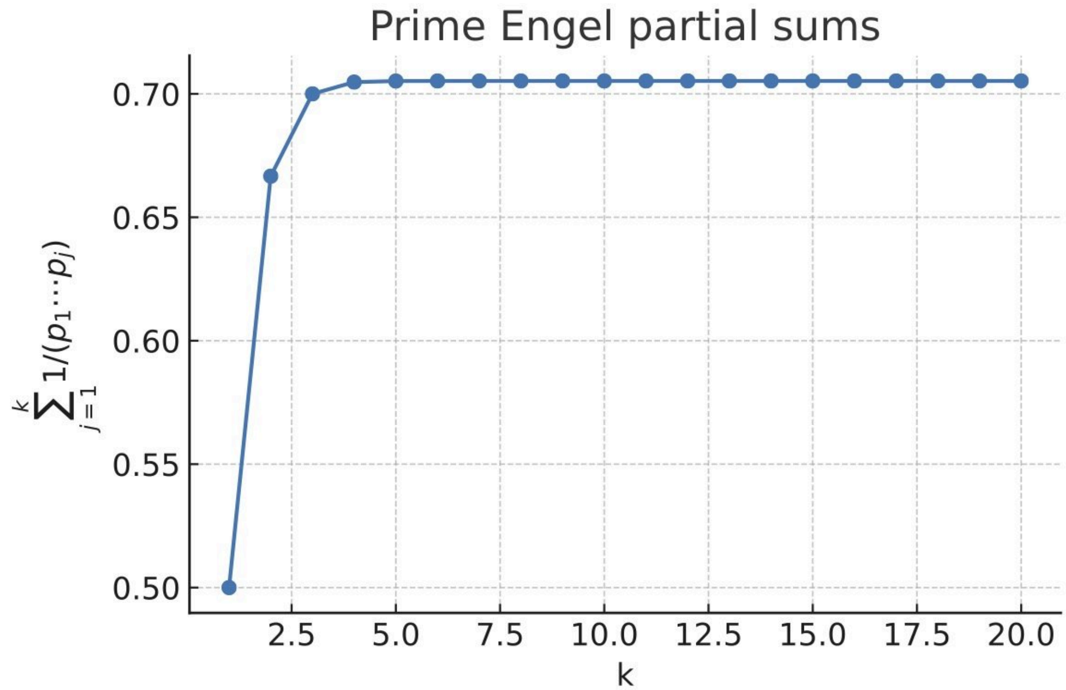

Figure 4. Partial sums of the prime Engel series $\sum_{k\geq 1} (p_1 \cdots p_k)^{-1}$.

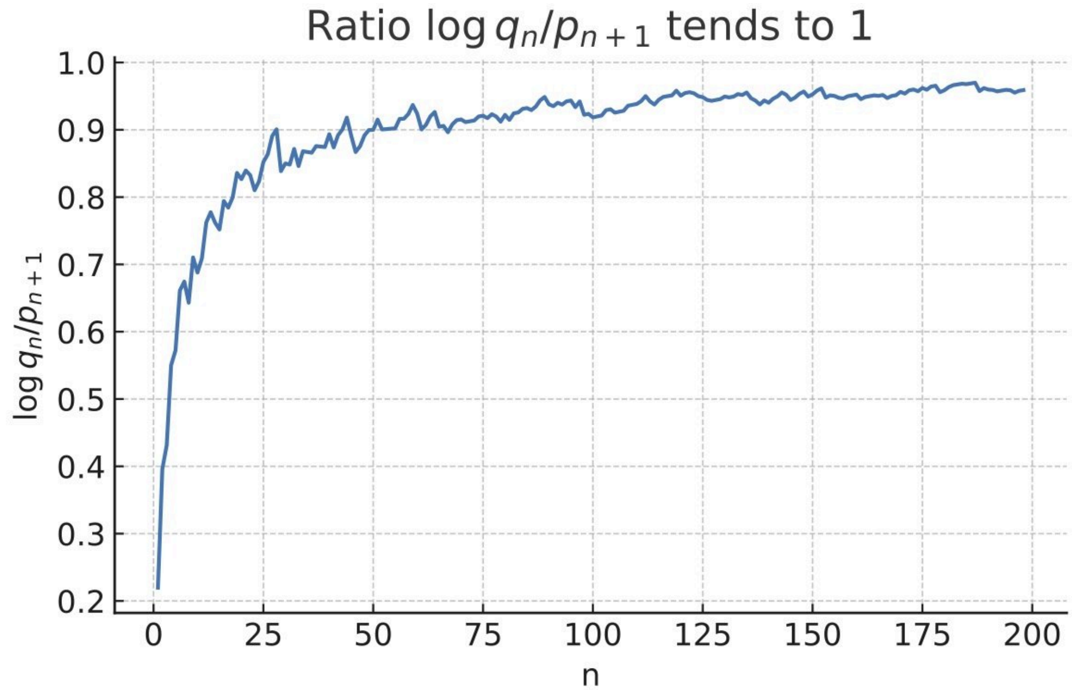

Figure 5. The ratio $\log Q_n/p_{n+1}$ tends to one.

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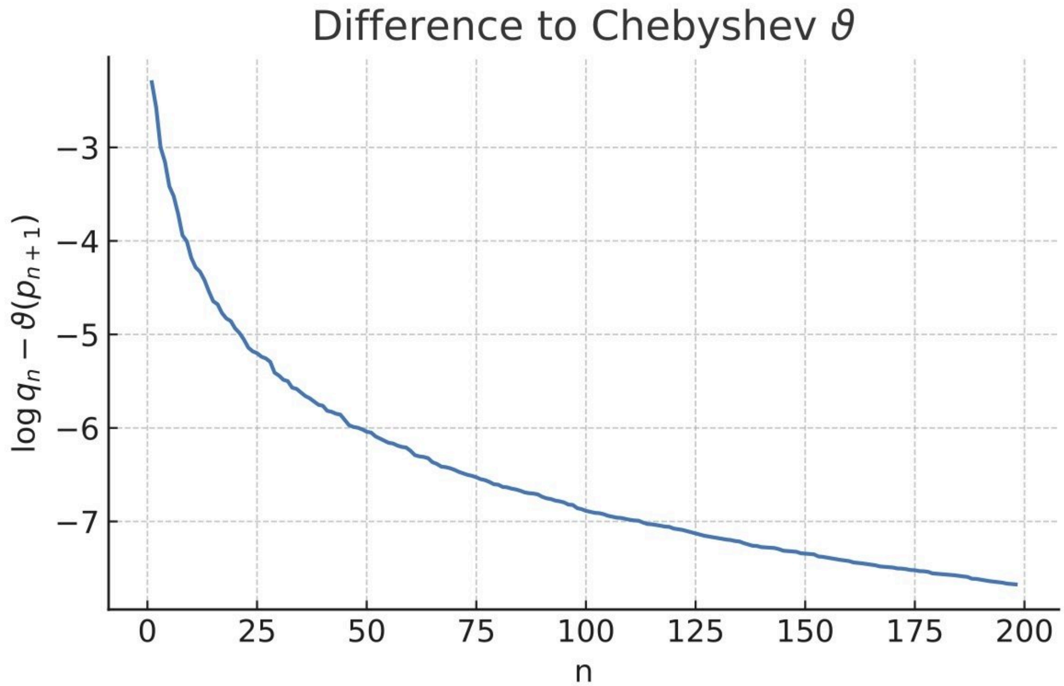

Figure 6. The difference $\log Q_n - \vartheta(p_{n+1})$.

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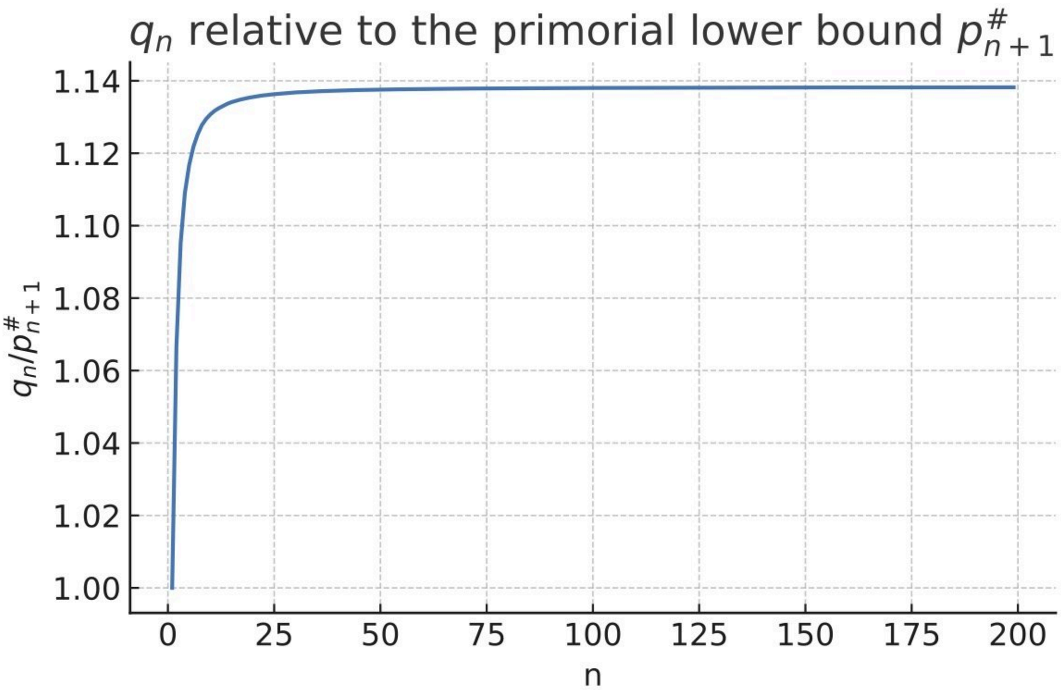

Figure 7. The ratio $Q_n/p^{\#}_{n+1}$ measures how far Q_n exceeds the product lower bound.

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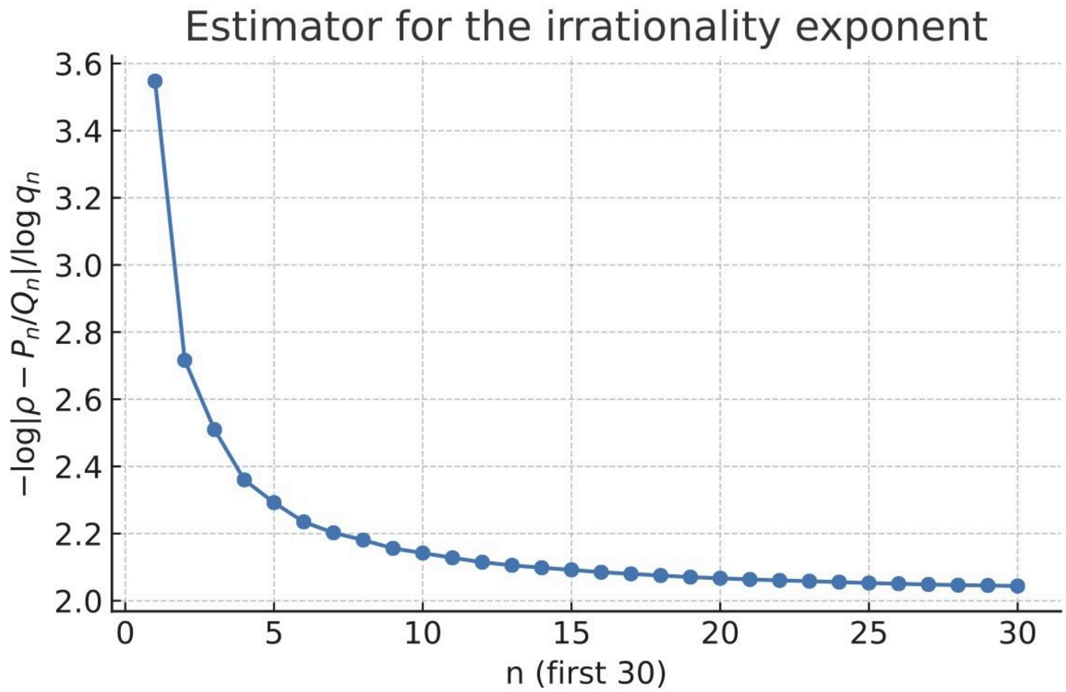

Figure 8. The estimator $-\log|\rho - P_n/Q_n|/\log Q_n$ is consistent with $\mu(\rho) = 2$.

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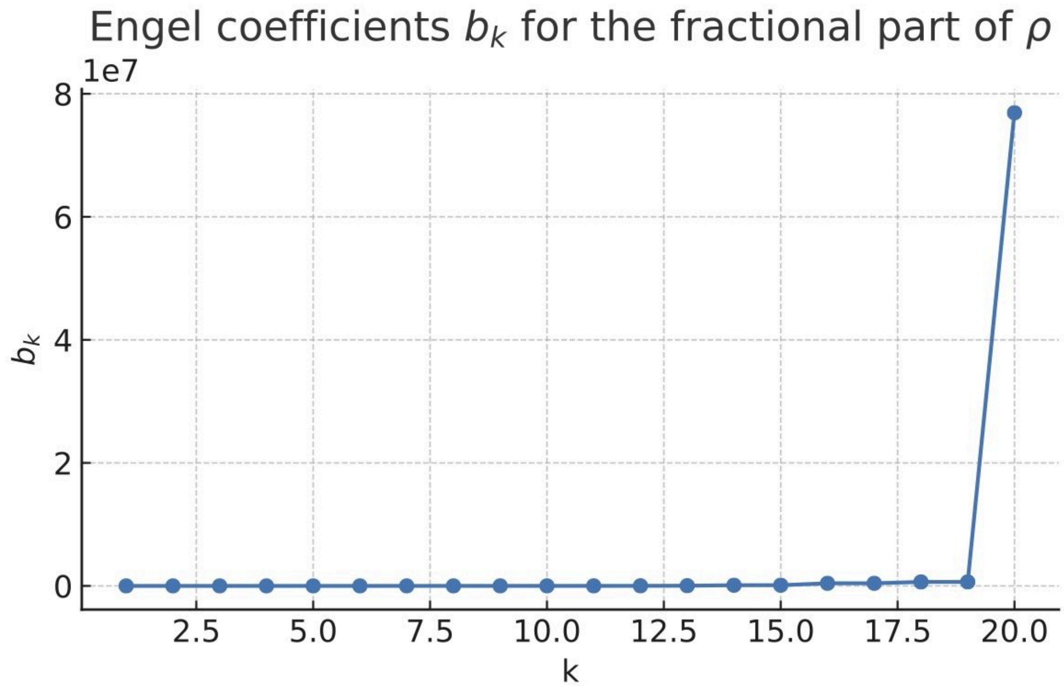

Figure 9. Engel coefficients b_k for the fractional part of ρ.

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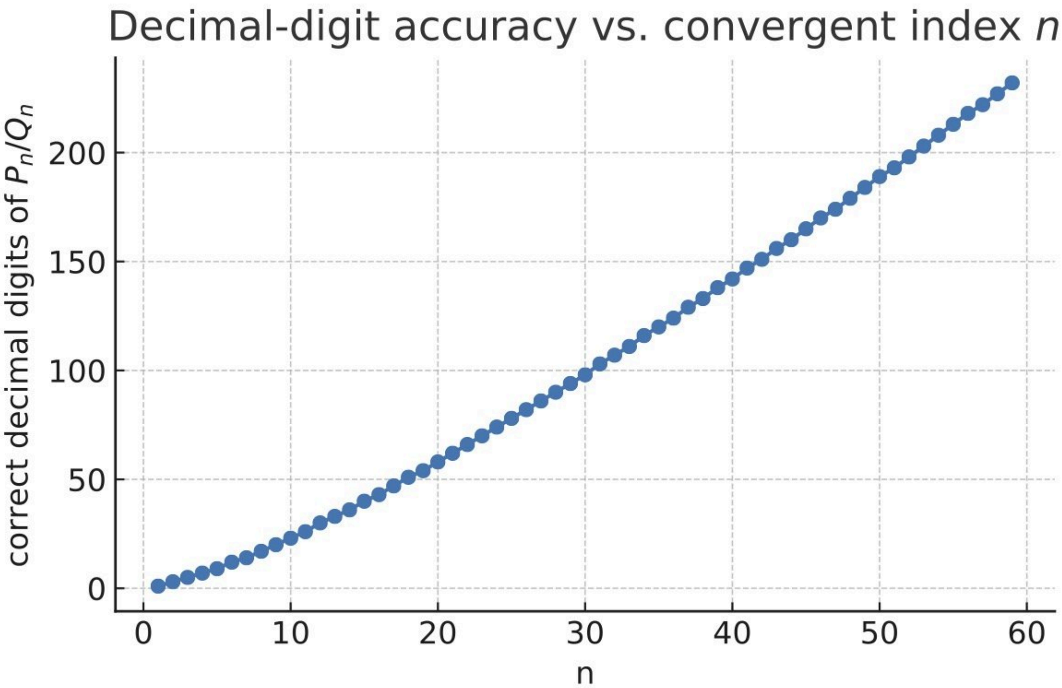

Figure 10. Decimal-digit accuracy of P_n/Q_n versus n.

High-Precision Value of ρ

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rho = 2.

  1-  50: 31303673643358290638395160264178247639668977180325

 51- 100: 63402101244421445647317762722436953220172383281745

101- 150: 30158200723602166215392264873100537277015671222068

151- 200: 54946746553641875807695292751508271179331644728572

201- 250: 14429997248718089692883287803610158268658025452518

251- 300: 75959989587338751034270027785201911407970444026598

301- 350: 54968883397536235101296872043310332602431368987056

351- 400: 13712460765398960478704029682329800625385198689759

401- 450: 32966072061038015630107111205800232441470113110012

451- 500: 44249363921959250340123425818345639722631066987774

501- 550: 26003859722732393725224303567167114519346803236594

551- 600: 59599609957495635296346859696527563391292255481382

601- 650: 42452823629299976568199482983987394850487534226537
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651- 700: 34722161752661998377073949321591394060873758968063

701- 750: 50717251666928278867779875481370054294397482926188

751- 800: 16988987127052431869477725146807687459967819147955

801- 850: 30760521001893498316317819593191534212590367436090

851- 900: 63056632261007500643033737914774608051975643359577

901- 950: 51085919271779161718793444828412312892367479007849

951-1000: 41406876995540006973571769556635649100449963499117
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Statements and Declarations

Data and Code Availability

An anonymized reproduction bundle is included in code/. A public archival link will be added in the camera-ready version.

References

1. [a, b, c, d] *Khinchin AY (1997). Continued Fractions. New York: Dover.*
2. [a, b] *Olds CD (1963). Continued Fractions. Washington, D.C.: Mathematical Association of America.*
3. [a, b, c, d, e, f] *Rockett AM, Süsz P (1992). Continued Fractions. Singapore: World Scientific.*
4. [a, b, c] *Montgomery HL, Vaughan RC (2007). Multiplicative Number Theory I: Classical Theory. Cambridge: Cambridge University Press.*
5. [a, b, c] *Tenenbaum GT (2015). Introduction to Analytic and Probabilistic Number Theory. 3rd ed. Providence, RI: American Mathematical Society.*
6. [a, b] *Titchmarsh EC (1986). The Theory of the Riemann Zeta-Function. 2nd ed. Oxford: Oxford University Press.*
7. [a, b] *Schoenfeld L (1976). "Sharper Bounds for the Chebyshev Functions $\theta(x)$ and $\psi(x)$ II." Math Comp.* ***30****(134):337–360.*
8. [a, b] *Rosser JB, Schoenfeld L (1962). "Approximate Formulas for Some Functions of Prime Numbers." Illinois J Math.* ***6****:64–94.*
9. [a, b, c] *Hone ANW, Varona JL (2018). "Engel Expansions and a Farey Tree." J Integer Seq.* ***21****:Article 18.2.5.*
10. [a, b, c] *Eppstein D (1995). "Ten Algorithms for Egyptian Fractions." Mathematica Educ Res.* ***4****(2):5–15.*

 doi.org/10.32388/EUWD02

11. [a, b, c, d, e] *Wolf M (2010). "Continued Fractions Constructed from Prime Numbers." arXiv:1003.4015.*

12. [^] *Hardy GH, Wright EM (2008). An Introduction to the Theory of Numbers. 6th ed. Oxford: Oxford University Press.*

13. [^] *Mertens FM (1874). "Ein Beitrag zur analytischen Zahlentheorie" [A Contribution to Analytic Number Theory]. J Reine Angew Math [Journal for Pure and Applied Mathematics].* ***78****:46–62.*

Declarations

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