Research Article

Constructing a Set of Kronecker-Pauli Matrices

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In quantum physics, the choice of basis is crucial for formulation. The generalization of the Pauli matrices via the Kronecker product, known as Pauli strings, is typically restricted to 2^n dimensional systems. This paper explores extending this generalization to N-dimensional systems, where N is a prime integer, to construct $N \times N$-Kronecker-Pauli matrices. We begin by examining the specific cases of 3×3 and 5×5 Kronecker-Pauli matrices, with the goal of the purpose constructing a set of $N \times N$-Kronecker-Pauli matrices for any prime integer N. Another possible method for constructing a set of Kronecker-Pauli matrices is discussed.

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1. Introduction

In quantum physics the choice of basis for formulation is important. For higher level system, for having a basis, it is normal the generalization of the Pauli matrices by tensor or Kronecker product

$$\left(\boldsymbol{\sigma}_{j_1} \otimes \boldsymbol{\sigma}_{j_2} \otimes \ldots \boldsymbol{\sigma}_{j_n}\right)_{j_1, j_2, \ldots, j_n = 0,1,2,3}$$

where $\boldsymbol{\sigma}_0$ is the 2×2-unit matrix and

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices.

However the generalization in this sense applies only to 2^n level systems. These matrices, known as Pauli strings, are referred in this paper to as $2^n \times 2^n$-KPMs (Kronecker-Pauli matrices). The work in [1] extended some of these matrices' properties [2] to three-dimensional systems, leading to the 3×3-

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KPMs and offering a path to generalizing this for any dimension. In this road to generalization it has been demonstrated that for any integers p and q, $p, q > 1$, the tensor product of set of $p \times p$-KPMs with set of $q \times q$-KPMs is a set of $pq \times pq$-KPMs [1]. That reduces the problem to the construction of $N \times N$-KPMs with N a prime integer, $N > 2$.

The set $\mathcal{K}_3$ of 3×3-KPMs which are not traceless, does not form a group which excludes it from being considered as a Pauli group. Neverthless, the set of traceless matrices $\mathcal{K}_3 \times \mathcal{K}_3 \otimes \left\{1, \omega, \omega^2\right\} = \boldsymbol{\tau}\mathcal{K}_3 \otimes \left\{1, \omega, \omega^2\right\}$, for $\boldsymbol{\tau} \in \mathcal{K}_3$ with $\omega = e^{\frac{2i\pi}{3}}$ and $\omega^2 = e^{\frac{4i\pi}{3}}$ forms a group. This group corresponds to the Weyl-Heisenberg group for the three-dimensional case, according to its definition in [3].

The objective of this paper is to demonstrate that the method for constructing a set of 5×5-KPMs described in [1] can be extended to produce a set of $N \times N$-KPMs for any prime integer N. In other words, we aim to define a set $(\boldsymbol{\Sigma}_k)_{0 \leq k \leq N^2-1}$ of N^2 matrices that satisfy the following properties:

i. $\mathbf{S}_{N \otimes N} = \frac{1}{N} \sum_{i=0}^{N^2-1} \boldsymbol{\Sigma}_i \otimes \boldsymbol{\Sigma}_i$ is the $N \otimes N$-swap operator;
ii. $\boldsymbol{\Sigma}_i^\dagger = \boldsymbol{\Sigma}_i$, for any $i \in \left\{0,\ 1, \ldots, N^2-1\right\}$ (hermiticity);
iii. $\boldsymbol{\Sigma}_i^2 = \mathbf{I}_N$, for any $i \in \left\{0,\ 1, \ldots, N^2-1\right\}$ (square root of the unit);
iv. $\mathrm{Tr}\left(\boldsymbol{\Sigma}_j^\dagger \boldsymbol{\Sigma}_k\right) = N\delta_{jk}$ for any $j, k \in \left\{0,\ 1, \ldots, N^2-1\right\}$ (orthogonality).

We would like to point out that an analogous of the relationship i) of the swap operator or tensor commutation matrix with the KPMs is satisfied by the generalized Gell-Mann matrices and the unit matrix [4].

According to ii),iii) and iv), like the set of the generalized Gell-Mann matrices and the identity, the $N \times N$ -KPMs are elements of the unitary group $U(N)$ and generators of the Lie algebra $u(N)$ of the Lie group $U(N)$ of N-dimensional unitary matrices (See for example [5]).

For $\boldsymbol{\Sigma} \in \mathcal{K}_\mathcal{N}$,

i. the basis $\boldsymbol{\Sigma}\mathcal{K}_\mathcal{N} = (\boldsymbol{\Sigma}\boldsymbol{\Sigma}_i)_{0 \leq i \leq N^2-1}$ is an unitary basis, containing the identity matrix $\mathbf{I}$, and all elements, except the identity, are traceless;
ii. the elements of $\boldsymbol{\Sigma}\mathcal{K}_\mathcal{N}$ are mutually orthogonals.

Thus, the elements of $\boldsymbol{\Sigma}\mathcal{K}_\mathcal{N}$ satisfy the general properties required to be a matrix basis which is used for the Bloch vector decomposition of qudits [6].

As unitary matrices, the KPMs could serve as quantum gates in 1-qudit quantum circuit. For instance, three gates are defined as elementary gates [7][8] for 1-qutrit quantum circuit:

$$\mathbf{X}^{(01)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \mathbf{X}^{(02)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \mathbf{X}^{(12)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

These are among the 3×3-KPMs.

Given these considerations, it is clear that $N \times N$-KPMs merit further study.

The paper is organized as the following. In the second section a study of the 3×3-KPMs is given in comparing them with the matrices of the Weyl operator basis. In the third section we will expose the methods for constructing a set of KPMs. The fourth section is for the discussion.

2. 3×3-Kronecker-Pauli matrices

2.1. Weyl Operator Basis

In this subsection, we present what is Weyl operator basis (See for example, [9][10][11]) in order to show its relationship with the 3×3-KPMs in the case of 3-dimension, the qutrit case.

Definition 1. *The following d^2 operators*

$$\mathbf{U}_{nm} = \sum_{k=0}^{d-1} e^{\frac{2i\pi}{d}kn} |k\rangle \langle (k+m) \mathrm{mod} d|, n, m = 0, 1, 2, \ldots, d-1 \tag{2.1}$$

are called Weyl operators.

For the case of 3-dimension the matrices of the Weyl operators are the following $\mathbf{U}_{00} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$\mathbf{U}_{01} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \mathbf{U}_{02} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{U}_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \mathbf{U}_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix}, \mathbf{U}_{12} = \begin{pmatrix} 0 & 0 & 1 \\ \omega & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix},$$

$$\mathbf{U}_{20} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \mathbf{U}_{21} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega^2 \\ \omega & 0 & 0 \end{pmatrix}, \mathbf{U}_{22} = \begin{pmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}.$$

2.2. 3×3-KPMs and the Weyl Operator Basis

In this subsection we compare the 3×3-Kronecker-Pauli matrices, formed by the cubic roots of unit that are inverse-symmetric matrices

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$$\boldsymbol{\tau}_1=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\boldsymbol{\tau}_2=\begin{pmatrix}1&0&0\\0&0&\omega\\0&\omega^2&0\end{pmatrix},\boldsymbol{\tau}_3=\begin{pmatrix}1&0&0\\0&0&\omega^2\\0&\omega&0\end{pmatrix},$$
$$\boldsymbol{\tau}_4=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},\boldsymbol{\tau}_5=\begin{pmatrix}0&0&\omega\\0&1&0\\\omega^2&0&0\end{pmatrix},\boldsymbol{\tau}_6=\begin{pmatrix}0&0&\omega^2\\0&1&0\\\omega&0&0\end{pmatrix},$$
$$\boldsymbol{\tau}_7=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\boldsymbol{\tau}_8=\begin{pmatrix}0&\omega&0\\\omega^2&0&0\\0&0&1\end{pmatrix},\boldsymbol{\tau}_9=\begin{pmatrix}0&\omega^2&0\\\omega&0&0\\0&0&1\end{pmatrix}.$$

with the matrices of the Weyl operator basis.

For $\boldsymbol{\tau}_k$, $k=1,2,\ldots,9$, we can check that the set $\boldsymbol{\tau}_k\mathcal{K}_3$ contains the unit matrix and is equal to the set of the matrices in the Weyl operator basis up to the phases $\omega=e^{\frac{2i\pi}{3}}$ and $\omega^2=e^{\frac{4i\pi}{3}}$. For example ($k=1$),

$\boldsymbol{\tau}_1\boldsymbol{\tau}_1=\mathbf{U}_{00},\boldsymbol{\tau}_1\boldsymbol{\tau}_2=\mathbf{U}_{20},\boldsymbol{\tau}_1\boldsymbol{\tau}_3=\mathbf{U}_{10},\boldsymbol{\tau}_1\boldsymbol{\tau}_4=\mathbf{U}_{02},\boldsymbol{\tau}_1\boldsymbol{\tau}_5=\omega\mathbf{U}_{12},\boldsymbol{\tau}_1\boldsymbol{\tau}_6=\omega\mathbf{U}_{22}$

$\boldsymbol{\tau}_1\boldsymbol{\tau}_7=\mathbf{U}_{01},\boldsymbol{\tau}_1\boldsymbol{\tau}_8=\omega^2\mathbf{U}_{11},\boldsymbol{\tau}_1\boldsymbol{\tau}_9=\omega^2\mathbf{U}_{21}$

Therefore, in multiplying the Weyl operators by an element of the set $\mathcal{K}_3$ of the 3×3-KPMs, for example $\boldsymbol{\tau}_4$, which is inverse of itself, we have the elements of $\mathcal{K}_3$ up to phase factor. It follows that

$$\mathcal{K}_3=\Bigg\{\frac{\sqrt{3}}{\left[\mathrm{Tr}\left((\boldsymbol{\tau}_4\mathbf{U}_{00})^2\right)\right]^{\frac{1}{2}}}\boldsymbol{\tau}_4\mathbf{U}_{00},\frac{\sqrt{3}}{\left[\mathrm{Tr}\left((\boldsymbol{\tau}_4\mathbf{U}_{01})^2\right)\right]^{\frac{1}{2}}}\boldsymbol{\tau}_4\mathbf{U}_{01},\frac{\sqrt{3}}{\left[\mathrm{Tr}\left((\boldsymbol{\tau}_4\mathbf{U}_{02})^2\right)\right]^{\frac{1}{2}}}\boldsymbol{\tau}_4\mathbf{U}_{02},$$
$$\frac{\sqrt{3}}{\left[\mathrm{Tr}\left((\boldsymbol{\tau}_4\mathbf{U}_{10})^2\right)\right]^{\frac{1}{2}}}\boldsymbol{\tau}_4\mathbf{U}_{10},\frac{\sqrt{3}}{\left[\mathrm{Tr}\left((\boldsymbol{\tau}_4\mathbf{U}_{11})^2\right)\right]^{\frac{1}{2}}}\boldsymbol{\tau}_4\mathbf{U}_{11},\frac{\sqrt{3}}{\left[\mathrm{Tr}\left((\boldsymbol{\tau}_4\mathbf{U}_{12})^2\right)\right]^{\frac{1}{2}}}\boldsymbol{\tau}_4\mathbf{U}_{12},\frac{\sqrt{3}}{\left[\mathrm{Tr}\left((\boldsymbol{\tau}_4\mathbf{U}_{20})^2\right)\right]^{\frac{1}{2}}}\boldsymbol{\tau}_4\mathbf{U}_{20},$$
$$\frac{\sqrt{3}}{\left[\mathrm{Tr}\left((\boldsymbol{\tau}_4\mathbf{U}_{21})^2\right)\right]^{\frac{1}{2}}}\boldsymbol{\tau}_4\mathbf{U}_{21},\frac{\sqrt{3}}{\left[\mathrm{Tr}\left((\boldsymbol{\tau}_4\mathbf{U}_{22})^2\right)\right]^{\frac{1}{2}}}\boldsymbol{\tau}_4\mathbf{U}_{22}\Bigg\}$$

with $\frac{1}{\sqrt{3}}\left[\mathrm{Tr}\left((\boldsymbol{\tau}_4\mathbf{U}_{00})^2\right)\right]^{\frac{1}{2}}$, $\frac{1}{\sqrt{3}}\left[\mathrm{Tr}\left((\boldsymbol{\tau}_4\mathbf{U}_{01})^2\right)\right]^{\frac{1}{2}}$, ... are the phase factors.

Thus, we can reconstruct the set of the 3×3-KPMs from the 3-dimensional Weyl operators.

3. Constructing a set of KPMs

Definition 2. *A set $\mathcal{K}_d=(\boldsymbol{\Sigma}_k)_{0\le k\le d^2-1}$ of d^2 matrices that satisfy the following properties:*

i. $\mathbf{S}_{d\otimes d}=\frac{1}{d}\sum_{i=0}^{d^2-1}\boldsymbol{\Sigma}_i\otimes\boldsymbol{\Sigma}_i$ *is the $d\otimes d$-swap operator;*

ii. $\boldsymbol{\Sigma}_i^\dagger=\boldsymbol{\Sigma}_i$, *for any $i\in\{0,\ 1,\ldots,d^2-1\}$ (hermiticity);*

iii. $\boldsymbol{\Sigma}_i^2=\mathbf{I}_d$, *for any $i\in\{0,\ 1,\ldots,d^2-1\}$ (square root of the unit);*

iv. *Tr*$\left(\boldsymbol{\Sigma}_j^\dagger\boldsymbol{\Sigma}_k\right)=d\delta_{jk}$ *for any $j,k\in\{0,\ 1,\ldots,d^2-1\}$ (orthogonality).*

is called a set of $d\times d$-KPMs.

Proposition 1. *If $\mathcal{K}_p$ and $\mathcal{K}_q$ are sets of KPMs, then $\mathcal{K}_p\otimes\mathcal{K}_q$ is also a set of KPMs.*

Proof. Suppose that $\mathcal{K}_p = (\mathbf{\Lambda}_j)_{0\le j\le p^2-1}$ and $\mathcal{K}_q = (\mathbf{\Sigma}_k)_{0\le k\le q^2-1}$ are sets of KPMs. Let $\mathbf{\Lambda}_j$ and $\mathbf{\Sigma}_k$ be respectively elements of $\mathcal{K}_p$ and $\mathcal{K}_q$, then $(\mathbf{\Lambda}_j \otimes \mathbf{\Sigma}_k)^2 = \mathbf{\Lambda}_j^2 \otimes \mathbf{\Sigma}_k^2 = \mathbf{I}_p \otimes \mathbf{I}_q = \mathbf{I}_{pq}$, $(\mathbf{\Lambda}_j \otimes \mathbf{\Sigma}_k)^\dagger = \mathbf{\Lambda}_j^\dagger \otimes \mathbf{\Sigma}_k^\dagger = \mathbf{\Lambda}_j \otimes \mathbf{\Sigma}_k$.

Let $\mathbf{\Lambda}$ and $\mathbf{\Sigma}$ be respectively elements of $\mathcal{K}_p$ and $\mathcal{K}_q$.

$\mathrm{Tr}\Big((\mathbf{\Lambda}_j \otimes \mathbf{\Sigma}_k)^\dagger (\mathbf{\Lambda} \otimes \mathbf{\Sigma})\Big)$ $=\mathrm{Tr}(\mathbf{\Lambda}_j\mathbf{\Lambda} \otimes \mathbf{\Sigma}_k\mathbf{\Sigma})$ $=\mathrm{Tr}\Big(\mathbf{\Lambda}_j^\dagger\mathbf{\Lambda}\Big)\mathrm{Tr}\Big(\mathbf{\Sigma}_k^\dagger\mathbf{\Sigma}\Big)$. If $\mathbf{\Lambda}_j \otimes \mathbf{\Sigma}_k = \mathbf{\Lambda} \otimes \mathbf{\Sigma}$, then Tr $\Big((\mathbf{\Lambda}_j \otimes \mathbf{\Sigma}_k)^\dagger (\mathbf{\Lambda} \otimes \mathbf{\Sigma})\Big) = pq$. If $\mathbf{\Lambda}_j \otimes \mathbf{\Sigma}_k \neq \mathbf{\Lambda} \otimes \mathbf{\Sigma}$, then $\mathbf{\Lambda}_j \neq \mathbf{\Lambda}$ or $\mathbf{\Sigma}_k \neq \mathbf{\Sigma}$ and Tr $\Big((\mathbf{\Lambda}_j \otimes \mathbf{\Sigma}_k)^\dagger (\mathbf{\Lambda} \otimes \mathbf{\Sigma})\Big) = 0$. That proves iv).

To finish, as $\mathbf{S}_{p\otimes p} = \frac{1}{p}\sum_{j=0}^{p^2-1} \mathbf{\Lambda}_j \otimes \mathbf{\Lambda}_j$ and $\mathbf{S}_{q\otimes q} = \frac{1}{q}\sum_{k=0}^{q^2-1} \mathbf{\Sigma}_k \otimes \mathbf{\Sigma}_k$, then [1]

$$\mathbf{S}_{pq\otimes pq} = \frac{1}{pq}\sum_{j=0}^{p^2-1}\sum_{k=0}^{q^2-1} (\mathbf{\Lambda}_j \otimes \mathbf{\Sigma}_k) \otimes (\mathbf{\Lambda}_j \otimes \mathbf{\Sigma}_k)$$

That proves i). □

According to this Proposition 1, it remains for us to construct the $N \times N$-KPMs, for N prime integer. After that, we will, by Kronecker product, have set of $n \times n$-KPMs for any integer $n, n > 1$.

Example 1. *$\mathcal{K}_2 = \{\boldsymbol{\sigma}_0, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$ set of 2×2-KPMs,*

$\mathcal{K}_3 = \{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\tau}_3, \boldsymbol{\tau}_4, \boldsymbol{\tau}_5, \boldsymbol{\tau}_6, \boldsymbol{\tau}_7, \boldsymbol{\tau}_8, \boldsymbol{\tau}_9\}$ set of 3×3-KPMs. Then, $\mathcal{K}_2 \otimes \mathcal{K}_3 = (\boldsymbol{\sigma}_j \otimes \boldsymbol{\tau}_k)_{0\le j\le 3, 1\le k\le 9}$ is set of 6×6-KPMs.

3.1. Inverse-symmetric Matrices

To construct a set of $N \times N$-KPMs, with N prime integer, we first introduce the concept of inverse-symmetric matrices and some of their properties [1].

Definition 3. *Let us call inverse-symmetric an invertible complex matrix $\mathbf{A} = \Big(A_j^i\Big)$ such that*

$$A_i^j = \frac{1}{A_j^i}\,\mathrm{if} A_j^i \neq 0$$

If a permutation matrix is symmetric, then it is inverse-symmetric.

Proposition 2. *The Kronecker product of two inverse-symmetric matrices is itself inverse-symmetric.*

Proposition 3. *For any $n \times n$ inverse-symmetric matrix $\mathbf{A}$, with only n non zero elements, $\mathbf{A}^2 = \mathbf{I}_n$.*

Following this Proposition, taking inverse-symmetric matrices built with the N-th roots of the unit is an appropriate choice to satisfy the properties ii) and iii) of the $N \times N$-KPMs.

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3.2. Constructing a set of $N \times N$-KPMs for N prime integer

Example 2. *To construct a set of* 5×5*-KPMs, we begin by decomposing* 5×5*-ones matrix into the sum of five symmetric permutation matrices, each having only one entry of unit ("1") in the diagonal, as the follows:*

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then, for each term of this sum, by replacing the "ones" by the quintic roots of the unit: 1, $\eta = e^{\frac{2i\pi}{5}}$, $\eta^2 = e^{\frac{4i\pi}{5}}$, $\eta^3 = e^{\frac{6i\pi}{5}}$, $\eta^4 = e^{\frac{8i\pi}{5}}$, in keeping the only unit in the diagonal and in keeping that the matrices are inverse-symmetrics, we have additional four matrices i.e five matrices with the matrix taken from the sum above. Thus, we have twenty five inverse-symmmetric matrices.

$$\boldsymbol{\chi}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \boldsymbol{\chi}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 0 & \eta^3 & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \end{pmatrix}, \boldsymbol{\chi}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 0 & \eta^4 & 0 \\ 0 & 0 & \eta & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{\chi}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 0 & \eta & 0 \\ 0 & 0 & \eta^4 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \end{pmatrix}, \boldsymbol{\chi}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & \eta^2 & 0 \\ 0 & 0 & \eta^3 & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{\chi}_6 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \boldsymbol{\chi}_7 = \begin{pmatrix} 0 & 0 & \eta & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 0 & \eta^3 & 0 \end{pmatrix}, \boldsymbol{\chi}_8 = \begin{pmatrix} 0 & 0 & \eta^3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & \eta^4 & 0 \end{pmatrix},$$

$$\boldsymbol{\chi}_9 = \begin{pmatrix} 0 & 0 & \eta^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 0 & \eta & 0 \end{pmatrix}, \boldsymbol{\chi}_{10} = \begin{pmatrix} 0 & 0 & \eta^4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 0 & \eta^2 & 0 \end{pmatrix},$$

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$$\boldsymbol{\chi}_{11}=\begin{pmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\end{pmatrix},\boldsymbol{\chi}_{12}=\begin{pmatrix}0&0&0&0&\eta\\0&0&0&\eta^2&0\\0&0&1&0&0\\0&\eta^3&0&0&0\\\eta^4&0&0&0&0\end{pmatrix},\boldsymbol{\chi}_{13}=\begin{pmatrix}0&0&0&0&\eta^3\\0&0&0&\eta&0\\0&0&1&0&0\\0&\eta^4&0&0&0\\\eta^2&0&0&0&0\end{pmatrix},$$

$$\boldsymbol{\chi}_{14}=\begin{pmatrix}0&0&0&0&\eta^2\\0&0&0&\eta^4&0\\0&0&1&0&0\\0&\eta&0&0&0\\\eta^3&0&0&0&0\end{pmatrix},\boldsymbol{\chi}_{15}=\begin{pmatrix}0&0&0&0&\eta^4\\0&0&0&\eta^3&0\\0&0&1&0&0\\0&\eta^2&0&0&0\\\eta&0&0&0&0\end{pmatrix},$$

$$\boldsymbol{\chi}_{16}=\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix},\boldsymbol{\chi}_{17}=\begin{pmatrix}0&\eta&0&0&0\\\eta^4&0&0&0&0\\0&0&0&0&\eta^2\\0&0&0&1&0\\0&0&\eta^3&0&0\end{pmatrix},\boldsymbol{\chi}_{18}=\begin{pmatrix}0&\eta^3&0&0&0\\\eta^2&0&0&0&0\\0&0&0&0&\eta\\0&0&0&1&0\\0&0&\eta^4&0&0\end{pmatrix},$$

$$\boldsymbol{\chi}_{19}=\begin{pmatrix}0&\eta^2&0&0&0\\\eta^3&0&0&0&0\\0&0&0&0&\eta^4\\0&0&0&1&0\\0&0&\eta&0&0\end{pmatrix},\boldsymbol{\chi}_{20}=\begin{pmatrix}0&\eta^4&0&0&0\\\eta&0&0&0&0\\0&0&0&0&\eta^3\\0&0&0&1&0\\0&0&\eta^2&0&0\end{pmatrix},$$

$$\boldsymbol{\chi}_{21}=\begin{pmatrix}0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\end{pmatrix},\boldsymbol{\chi}_{22}=\begin{pmatrix}0&0&0&\eta&0\\0&0&\eta^2&0&0\\0&\eta^3&0&0&0\\\eta^4&0&0&0&0\\0&0&0&0&1\end{pmatrix},\boldsymbol{\chi}_{23}=\begin{pmatrix}0&0&0&\eta^2&0\\0&0&\eta^4&0&0\\0&\eta&0&0&0\\\eta^3&0&0&0&0\\0&0&0&0&1\end{pmatrix},$$

$$\boldsymbol{\chi}_{24}=\begin{pmatrix}0&0&0&\eta^3&0\\0&0&\eta&0&0\\0&\eta^4&0&0&0\\\eta^2&0&0&0&0\\0&0&0&0&1\end{pmatrix},\boldsymbol{\chi}_{25}=\begin{pmatrix}0&0&0&\eta^4&0\\0&0&\eta^3&0&0\\0&\eta^2&0&0&0\\\eta&0&0&0&0\\0&0&0&0&1\end{pmatrix}.$$

We have verified, with the help of SCILAB free software [1], that the set of these twenty- five matrices constitutes a set of 5×5-KPMs. But we can have another decomposition of the 5×5-ones matrix as the sum of five symmetric permutation matrices with only one unit in the diagonal, namely

$$\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{pmatrix}=\begin{pmatrix}1&0&0&0&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix}+\begin{pmatrix}0&0&0&0&1\\0&1&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\1&0&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix}+$$

$$\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix}+\begin{pmatrix}0&0&1&0&0\\0&0&0&1&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&1\end{pmatrix}$$

If we replace the "ones" in these symmetric permutations matrices with the five quintic roots of the unit, but in keeping that they are inverse symmetric, it is obvious that the properties ii) and iii) of hermiticity and the square root of the unit are satisfied by the twenty-five obtained matrices. It remains for us to study the properties i) and iv) of the relationship with the swap operator and the orthogonality. But to study it, let us consider the general case for any N dimension, with N a prime integer. Instead of studying the relationship between matrices let us study the relationship between operators whose matrices with respect to the standard basis $(|0\rangle, |1\rangle, \ldots, |N^2-1\rangle)$ are the matrices in question.

The following lemma (See, for example, [12]) for the swap operator $S_{N\otimes N}$ should be helpful for this study.

Lemma 1.

$$\mathbf{S}_{N\otimes N} = \sum_{(i,j)} |i\rangle\langle j| \otimes |j\rangle\langle i|$$

In order to make the presentation of the following theorem more shorter we take that the matrices of the operators are their matrices with respect to the standard basis $(|0\rangle, |1\rangle, \ldots, |N^2-1\rangle)$.

Proposition 4. *Let $\mathbf{P}_1$, $\mathbf{P}_2$, ..., $\mathbf{P}_N$ be $N \times N$ operators whose matrices are symmetric permutation matrices with only one unit in the diagonal.*

$\mathbf{\Sigma}_0 = \mathbf{P}_1$ and $\mathbf{\Sigma}_1$, $\mathbf{\Sigma}_2$, ..., $\mathbf{\Sigma}_{N-1}$ are operators whose matrices are obtained in replacing the "ones" in $\mathbf{\Sigma}_0 = \mathbf{P}_1$ by the N-th roots of unit in keeping that they are inverse-symmetric. We do the same to the operators $\mathbf{P}_2$, ..., $\mathbf{P}_N$ in order to have the operators

$\mathbf{\Sigma}_N = \mathbf{P}_2$ and $\mathbf{\Sigma}_{N+1}$, $\mathbf{\Sigma}_{N+2}$, ..., $\mathbf{\Sigma}_{2N-1}$

. . . .

$\mathbf{\Sigma}_{N^2-N} = \mathbf{P}_N$ and $\mathbf{\Sigma}_{N^2-N+1}$, $\mathbf{\Sigma}_{N^2-N+2}$, ..., $\mathbf{\Sigma}_{N^2-1}$

whose matrices are inverse-symmetrics.

If

1. *the sum $\mathbf{P}_1 + \mathbf{P}_2 + \ldots, +\mathbf{P}_N$ is equal to the operator whose matrices is the $N \times N$ ones matrix;*
2. *for any $l \in \{0,\ 1, \ldots, N-1\}$, for any $k, j \in \{lN+1, \ldots, lN+N-1\}$, for any two places in a $N \times N$ -matrix, non symmetrics with respect to the diagonal where the elements of $\mathbf{\Sigma}_k$ are $e^{\frac{2i\pi p_k}{N}}$ and $e^{\frac{2i\pi r_k}{N}}$ and the elements of $\mathbf{\Sigma}_j$ are $e^{\frac{2i\pi p_j}{N}}$ and $e^{\frac{2i\pi r_j}{N}}$ such that*

$$e^{\frac{2i\pi(r_k+p_k)}{N}} \neq e^{\frac{2i\pi(r_j+p_j)}{N}}$$

then

$$\mathbf{S}_{N\otimes N}=\frac{1}{N}\sum_{j=0}^{N^2-1}\mathbf{\Sigma}_j\otimes\mathbf{\Sigma}_j\text{is the-}N\otimes N\text{-swap operators and}\mathrm{Tr}\left(\mathbf{\Sigma}_j^\dagger\mathbf{\Sigma}_k\right)=N\delta_{jk}$$

Proof. Let us take the operator $\mathbf{\Sigma}_j$, with $j\in\{0,1,2,\ldots,N-1\}$. $\mathbf{\Sigma}_j$ can be decomposed as sum of N non-zero elementary operators of the form

$$e^{\frac{2i\pi p_j}{N}}\left|k\right\rangle\left\langle l\right|$$

with $p_j\in\{0,1,2,\ldots,N-1\}$. Thus $\mathbf{\Sigma}_j\otimes\mathbf{\Sigma}_j$ is the sum of N^2 non-zero elementary operators of the form

$$e^{\frac{2i\pi(p_j+r_j)}{N}}\left|k\right\rangle\left\langle l\right|\otimes\left|m\right\rangle\left\langle n\right|$$

with $r_j\in\{0,1,2,\ldots,N-1\}$. If $k=n$ and $l=m$, then

$$e^{\frac{2i\pi(p_j+r_j)}{N}}\left|k\right\rangle\left\langle l\right|\otimes\left|m\right\rangle\left\langle n\right|=\left|k\right\rangle\left\langle l\right|\otimes\left|l\right\rangle\left\langle k\right|$$

due to the inverse-symmetry. A non-zero term of the sum $\sum_{j=0}^{N-1}\mathbf{\Sigma}_j\otimes\mathbf{\Sigma}_j$ is

$$\sum_{j=0}^{N-1}e^{\frac{2i\pi(p_j+r_j)}{N}}\left|k\right\rangle\left\langle l\right|\otimes\left|m\right\rangle\left\langle n\right|=N\left|k\right\rangle\left\langle l\right|\otimes\left|l\right\rangle\left\langle k\right|$$

If $k\neq n$ or $l\neq m$, for the case where $k=l$, then $p_j=0$, because on the diagonal, only one entry is unit and the others are zeros. It follows that

$$e^{\frac{2i\pi(p_j+r_j)}{N}}\left|k\right\rangle\left\langle l\right|\otimes\left|m\right\rangle\left\langle n\right|=e^{\frac{2i\pi r_j}{N}}\left|k\right\rangle\left\langle l\right|\otimes\left|m\right\rangle\left\langle n\right|$$

and in the sum $\sum_{j=0}^{N-1}\mathbf{\Sigma}_j\otimes\mathbf{\Sigma}_j$, there is the following sum

$$\left(\sum_{j=0}^{N-1}e^{\frac{2i\pi r_j}{N}}\right)\left|k\right\rangle\left\langle l\right|\otimes\left|m\right\rangle\left\langle n\right|=0$$

according to the hypothesis (2) and as the sum of the five quintic roots of unit is equal to zero.

For the case where $m=n$, then $r_j=0$

$$e^{\frac{2i\pi(p_j+r_j)}{N}}\left|k\right\rangle\left\langle l\right|\otimes\left|m\right\rangle\left\langle n\right|=e^{\frac{2i\pi p_j}{N}}\left|k\right\rangle\left\langle l\right|\otimes\left|m\right\rangle\left\langle n\right|$$

and in the sum $\sum_{j=0}^{N-1}\mathbf{\Sigma}_j\otimes\mathbf{\Sigma}_j$, there is the following sum

$$\sum_{j=0}^{N-1}e^{\frac{2i\pi p_j}{N}}\left|k\right\rangle\left\langle l\right|\otimes\left|m\right\rangle\left\langle n\right|=0.$$

For the case where $k\neq l$ and $m\neq n$, a term of the sum giving $\mathbf{\Sigma}_j\otimes\mathbf{\Sigma}_j$ is of the form

$$e^{\frac{2i\pi(p_j+r_j)}{N}}\left|k\right\rangle\left\langle l\right|\otimes\left|m\right\rangle\left\langle n\right|$$

then in the sum $\sum_{j=0}^{N-1}\mathbf{\Sigma}_j\otimes\mathbf{\Sigma}_j$, there is the following sum

$$\sum_{j=0}^{N-1}e^{\frac{2i\pi(p_j+r_j)}{N}}\left|k\right\rangle\left\langle l\right|\otimes\left|m\right\rangle\left\langle n\right|$$

which equal to the null operator 0, according to the hypothesis (2).

Then, we can conclude that only the elementary operators which are non null operators in the decomposition of the sum $\sum_{j=0}^{N^2-1}\mathbf{\Sigma}_j\otimes\mathbf{\Sigma}_j$ are the elementary operators of the form $N\left|k\right\rangle\left\langle l\right|\otimes\left|l\right\rangle\left\langle k\right|$. Hence, According to the lemma the first part of the conclusion of the proposition is obtained.

Now, let us move on to the second part. For $j,k\in\{0,1,2,\ldots,N-1\}$ with $j\neq k$, for $p\neq r$, $p,r\in\{0,1,2,\ldots,N\}$, the operator $\mathbf{\Sigma}_j\mathbf{\Sigma}_k$ contains two terms $e^{\frac{2i\pi}{N}(p_j+p_k)}\left|p\right\rangle\left\langle p\right|$ and $e^{\frac{2i\pi}{N}(r_j+r_k)}\left|r\right\rangle\left\langle r\right|$, with $e^{\frac{2i\pi}{N}p_j}$ and $e^{\frac{2i\pi}{N}r_j}$ are elements of the matrix of $\mathbf{\Sigma}_j$. Suppose

$$e^{\frac{2i\pi}{N}(p_j+p_k)}=e^{\frac{2i\pi}{N}(r_j+r_k)}$$

Then,

$$e^{\frac{2i\pi}{N}(p_j-r_j)}=e^{\frac{2i\pi}{N}(r_k-p_k)}$$

However, the elements $e^{\frac{2i\pi}{N}p_j}$ and $e^{\frac{2i\pi}{N}(-r_j)}$ of the matrix of $\mathbf{\Sigma}_j$ are respectively in the same places as the elements $e^{\frac{2i\pi}{N}(-p_k)}$ and $e^{\frac{2i\pi}{N}r_k}$ of the matrix of $\mathbf{\Sigma}_k$. That is in contradiction with the hypothesis (2). Thus, the diagonal of the matrix of $\mathbf{\Sigma}_j^\dagger\mathbf{\Sigma}_k$ is formed by the N-th roots of units. Hence, for $j\neq k$, Tr $\left(\mathbf{\Sigma}_j^\dagger\mathbf{\Sigma}_k\right)=0$.

For $l_1,l_2\in\{0,\ 1,\ldots,N-1\}$, with $l_1\neq l_2$, for $j\in\{l_1N+1,\ldots,l_1N+N-1\}$, $k\in\{l_2N+1,\ldots,l_2N+N-1\}$ it is obvious that all elements in the diagonal of $\mathbf{\Sigma}_j^\dagger\mathbf{\Sigma}_k$ are equals to zero. Thus, $\mathrm{Tr}\left(\mathbf{\Sigma}_j^\dagger\mathbf{\Sigma}_k\right)=0$. $\square$

We can remark that the theorem help us how to build a set of $N\times N$-Kronecker-Pauli matrices, for a prime integer N. Let us take as an example the continuation of the construction of 5×5-KPMs above.

Example 3. *Let us take one by one the permutation matrices terms of the decomposition of the* 5×5*-ones matrix above. For each term, we add four inverse-symmetric matrices obtained in replacing the five units with the quintic roots of the unit, in keeping that they are inverse-symmetric, but according to the hypothesis (2) of the Proposition 4 above. Proposition 3 ensures property iii) of a set of KPMs. To satisfy property ii) of hermiticity, the construction of a family of inverse-symmetric matrices whose elements are N-th roots of the unit is needed. The hypotheses of the Proposition 4 guarantees properties i) and iv).*

$$\mathbf{\Sigma}_1=\begin{pmatrix}1&0&0&0&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix},\quad \mathbf{\Sigma}_2=\begin{pmatrix}1&0&0&0&0\\0&0&\eta&0&0\\0&\eta^4&0&0&0\\0&0&0&0&\eta^2\\0&0&0&\eta^3&0\end{pmatrix},\quad \mathbf{\Sigma}_3=\begin{pmatrix}1&0&0&0&0\\0&0&\eta^2&0&0\\0&\eta^3&0&0&0\\0&0&0&0&\eta^4\\0&0&0&\eta&0\end{pmatrix},$$

$$\mathbf{\Sigma}_4=\begin{pmatrix}1&0&0&0&0\\0&0&\eta^3&0&0\\0&\eta^2&0&0&0\\0&0&0&0&\eta\\0&0&0&\eta^4&0\end{pmatrix},\mathbf{\Sigma}_5=\begin{pmatrix}1&0&0&0&0\\0&0&\eta^4&0&0\\0&\eta&0&0&0\\0&0&0&0&\eta^3\\0&0&0&\eta^2&0\end{pmatrix}$$

$$\mathbf{\Sigma}_6=\begin{pmatrix}0&0&0&0&1\\0&1&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\1&0&0&0&0\end{pmatrix},\quad \mathbf{\Sigma}_7=\begin{pmatrix}0&0&0&0&\eta^4\\0&1&0&0&0\\0&0&0&\eta^3&0\\0&0&\eta^2&0&0\\\eta&0&0&0&0\end{pmatrix},\quad \mathbf{\Sigma}_8=\begin{pmatrix}0&0&0&0&\eta^3\\0&1&0&0&0\\0&0&0&\eta&0\\0&0&\eta^4&0&0\\\eta^2&0&0&0&0\end{pmatrix},$$

$$\mathbf{\Sigma}_9=\begin{pmatrix}0&0&0&0&\eta^2\\0&1&0&0&0\\0&0&0&\eta^4&0\\0&0&\eta&0&0\\\eta^3&0&0&0&0\end{pmatrix},\mathbf{\Sigma}_{10}=\begin{pmatrix}0&0&0&0&\eta\\0&1&0&0&0\\0&0&0&\eta^2&0\\0&0&\eta^3&0&0\\\eta^4&0&0&0&0\end{pmatrix},$$

$$\mathbf{\Sigma}_{11}=\begin{pmatrix}0&0&0&1&0\\0&0&0&0&1\\0&0&1&0&0\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix},\quad \mathbf{\Sigma}_{12}=\begin{pmatrix}0&0&0&\eta^4&0\\0&0&0&0&\eta^3\\0&0&1&0&0\\\eta&0&0&0&0\\0&\eta^2&0&0&0\end{pmatrix},\quad \mathbf{\Sigma}_{13}=\begin{pmatrix}0&0&0&\eta^3&0\\0&0&0&0&\eta\\0&0&1&0&0\\\eta^2&0&0&0&0\\0&\eta^4&0&0&0\end{pmatrix},$$

$$\mathbf{\Sigma}_{14}=\begin{pmatrix}0&0&0&\eta^2&0\\0&0&0&0&\eta^4\\0&0&1&0&0\\\eta^3&0&0&0&0\\0&\eta&0&0&0\end{pmatrix},\mathbf{\Sigma}_{15}=\begin{pmatrix}0&0&0&\eta&0\\0&0&0&0&\eta^2\\0&0&1&0&0\\\eta^4&0&0&0&0\\0&\eta^3&0&0&0\end{pmatrix},$$

$$\mathbf{\Sigma}_{15}=\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix},\quad \mathbf{\Sigma}_{16}=\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix},\quad \mathbf{\Sigma}_{17}=\begin{pmatrix}0&\eta^4&0&0&0\\\eta&0&0&0&0\\0&0&0&0&\eta^3\\0&0&0&1&0\\0&0&\eta^2&0&0\end{pmatrix},$$

$$\mathbf{\Sigma}_{18}=\begin{pmatrix}0&\eta^3&0&0&0\\\eta^2&0&0&0&0\\0&0&0&0&\eta\\0&0&0&1&0\\0&0&\eta^4&0&0\end{pmatrix},\mathbf{\Sigma}_{19}=\begin{pmatrix}0&\eta^2&0&0&0\\\eta^3&0&0&0&0\\0&0&0&0&\eta^4\\0&0&0&1&0\\0&0&\eta&0&0\end{pmatrix},\mathbf{\Sigma}_{20}=\begin{pmatrix}0&\eta&0&0&0\\\eta^4&0&0&0&0\\0&0&0&0&\eta^2\\0&0&0&1&0\\0&0&\eta^3&0&0\end{pmatrix},$$

$$\mathbf{\Sigma}_{21} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{\Sigma}_{22} = \begin{pmatrix} 0 & 0 & \eta^4 & 0 & 0 \\ 0 & 0 & 0 & \eta^3 & 0 \\ \eta & 0 & 0 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{\Sigma}_{23} = \begin{pmatrix} 0 & 0 & \eta^3 & 0 & 0 \\ 0 & 0 & 0 & \eta & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{\Sigma}_{24} = \begin{pmatrix} 0 & 0 & \eta^2 & 0 & 0 \\ 0 & 0 & 0 & \eta^4 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{\Sigma}_{25} = \begin{pmatrix} 0 & 0 & \eta & 0 & 0 \\ 0 & 0 & 0 & \eta^2 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Counter-example 1.

$$\mathbf{\Sigma}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \mathbf{\Sigma}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 0 & \eta^3 & 0 \end{pmatrix},$$

$$\mathbf{\Sigma}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & \eta^4 & 0 \end{pmatrix}, \ldots$$

are forbidden to be in the same set of 5×5-KPMs, because they does not satisfy the hypothesis (2) of the Proposition 4 above, even though they are inverse-symmetric. Actually, the property iv) of the definition of a set of KPMs is not satisfied because $\mathrm{Tr}\left(\mathbf{\Sigma}_2^\dagger \mathbf{\Sigma}_3\right) \neq 0$.

4. Discussion

According to the Proposition 1, the expression of the Pauli matrices σ_1 and the method developped for constructing a set of KPMs we can assert that for any integer n, $n > 1$, there is a set of $n \times n$-KPMs containing the matrix

$$\mathbf{\Delta}_n = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

whose entries on the second diagonal are units and the other entries are zeros. For N prime integer, $N > 2$ the KPMs can be built with the N-th roots of unit. It is not the case for the Pauli matrices, the $2^p \times 2^p$-KPMs, with $p > 2$, obtained by the Kronecker product of the Pauli matrices and those obtained by Kronecker product of two KPMs, according to the Proposition 1. However, if we can generalise the relationship between the 3×3-KPMs and the 3-dimensional Weyl operators in the Section 2.2, then we will get that, for 6-dimensional

$$\mathbf{\Delta}_6\mathbf{U}_{01}=e^{\frac{5i\pi}{3}}\begin{pmatrix}1&0&0&0&0&0\\0&0&0&0&0&e^{-\frac{i\pi}{3}}\\0&0&0&0&e^{-\frac{2i\pi}{3}}&0\\0&0&0&-1&0&0\\0&0&e^{-\frac{4i\pi}{3}}&0&0&0\\0&e^{-\frac{5i\pi}{3}}&0&0&0&0\end{pmatrix},\tag{4.1}$$

with $\mathbf{U}_{01}$ is the 6-dimensional Weyl operator for $n=0$ and $m=1$, will be an element of a set of 6×6-KPMs, up to phase factor. But we can see that the matrix (4.1) is far from to be a Kronecker product of a Pauli matrix with an element of the set of 3×3-KPMs, even up to phase factor. Thus, we can conclude that if the generalisation of the relations in the Section 2.2 are valide, then there will be, for a composite integer n, other set of $n\times n$-KPMs than those obtained by Kronecker product of sets of KPMs.

We can notice that the phase factors will be eliminated in taking into account that the KPMs should be inverse-symmetric matrices.

To make it clearer, let us take the 2-dimensional case, $d=2$ where

$\mathbf{U}_{00}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\mathbf{U}_{01}=\begin{pmatrix}0&1\\1&0\end{pmatrix},\mathbf{U}_{10}=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\mathbf{U}_{11}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$

Then

$\boldsymbol{\sigma}_1\mathbf{U}_{00}=\begin{pmatrix}0&1\\1&0\end{pmatrix},\boldsymbol{\sigma}_1\mathbf{U}_{01}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\boldsymbol{\sigma}_1\mathbf{U}_{10}=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\boldsymbol{\sigma}_1\mathbf{U}_{11}=\begin{pmatrix}-1&0\\0&1\end{pmatrix}.$

In order that $\boldsymbol{\sigma}_1\mathbf{U}_{10}$ can become inverse-symmetric matrix, we multiply it by the phase factor i. It follows that we have the set $\mathcal{K}_2=\{\boldsymbol{\sigma}_0,\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3\}$ of 2×2-KPMs.

5. Conclusion

In conclusion, we have shown that for any given element of the 3×3-KPMs, its products with other elements generate a basis equivalent to the Weyl operator basis, up to phase factors $\omega=e^{\frac{2i\pi}{3}}$ and $\omega^2=e^{\frac{4i\pi}{3}}$. Reciprocally, multiplying the 3-dimensional Weyl operators basis by a 3×3-symmetric permutation matrix, we have the set of 3×3-KPMs, in eliminating phase factors. We have noticed that the generalisation to any dimension will be another method for constructing a set of KPMs. For any prime integer N, we have demonstrated a method for constructing a set of $N\times N$-KPMs. It starts in decomposing $N\times N$-ones matrix as sum of symmetric permutation matrices. The fact that such decomposition of $N\times N$-ones matrix is not unique shows that the set of $N\times N$-KPMs is not unique too. Actually, we have two examples of sets of 5×5-KPMs.

For a composite integer n, a set of $n \times n$-KPMs can be constructed by Kronecker product.

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Declarations

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