Research Article

σ-Sets and σ-Antisets

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In this paper we present a brief study of the σ-set-σ-antiset duality that occurs in σ-set theory and we also present the development of the integer space $3^A = \left\langle 2^A, 2^{A^-} \right\rangle$ for the cardinals $|A| = 2, 3$ together with its algebraic properties. In this article, we also develop a presentation of some of the properties of fusion of σ-sets and finally we present the development and definition of a type of equations of one σ-set variable.

1. σ-Sets and σ-Antisets

As we have seen in [1], an σ-antiset is defined as follows:

Definition 1.1. *Let A be a σ-set, then B is said to be the σ-antiset of A if and only if $A \oplus B = \varnothing$, where $\oplus$ is the fusion of σ-sets.*

We must observe that given the definition of the fusion operator $\oplus$ in [1] it is clear that it is commutative and therefore if B is an σ-antiset of A, then it will be necessary that A is also the σ-antiset of B. On the other hand, following the Blizard notation, [2] p. 347, we will denote B the σ-antiset of A as $B = A^-$, in this way we will have $A = (A^-)^-$.

Continuing with the development of the σ-sets we have constructed three primary σ-sets, which are:

Natural Numbers	$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$
0-**Natural Numbers**	$\mathbb{N}^0 = \{1_0, 2_0, 3_0, 4_0, 5_0, 6_0, 7_0, 8_0, 9_0, 10_0, \ldots\}$
Antinatural Numbers	$\mathbb{N}^- = \{1^*, 2^*, 3^*, 4^*, 5^*, 6^*, 7^*, 8^*, 9^*, 10^*, \ldots\}$

where $1 = \{\alpha\}$, $1_0 = \{\varnothing\}$ and $1^* = \{\omega\}$, we must clarify that we have changed the letter β for the letter ω for symmetry reasons, we must also remember that:

$$\ldots \in \alpha_{-2} \in \alpha_{-1} \in \alpha \in \alpha_1 \in \alpha_2 \ldots$$

and

$$\ldots \in \omega_{-2} \in \omega_{-1} \in \omega \in \omega_1 \in \omega_2 \in \ldots$$

where both ϵ-chains have the linear ϵ-root property and are totally different, i.e. they do not have a link-intersection. These definitions can be found in [1] Definition 3.13, 3.14 and 3.16.

On the other hand, we must remember the definition of the space generated by two σ-sets A and B which is:

Definition 1.2. *Let A and B be two σ-sets. The Generated space by A and B is given by*

$$\left\langle 2^A, 2^B \right\rangle = \{x \oplus y : x \in 2^A \wedge y \in 2^B\},$$

where $\oplus$ is the fusion operator.

Let us recall a few things about the fusion operator $\oplus$. In this brief analysis, we must observe that given x, y two σ-sets, if $\{x\} \cup \{y\} = \varnothing$ then it will be said that y is the antielement of x and x the antielement of y, where the union of pairs $\cup$ axiomatized within the theory of σ-sets is used, in particular in the completion axioms A and B, which we will call annihilation axioms from now on.

Notation 1.3. *Let x be an element of some σ-set, then we will denote by x^* the anti-element of x, if it exists.*

Now we move on to define the new operations with σ-sets which will help us define the fusion of σ-sets $\oplus$.

Definition 1.4. *Let A and B be two σ-sets, then we define the $*$-intersection of A with B by*

$$A \hat{\cap} B = \{x \in A : x^* \in B\}.$$

Example 1.5. *Let $A = \{1, 2, 3^*, 4\}$ and $B = \{2, 3, 4^*\}$ be two σ-sets, then we have that:*

$$A \hat{\cap} B = \{3^*, 4\}$$

and

$$B \hat{\cap} A = \{3, 4^*\},$$

it is clear that the $*$-intersection operator is not commutative.

Theorem 1.6. *Let A be a σ-set, then $A \hat{\cap} A = \varnothing$.*

Proof. Let A be a σ-set, by definition we will have that

$$A \hat{\cap} A = \{x \in A : x^* \in A\}.$$

Suppose now that $A \hat{\cap} A \neq \varnothing$, then there exists an $x \in A$ such that $x^* \in A$, therefore we will have that $x, x^* \in A$, which is a contradiction with Theorem 3.39 (Exclusion of inverses) from [1], so if A is a σ-set then

$$A \hat{\cap} A = \varnothing.$$

□

Example 1.7. *Let $A = \{1, 2, 3^*, 4\}$, then*

$$A \hat{\cap} A = \{1, 2, 3^*, 4\} \hat{\cap} \{1, 2, 3^*, 4\},$$

$$A \hat{\cap} A = \{x \in \{1, 2, 3^*, 4\} : x^* \in \{1, 2, 3^*, 4\}\},$$

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$$A\hat{\cap}A = \varnothing.$$

Regarding Theorem 1.6, we can observe that given a σ-set A, the σ-set theory does not allow the coexistence of a σ-element x and its σ-antielement in the same σ-set A, and this is because A is a σ-set. However, since σ-set theory is a σ-class theory, one can find the σ-elements together with the σ-antielements coexisting without problems in what we call the proper σ-class, in this way one will have that $\{x, x^*\}$ is a proper σ-class and not a σ-set.

Theorem 1.8. *Let A be a σ-set, then $A\hat{\cap}\varnothing = \varnothing$ and $\varnothing\hat{\cap}A = \varnothing$.*

Proof. Let A be a σ-set, by definition we will have that

$$A\hat{\cap}\varnothing = \{x \in A : x^* \in \varnothing\}.$$

Now suppose that $A\hat{\cap}\varnothing \neq \varnothing$, then there exists an $x \in A$ such that $x^* \in \varnothing$, which is a contradiction, hence $A\hat{\cap}\varnothing = \varnothing$. On the other hand, $\varnothing\hat{\cap}A \subseteq \varnothing$ thus we will have to $\varnothing\hat{\cap}A = \varnothing$. □

On the other hand, we will define the $*$-difference between σ-sets, a fundamental operation to be able to define the fusion between σ-sets.

Definition 1.9. *Let A and B be two σ-sets, then we define the $*$-difference between A y B by*

$$A \circledast B = A - (A\hat{\cap}B),$$

where $A - B = \{x \in A : x \notin B\}$.

Example 1.10. *Let $A = \{1, 2, 3^*, 4\}$ and $B = \{2, 3, 4^*\}$, then we have that:*

$$A\hat{\cap}B = \{3^*, 4\},$$

therefore

$$A \circledast B = A - (A\hat{\cap}B) = \{1, 2, 3^*, 4\} - \{3^*, 4\} = \{1, 2\}$$

$$A \circledast B = \{1, 2\}.$$

We also have to

$$B\hat{\cap}A = \{3, 4^*\}$$

therefore

$$B \circledast A = B - (B\hat{\cap}A) = \{2, 3, 4^*\} - \{3, 4^*\} = \{2\}$$

$$B \divideontimes A = \{2\}.$$

Corollary 1.11. *Let A be a σ-set. Then $A \divideontimes A = A$.*

Proof. Let A be a σ-set, then by Theorem 1.6 we will have that $A \hat{\cap} A = \varnothing$ therefore

$$A \divideontimes A = A - (A \hat{\cap} A) = A - \varnothing = A.$$

□

Corollary 1.12. *Let A be a σ-set. Then $A \divideontimes \varnothing = A$ and $\varnothing \divideontimes A = \varnothing$.*

Proof. Let A be a σ-set, then by Theorem 1.8 we will have that $A \hat{\cap} \varnothing = \varnothing \hat{\cap} A = \varnothing$ therefore

$$A \divideontimes \varnothing = A - (A \hat{\cap} \varnothing) = A - \varnothing = A$$

and

$$\varnothing \divideontimes A = \varnothing - (\varnothing \hat{\cap} A) = \varnothing - \varnothing = \varnothing.$$

□

Now after defining the $*$-intersection and the $*$-difference we can define the fusion of σ-sets as follows:

Definition 1.13. *Let A and B be two σ-sets, then we define the fusion of A and B by*

$$A \oplus B = \{x : x \in A \divideontimes B \vee x \in B \divideontimes A\}.$$

It is clear that the fusion of σ-sets is commutative by definition. Now, let us show an example

Example 1.14. *Let $A = \{1, 2, 3^*, 4\}$ y $B = \{2, 3, 4^*\}$, then we have that:*

$$A \oplus B = \{x : x \in A \divideontimes B \vee x \in B \divideontimes A\},$$

$$A \oplus B = \{x : x \in \{1, 2\} \vee x \in \{2\}\},$$

$$A \oplus B = \{1, 2\},$$

therefore we have that

$$\{1, 2, 3^*, 4\} \oplus \{2, 3, 4^*\} = \{2, 3, 4^*\} \oplus \{1, 2, 3^*, 4\} = \{1, 2\}.$$

Corollary 1.15. *Let A be a σ-set, then $A \oplus A = A$.*

Proof. Let A be a σ-set, by definition we have that,

$$A \oplus A = \{x : x \in A \divideontimes A \vee x \in A \divideontimes A\}.$$

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Now by corollary 1.11, we have that

$$A \oplus A = \{x : x \in A \vee x \in A\},$$

$$A \oplus A = \{x : x \in A\},$$

therefore it is clear that $A \subset A \oplus A$ and that $A \oplus A \subset A$, therefore $A \oplus A = A$.

□

Corollary 1.16. *Let A be a σ-set, then $A \oplus \varnothing = \varnothing \oplus A = A$.*

Proof. First we will show that $A \oplus \varnothing = A$. By definition we will have that,

$$A \oplus \varnothing = \{x : x \in A \circledast \varnothing \vee x \in \varnothing \circledast A\}.$$

Now by the corollary 1.12, we will have that

$$A \oplus \varnothing = \{x : x \in A \vee x \in \varnothing\},$$

$$A \oplus \varnothing = \{x : x \in A\},$$

from this it is clear that $A \subset A \oplus \varnothing$ and that $A \oplus \varnothing \subset A$, in this way $A \oplus \varnothing = A$.

Second, we will show that $\varnothing \oplus A = A$. By definition we will have that,

$$\varnothing \oplus A = \{x : x \in \varnothing \circledast A \vee x \in A \circledast \varnothing\}.$$

Now by the corollary 1.12, we will have that

$$\varnothing \oplus A = \{x : x \in \varnothing \vee x \in A\},$$

$$A \oplus \varnothing = \{x : x \in A\},$$

from this it is clear that $A \subset \varnothing \oplus A$ and that $\varnothing \oplus A \subset A$, in this way $\varnothing \oplus A = A$. □

Theorem 1.17. *Let X be a σ-set, then for all $A, B \in 2^X$, we have that:*

$$A \oplus B = A \cup B,$$

where $A \cup B = \{x : x \in A \vee x \in B\}$.

Proof. Let X be a σ-set and $A, B \in 2^X$. Then, by theorem 3.39 of [1] we have that

$$A \hat{\cap} B = B \hat{\cap} A = \varnothing,$$

in this way

$$A \divideontimes B = A \wedge B \divideontimes A = B.$$

Finally $A \oplus B = \{x : x \in A \vee x \in B\} = A \cup B.$ $\square$

Example 1.18. *Let $X = \{1, 2, 3\}$, $A = \{1, 2\}$ and $B = \{2, 3\}$, it is clear that $A, B \in 2^X$. Now we apply the fusion operator $\oplus$.*

$$A \oplus B = \{x : x \in A \divideontimes B \vee x \in B \divideontimes A\},$$

$$A \oplus B = \{x : x \in A \vee x \in B\},$$

$$A \oplus B = A \cup B = \{1, 2, 3\}.$$

Corollary 1.19. *Let X be a σ-set, then for all $A \in 2^X$, we have that:*

$$A \oplus X = X.$$

Proof. Let X be a σ-set and $A \in 2^X$. Then by theorem 1.17 we have that

$$A \oplus X = A \cup X.$$

Now as $A \subset X$, then $A \cup X = X$, therefore

$$A \oplus X = X.$$

$\square$

Example 1.20. *Let $X = \{1, 2, 3, 4\}$ and $A = \{1, 2, 3\}$, it is clear that $A \in 2^X$. Now we apply the fusion operator $\oplus$.*

$$A \oplus X = \{x : x \in A \divideontimes X \vee x \in X \divideontimes A\},$$

$$A \oplus B = \{x : x \in A \vee x \in X\},$$

$$A \oplus X = A \cup X = \{1, 2, 3, 4\} = X.$$

As we said before, the fusion of σ-sets $\oplus$ is commutative by definition but as we demonstrated in [1][3][4] this operation is not associative.

Example 1.21. *Let $A = \{1^*, 2^*\}$, $B = \{1, 2\}$ y $C = \{1\}$, then*

$$(A \oplus B) \oplus C = \varnothing \oplus C = C$$

and

$$A \oplus (B \oplus C) = A \oplus B = \varnothing,$$

therefore we have that

$$(A \oplus B) \oplus C \neq A \oplus (B \oplus C).$$

2. Generated space

As we have already indicated in the definition 1.2 we will have that the space generated by two σ-sets A and B is:

$$\left\langle 2^A, 2^B \right\rangle = \{x \oplus y : x \in 2^A \wedge y \in 2^B\}.$$

Now taking into account the duality σ-set, σ-antiset we could consider the following example.

Example 2.1. *We consider the σ-set $A = \{1, 2, 3\}$ and its σ-antiset $A^- = \{1^*, 2^*, 3^*\}$ then we obtain the integer space 3^A where,*

$$3^A = \left\langle 2^A, 2^{A^-} \right\rangle.$$

Is important to observe that

$$2^A = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 2\}, \{2, 3\}, A\}$$

and

$$2^{A^-} = \{\varnothing^-, \{1^*\}, \{2^*\}, \{3^*\}, \{1^*, 2^*\}, \{1^*, 2^*\}, \{2^*, 3^*\}, A^-\}.$$

Also is important to observe that $\varnothing = \varnothing^-$, which is very important for the construction of 3^A.

Now considering the definition of generated space,

$$3^A = \left\langle 2^A, 2^{A^-} \right\rangle = \{X \oplus Y : X \in 2^A \wedge Y \in 2^{A^-}\},$$

where the operator $\oplus$ is the fusion of σ-sets, we will obtain the following matrix:

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$\oplus$	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	A
$\emptyset^-$	$\emptyset_0^0$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	A
$\{1^*\}$	$\{1^*\}$	$\emptyset_1^1$	$\{1^*,2\}$	$\{1^*,3\}$	$\{2\}$	$\{3\}$	$\{1^*,2,3\}$	$\{2,3\}$
$\{2^*\}$	$\{2^*\}$	$\{1,2^*\}$	$\emptyset_1^2$	$\{2^*,3\}$	$\{1\}$	$\{1,2^*,3\}$	$\{3\}$	$\{1,3\}$
$\{3^*\}$	$\{3^*\}$	$\{1,3^*\}$	$\{2,3^*\}$	$\emptyset_1^3$	$\{1,2,3^*\}$	$\{1\}$	$\{2\}$	$\{1,2\}$
$\{1^*,2^*\}$	$\{1^*,2^*\}$	$\{2^*\}$	$\{1^*\}$	$\{1^*,2^*,3\}$	$\emptyset_2^4$	$\{2^*,3\}$	$\{1^*,3\}$	$\{3\}$
$\{1^*,3^*\}$	$\{1^*,3^*\}$	$\{3^*\}$	$\{1^*,2,3^*\}$	$\{1^*\}$	$\{2,3^*\}$	$\emptyset_2^5$	$\{1^*,3\}$	$\{2\}$
$\{2^*,3^*\}$	$\{2^*,3^*\}$	$\{1,2^*,3^*\}$	$\{3^*\}$	$\{2^*\}$	$\{1,3^*\}$	$\{1,2^*\}$	$\emptyset_2^6$	$\{1\}$
A^-	A^-	$\{2^*,3^*\}$	$\{1^*,3^*\}$	$\{1^*,2^*\}$	$\{3^*\}$	$\{2^*\}$	$\{1^*\}$	$\emptyset_3^7$

Table 1. Integer Space.

It is important to note that from the perspective of σ-sets we have that $\varnothing = \varnothing^- = \varnothing_j^i$ with $i \in \{0,1,2,3,4,5,6,7\}$ and $j \in \{0,1,2,3\}$, where the difference of the σ-emptysets $\varnothing_j^i$ is given by annihilation, which comes from equation $A \oplus A^- = \varnothing$.

From the matrix representation of the integer space 3^A, we can present another representation of the same integer space. This representation of the integer space 3^A is a graphical representation which we show in figure 1.

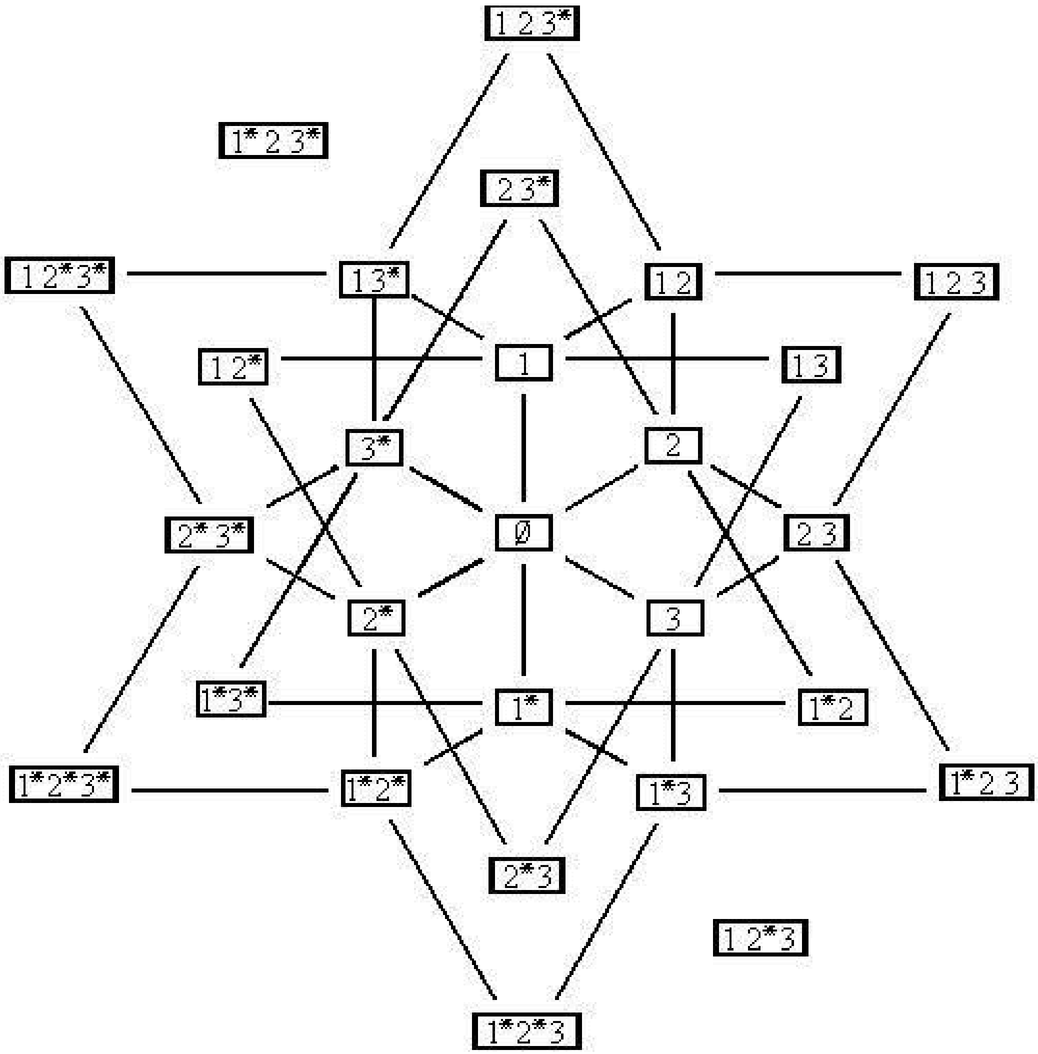

Figure 1. Integer Space 3^A.

Finally, as a theoretical result, we have a cardinal theorem:

Theorem 2.2. *Let* $A = \{1, 2, 3\}$*, then* $\left|3^A\right| = \left|\left\langle 2^A, 2^{A^-} \right\rangle\right| = 3^3 = 27$.

Proof. Let $A = \{1, 2, 3\}$, the proof is the same fusion matrix for this σ-set. □

We should also note that we have obtained other cardinal results for the integer space 3^A with $|A| \in \{0, 1, 2, 3, 4, 5\}$. The cardinal results are as follows:

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σ-Set	σ-Antiset	Generated	Cardinal
$A = \varnothing$	$A^- = \varnothing^-$	$\left\langle 2^A, 2^{A^-} \right\rangle$	$3^0 = 1$
$A = \{1\}$	$A^- = \{1^*\}$	$\left\langle 2^A, 2^{A^-} \right\rangle$	$3^1 = 3$
$A = \{1, 2\}$	$A^- = \{1^*, 2^*\}$	$\left\langle 2^A, 2^{A^-} \right\rangle$	$3^2 = 9$
$A = \{1, 2, 3\}$	$A^- = \{1^*, 2^*, 3^*\}$	$\left\langle 2^A, 2^{A^-} \right\rangle$	$3^3 = 27$
$A = \{1, 2, 3, 4\}$	$A^- = \{1^*, 2^*, 3^*, 4^*\}$	$\left\langle 2^A, 2^{A^-} \right\rangle$	$3^4 = 81$
$A = \{1, 2, 3, 4, 5\}$	$A^- = \{1^*, 2^*, 3^*, 4^*, 5^*\}$	$\left\langle 2^A, 2^{A^-} \right\rangle$	$3^5 = 243$

From these calculations made with the fusion matrix we can obtain the following conjecture.

Conjecture 2.3. *Let A be a σ-set such that $|A| = n$, then $\left|3^A\right| = \left|\left\langle 2^A, 2^{A^-} \right\rangle\right| = 3^n$.*

On the other hand, as we have already said, we are going to change the notation of 1_Θ to 1_0, in this way we will have the σ-set of 0-natural numbers defined as follows:

$1_0 = \{\varnothing\}$

$2_0 = \{\varnothing, 1_0\}$

$3_0 = \{\varnothing, 1_0, 2_0\}$

$4_0 = \{\varnothing, 1_0, 2_0, 3_0\}$

and so on, forming the 0-natural numbers

$$\mathbb{N}^0 = \{1_0, 2_0, 3_0, 4_0, 5_0, 6_0, 7_0, 8_0, 9_0, 10_0, \ldots\},$$

where one of the important properties of this σ-set is that it does not annihilate with the natural numbers $\mathbb{N}$ nor with the antinatural numbers $\mathbb{N}^-$, in this way we can consider the following example for the generated space.

Example 2.4. *We consider the σ-sets $A = \{1_0, 2_0\}$ and $B = \{1, 2\}$, therefore the space generated by $A \oplus B$ and $A \oplus B^-$ will be:*

$\left\langle 2^{A\oplus B}, 2^{A\oplus B^-} \right\rangle = \{x \oplus y : x \in 2^{A\oplus B} \wedge y \in 2^{A\oplus B^-}\}$

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$$\left\langle 2^{A\oplus B}, 2^{A\oplus B^-}\right\rangle = \{\varnothing, \{1_0\}, \{1\}, \{1^*\}, \{2_0\}, \{2\}, \{2^*\}, \{1_0, 2_0\}, \{1_0, 1\}, \{1_0, 1^*\},$$
$$\{1_0, 2\}, \{1_0, 2^*\}, \{2_0, 1\}, \{2_0, 1^*\}, \{2_0, 2\}, \{2_0, 2^*\},$$
$$\{1, 2\}, \{1, 2^*\}, \{1^*, 2\}, \{1^*, 2^*\}, \{1_0, 1, 2\}, \{1_0, 1, 2^*\}, \{1_0, 1^*, 2\}, \{1_0, 1^*, 2^*\},$$
$$\{2_0, 1, 2\}, \{2_0, 1, 2^*\}, \{2_0, 1^*, 2\}, \{2_0, 1^*, 2^*\}, \{1_0, 2_0, 1\}, \{1_0, 2_0, 1^*\}, \{1_0, 2_0, 2\},$$
$$\{1_0, 2_0, 2^*\}, \{1_0, 2_0, 1, 2\}, \{1_0, 2_0, 1, 2^*\}, \{1_0, 2_0, 1^*, 2\}, \{1_0, 2_0, 1^*, 2^*\}\}$$

In this case, the generated space becomes a meta-space generated by $A = \{1_0, 2_0\}$ *and* $B = \{1, 2\}$ *which can be ordered graphically as shown in figure 2.*

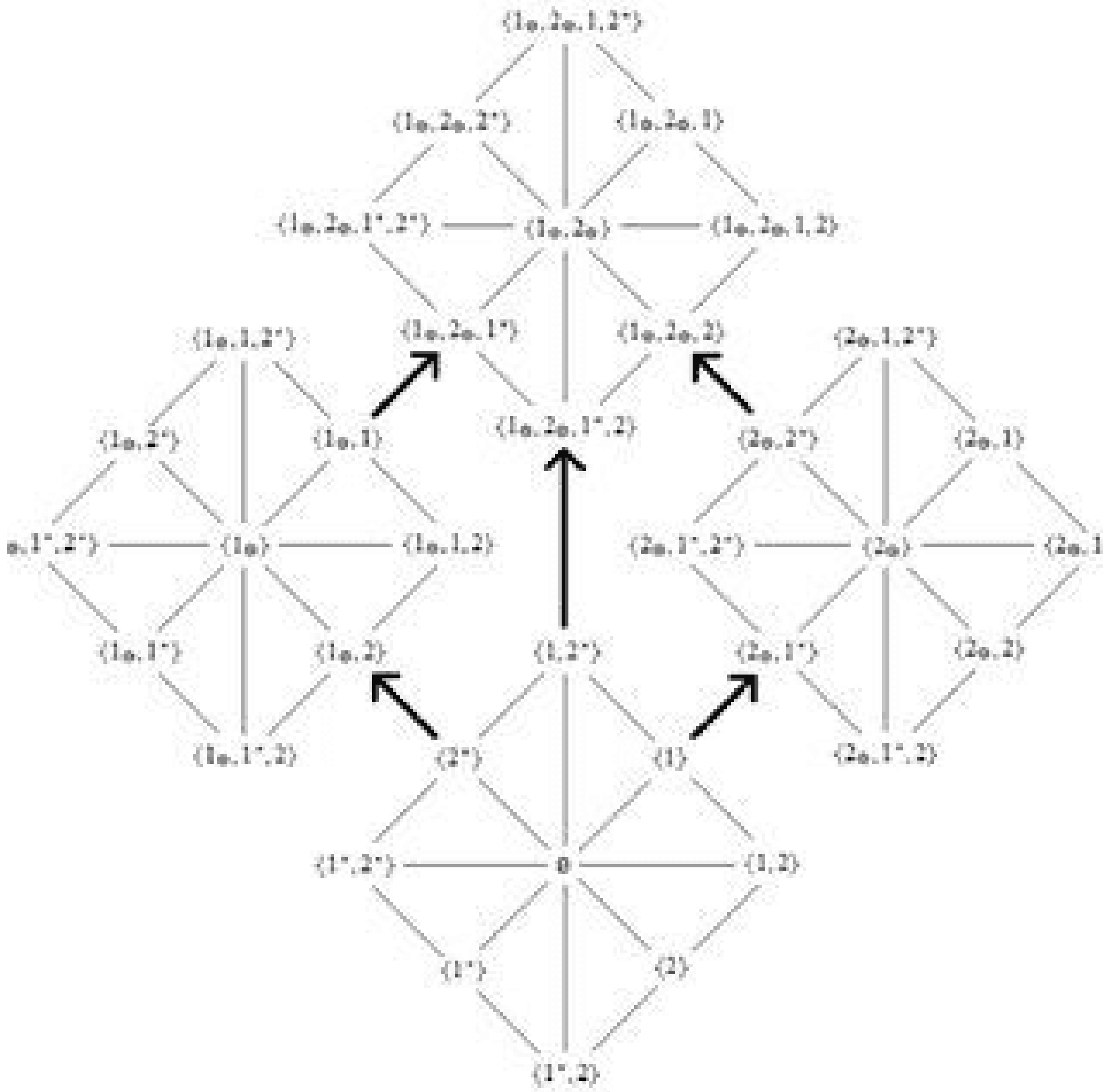

Figure 2. Meta-space $\left\langle 2^{A\oplus B}, 2^{A\oplus B^-}\right\rangle$.

Now, if we count the number of elements that the meta-space generated by $A = \{1_0, 2_0\}$ and $B = \{1, 2\}$ has, we will find that they are 36, where the prime decomposition of this number is $36 = 2^2 \cdot 3^2$ which is equivalent to the following multiplication of cardinals $36 = 2^{|A|} \cdot 3^{|B|}$, from where we can obtain the following conjecture:

Conjecture 2.5. *For all $A \in 2^{\mathbb{N}^0}$ and $B \in 2^{\mathbb{N}}$, then* $\left|\left\langle 2^{A\oplus B}, 2^{A\oplus B^-}\right\rangle\right| = 2^{|A|} \cdot 3^{|B|}$.

Example 2.6. *We consider $A = \{1_0\}$ and $B = \{1, 2\}$, then we obtain that*

$$\begin{aligned}\left\langle 2^{A\oplus B}, 2^{A\oplus B^-}\right\rangle = &\{\varnothing, \{1_0\}, \{1\}, \{1^*\}, \{2\}, \{2^*\}, \{1_0, 1\},\\ &\{1_0, 2\}, \{1_0, 1^*\}, \{1_0, 2^*\}, \{1, 2\}, \{1, 2^*\}, \{1^*, 2\},\\ &\{1^*, 2^*\}, \{1_0, 1, 2\}, \{1_0, 1, 2^*\}, \{1_0, 1^*, 2\}, \{1_0, 1^*, 2^*\}\}\end{aligned}$$

Thus, we have that $|A| = 1$ and $|B| = 2$ and $\left|\left\langle 2^{A\oplus B}, 2^{A\oplus B^-}\right\rangle\right| = 2^{|A|} \cdot 3^{|B|} = 2^1 \cdot 3^2 = 18.$

Example 2.7. *We consider $A = \varnothing$ and $B = \{1, 2\}$, then we obtain that*

$3^B = \{\varnothing, \{1\}, \{1^*\}, \{2\}, \{2^*\}, \{1, 2\}, \{1, 2^*\}, \{1^*, 2\}, \{1^*, 2^*\}\}$
Thus, we have that $|A| = 0$ and $|B| = 2$ and $\left|\left\langle 2^{A\oplus B}, 2^{A\oplus B^-}\right\rangle\right| = 2^{|A|} \cdot 3^{|B|} = 2^0 \cdot 3^2 = 9.$

3. Algebraic structure of integer space 3^A

With respect to the algebraic structure of the Integer Space 3^A for all $A \in 2^{\mathbb{N}}$ we think that these structures are related with structures called NAFIL (non-associative finite invertible loops)

Theorem 3.1. *Let $A = \{1, 2\}$, then $(3^A, \oplus)$ satisfies the following conditions:*

1. $(\forall X, Y \in 3^A)(X \oplus Y \in 3^A)$,
2. $(\exists! \varnothing \in 3^A)(\forall X \in 3^A)(X \oplus \varnothing = \varnothing \oplus X = X)$,
3. $(\forall X \in 3^A)(\exists! X^- \in 3^A)(X \oplus X^- = X^- \oplus X = \varnothing)$,
4. $(\forall X, Y \in 3^A)(X \oplus Y = Y \oplus X)$.

Proof. Let $A = \{1, 2\}$, then we quote the fusion matrix represented in table 2 for $3^{\{1,2\}}$.

From here it is clearly seen that conditions (1), (2), and (3) of theorem 3.1 are satisfied, where the condition (4) is obvious by definition.

We must clarify that since σ-set $\varnothing = \varnothing^-$, and also $\varnothing = \varnothing_0^0 = \varnothing_1^1 = \varnothing_1^2 = \varnothing_2^3$, from here we have condition (2) and the difference is in another dimension, the dimension of annihilation. Here we must clarify that the fusion operation $\oplus$ is not associative. Let $X = \{1^*, 2^*\}$, $Y = \{1, 2\}$ and $Z = \{1\}$ then we will have that $(\{1^*, 2^*\} \oplus \{1, 2\}) \oplus \{1\} = \varnothing \oplus \{1\} = \{1\}$

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on the other hand

$\{1^*, 2^*\} \oplus (\{1,2\} \oplus \{1\}) = \{1^*, 2^*\} \oplus \{1,2\} = \varnothing$

therefore we have that

$$(X \oplus Y) \oplus Z \neq X \oplus (Y \oplus Z),$$

which shows that the structure $(3^A, \oplus)$, is non-associative.

$\oplus$	$\varnothing$	$\{1\}$	$\{2\}$	$\{1,2\}$
$\varnothing^-$	$\varnothing_0^0$	$\{1\}$	$\{2\}$	$\{1,2\}$
$\{1^*\}$	$\{1^*\}$	$\varnothing_1^1$	$\{1^*, 2\}$	$\{2\}$
$\{2^*\}$	$\{2^*\}$	$\{1, 2^*\}$	$\varnothing_1^2$	$\{1\}$
$\{1^*, 2^*\}$	$\{1^*, 2^*\}$	$\{2^*\}$	$\{1^*\}$	$\varnothing_2^3$

Table 2. Integer Space $3^{\{1,2\}}$.

□

We now present a new conjecture.

Conjecture 3.2. *Let $A \in 2^{\mathbb{N}}$, then $(3^A, \oplus)$ satisfies the following conditions:*

1. $(\forall X, Y \in 3^A)(X \oplus Y \in 3^A)$,
2. $(\exists! \varnothing \in 3^A)(\forall X \in 3^A)(X \oplus \varnothing = \varnothing \oplus X = X)$,
3. $(\forall X \in 3^A)(\exists! X^- \in 3^A)(X \oplus X^- = X^- \oplus X = \varnothing)$,
4. $(\forall X, Y \in 3^A)(X \oplus Y = Y \oplus X)$.

4. σ-Sets Equations

Continuing with the analysis of the σ-sets, we now have the development of the equations of σ-sets of a σ-set variable, equations that play a very important role when solving a σ-set equation, now let's define and go deeper into the σ-sets variables.

We must remember that for every σ-set A and B, the fusion of both is defined as:

$$A \oplus B = \{x : x \in A \divideontimes B \vee x \in B \divideontimes A\}$$

Definition 4.1. *Let A be a σ-set, then A is said to be an entire σ-set if there exists the σ-antiset A^-.*

Example 4.2. *Let $A = \{1_0, 2_0, 3_0\}$, then this σ-set is not an integer, since A^- does not exist, on the other hand the σ-set $A = \{1, 2, 3, 4\}$, is an integer σ-set since $A^- = \{1^*, 2^*, 3^*, 4^*\}$ exists which is the σ-antiset of A.*

It is clear that if a σ-set A is integer, then by definition there exists the integer space 3^A. We should also note that if A is an integer σ-set, then $[A \cup A^-]$ is a proper σ-class, for example, consider $A = \{1, 2\}$, then $[A \cup A^-] = [1, 2, 1^*, 2^*]$, is a proper σ-class. We must observe that σ-set theory [1] is a theory of σ-classes, where σ-sets are characterized by axioms. We must also note that a proper σ-class is a σ-class that is not a σ-set. This difference is essential to give rise to the existence of antielements along with their respective elements.

Definition 4.3. *Let A be a integer σ-set such that $|A| = n$, then X is said to be a σ-set variable of 3^A, if and only if*

$$X = \{x_1, x_2, x_3, \ldots, x_m\},$$

where $m \leq n$ and x_i a variable of the proper class $[A \cup A^-]$.

Example 4.4. *Let $A = \{1, 2, 3\}$ be a σ-set, it is clear that A is an entire σ-set since there exists $A^- = \{1^*, 2^*, 3^*\}$ and therefore 3^A, in this way we will have that*

$$X = \varnothing,$$

$$X = \{x\},$$

$$X = \{x_1, x_2\},$$

$$X = \{x_1, x_2, x_3\},$$

are σ-sets variables of 3^A, where $x, x_1, x_2, x_3 \in [1, 2, 3, 1^*, 2^*, 3^*]$.

Lemma 4.5. *Let A be an integer σ-set and X a σ-set variable of 3^A, then $A \oplus X = A \cup X$, with $A \subset A \cup X$ and $X \subset A \cup X$.*

Proof.

Let A be an integer σ-set and X a σ-set variable of 3^A, then

$$A \oplus X = \{x : x \in A \divideontimes X \vee x \in X \divideontimes A\}$$

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Now we have that

$$A \divideontimes X = A$$

and

$$X \divideontimes A = X$$

since X is a σ-set variable, therefore we will have that

$$A \oplus X = \{x : x \in A \vee x \in X\} = A \cup X.$$

We can also observe that $A \cap X = \varnothing$ since X is a σ-set variable, therefore $A \subset A \cup X$ and $X \subset A \cup X$. □

Example 4.6. *Let $A = \{1, 2, 3\}$, and X be a σ-set variable of 3^A, that is,*

$$X = \varnothing$$

$$X = \{x\},$$

$$X = \{x_1, x_2\},$$

$$X = \{x_1, x_2, x_3\},$$

are σ-sets variable of 3^A, where $x, x_1, x_2, x_3 \in [1, 2, 3, 1^*, 2^*, 3^*]$. then

$$A \oplus X = \{1, 2, 3\},$$

$$A \oplus X = \{1, 2, 3, x\},$$

$$A \oplus X = \{1, 2, 3, x_1, x_2\},$$

$$A \oplus X = \{1, 2, 3, x_1, x_2, x_3\}$$

After the lemma 4.5 we proceed to analyze some equations of a σ-set variable and their solutions

Let A be an integer σ-set, X a σ-set variable and M and N two σ-sets of the integer space 3^A, then an equation of a σ-set variable will be

$$X \oplus M = N.$$

Now if $M = N$, then the equation becomes

$$X \oplus M = M,$$

and by the corollary 1.19 we have that the solutions are all $X \in 2^M$, where we naturally count $X = \varnothing$, hence we have an equation of a σ-set variable with multiple solutions.

Now consider $M \neq N$, then the σ-set equation becomes:

$$X \oplus M = N,$$

We must remember that the structure in general is not associative, therefore we cannot freely use this property, so to find the solution to the equation we must develop a previous theorem. To develop this theorem we will assume that for every integer σ-set A the generated space is $\left\langle 2^A, 2^{A^-} \right\rangle = 3^A$, and also that 3^A satisfies conjecture 3.2.

Theorem 4.7. *Let A be an integer σ-set, X be a σ-set variable of 3^A and $M \in 3^A$. Then*

$$(X \oplus M) \oplus M^- = X$$

Proof. Let A be an integer σ-set, X be a σ-set variable of 3^A and $M \in 3^A$, then by lemma 4.5 we have that $X \oplus M = X \cup M$, with $X \cap M = \varnothing$.

Therefore we have that

$$\circledast (X \oplus M) \oplus M^- = \{a : a \in (X \oplus M) \divideontimes M^- \vee a \in M^- \divideontimes (X \oplus M)\}$$

$$= \{a : a \in (X \cup M) \divideontimes M^- \vee a \in M^- \divideontimes (X \cup M)\}$$

so

$$(X \cup M) \divideontimes M^- = (X \cup M) - (X \cup M) \hat{\cap} M^- = (X \cup M) - M = X,$$

and

$$M^- \divideontimes (X \cup M) = M^- - M^- \hat{\cap} (X \cup M) = M^- - M^- = \varnothing.$$

Now replacing these calculations in $(\circledast)$ we will have that

$$(X \oplus M) \oplus M^- = \{a : a \in X \vee a \in \varnothing\}$$

$$(X \oplus M) \oplus M^- = \{a : a \in X\},$$

$$(X \oplus M) \oplus M^- = X.$$

□

Now, after theorem 4.7 has been proved, we can solve some σ-set equation for the integer σ-set $A = \{1, 2\}$, since the generated space is effectively equal to 3^A, that is, $\left\langle 2^A, 2^{A^-} \right\rangle = 3^A$, and also 3^A is a non-associative abelian loop.

Let $A = \{1, 2\}$ be an integer set and $M, N \in 3^A$, with $M \hat{\cap} N = \varnothing$, then the equation

$$X \oplus M = N,$$

has the following solution

$$X \oplus M = N \setminus \oplus M^-,$$

$$(X \oplus M) \oplus M^- = N \oplus M^-,$$

then by theorem 4.7 we will have that

$$X = N \oplus M^-.$$

Let us now show a concrete example for $A = \{1, 2\}$.

Example 4.8. *Let $A = \{1, 2\}$ be an integer σ-set, $M = \{1, 2^*\}$ and $N = \{1\}$, with $M \hat{\cap} N = \varnothing$, then the equation of a σ-set variable*

$$X \oplus \{1, 2^*\} = \{1\}$$

has the following solution.

$$X \oplus \{1, 2^*\} = \{1\} \setminus \oplus \{1^*, 2\},$$

$$(X \oplus \{1, 2^*\}) \oplus \{1^*, 2\} = \{1\} \oplus \{1^*, 2\},$$

$$X = \{2\}.$$

Here we can see that the equation has as solution the σ-set $S_1 = \{2\}$, since

$$\{2\} \oplus \{1, 2^*\} = \{1\},$$

but like the equation $X \oplus M = M$, this one does not have a unique solution since the σ-set $S_2 = \{1, 2\}$, is also a solution for the equation of a σ-set variable,

$$\{1, 2\} \oplus \{1, 2^*\} = \{1\}.$$

In this way we have two solutions for our equation of a σ-set variable which are:

$$S = \{S_1, S_2\} = \{\{2\}, \{1, 2\}\}.$$

Note that if $M\hat{\cap}N = \varnothing$ then the σ-set equation has a solution, but otherwise the σ-set equation has an empty solution.

Example 4.9. *Let $A = \{1, 2\}$ be an integer σ-set, $M = \{1^*\}$ and $N = \{1\}$, with $M\hat{\cap}N = \{1^*\}$, then the equation of a σ-set variable*

$$X \oplus \{1^*\} = \{1\}$$

There is no solution.

$$X \oplus \{1^*\} = \{1\} \setminus \oplus\{1\},$$

$$(X \oplus \{1^*\}) \oplus \{1\} = \{1\} \oplus \{1\},$$

$$X = \{1\},$$

which is a contradiction, because

$$\{1\} \oplus \{1^*\} = \{1\},$$

$$\varnothing = \{1\}.$$

Definition 4.10. *A σ-set equation $X \oplus M = N$ is said to be* **fusionable** *if $M\hat{\cap}N = \varnothing$.*

With this in mind, let us conclude with a bounded theorem to find some solutions of the σ-set equation.

Theorem 4.11. *Let A be an integer σ-set, X a σ-set variable of 3^A, and $M, N \in 3^A$, then two possible solutions $S = \{S_1, S_2\}$ of the fusionable equation*

$$X \oplus M = N,$$

are $S_1 = N \oplus R^-$ and $S_2 = R^-$, where $R := M \oplus N^-$.

Proof. For the first solution S_1 we have that

$$\begin{aligned} S_1 &= N \oplus R^- \\ &= N \oplus (M \oplus N^-)^- \\ &= N \oplus (N \oplus M^-) \\ &= N \oplus M^-, \end{aligned}$$

where $S_2 = (M \oplus N^-)^- = N \oplus M^-$ because of the result iteration seen above. Hence both results are actually a fusion solution for $X \oplus M = N$, where $S_2 = R^-$ is an exact solution and $S_1 = N \oplus R^-$ is an intersected rest solution. Because of $M\hat{\cap}N = \varnothing$ (Definition 4.10) as the equation $X \oplus M = N$ is fusionable, both $S_1 \oplus M$ and $S_2 \oplus M$ will be fusionable into another σ-set N. □

As we looked above, the solution space is reduced such that the solutions are indeed $N \oplus M^-$, being by consequence possible solutions for the fusionable equation $X \oplus M = N$.

Example 4.12. *Let $A = \{1, 2, 3, 4, 5, 6\}$ be an integer σ-set, $M = \{1, 2, 3^*, 4^*, 5, 6^*\}$ and $N = \{1, 2\}$, then the equation of a σ-set variable*

$$X \oplus \{1, 2, 3^*, 4^*, 5, 6^*\} = \{1, 2\},$$

which is fusionable because $M \hat{\cap} N = \{1, 2, 3^*, 4^*, 5, 6^*\} \hat{\cap} \{1, 2\} = \varnothing$.

Now, by using Theorem 4.11, let us first obtain

$$\begin{aligned} R^- &= (M \oplus N^-)^- \\ &= (\{1, 2, 3^*, 4^*, 5, 6^*\} \oplus \{1, 2\}^-)^- \\ &= (\{1, 2, 3^*, 4^*, 5, 6^*\} \oplus \{1^*, 2^*\})^- \\ &= (\{3^*, 4^*, 5, 6^*\})^- \\ &= \{3, 4, 5^*, 6\}, \end{aligned}$$

so we get $S_1 = N \oplus R^- = \{1, 2, 3, 4, 5^*, 6\}$ and $S_2 = R^- = \{3, 4, 5^*, 6\}$, which can be easily proved that both solutions gives $S_1 \oplus M = S_2 \oplus M = N$ as a resulting σ-set. Hence $S = \{\{1, 2, 3, 4, 5^*, 6\}, \{3, 4, 5^*, 6\}\}$ is a solution set for the fusionable equation $X \oplus M = N$.

5. Conclusions

One of the first conclusions we can draw is that the fusion operator $\oplus$ for σsets is equivalent to the union operator for sets within the context of the set of parts 2^A, which allows us to deduce that the fusion of σ-sets is an extension of the union for the generated space.

The fact that the integer space 3^A presents a cardinal of power 3, is very important for the development of the theory of transfinite numbers, since in general the power set 2^A that goes to the power of 2 is used; in this way our results can serve as an impetus for the development of the theory of transfinite numbers.

We can also conclude that the algebraic structure of the integer space $3^{1,2}$ is a loop, which leads us to conjecture that the integer space in general has a loop structure. This fact is relevant to σ-set theory since, if it were so, it would show that the fusion operator $\oplus$ is not associative which is relevant for solving set equations.

As a final conclusion, we can state that we can generate σ-set equations given the existence of inverses for the fusion operator $\oplus$ in the integer space, but in the general case, solutions are not given, so a condition must be imposed on the σ-sets of the equation. We have not yet conducted a detailed study on the number of solutions to each set equation, leaving this study for future research.

To see more works in which antisets or σ-antiset are used or in which equation $A \cup B = \varnothing$ is described, visit the references [4][3][5][1][6].

References

1. a, b, c, d, e, f, g, h *Gatica IA. σ-Set theory: introduction to the concepts of σ-antielement, σ-antiset and integer space [preprint]. arXiv:0906.3120v8; 2010.*
2. ^ *Blizard WD. The development of multiset theory. Mod Log. 1991;1(4):319-52.*
3. a, b *Bustamente A. Associativity of σ-sets for non-antielement σ-set group [preprint]. arXiv:1701.02993v1; 2016.*
4. a, b *Bustamente A. Link Algebra: a new approach to graph theory [preprint]. arXiv:1103.3539v2; 2011.*
5. ^ *Kuang C, Kuang G. Construction of smart sensor networks data system based on integration formalized BCCS model. Sens Transducers. 2013 Aug;155(8):98-106.*
6. ^ *Sengupta A. ChaNoXity: the nonlinear dynamics of nature [preprint]. arXiv:nlin/0408043v2; 2004.*

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