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Research Article

Generalized Mersenne Numbers and Prime Number Generation

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In this study, Mersenne numbers are generalized to include an infinite set defined by the sequence of odd primes. To implement a straightforward algorithm for generating prime numbers on a mainframe, similar to the construction of Mersenne numbers, this paper uses the transformation of sets of natural numbers into infinite-dimensional row vectors. This results in a compact formulation that describes how natural-number sequences are generated and, like Mersenne primes, includes some prime numbers of its own. However, unlike the Mersenne case, which produces a single sequence containing various primes (including the large known prime), the current process appears to produce infinitely many sequences with the same potential within the broader class of *generalized* Mersenne numbers. Furthermore, the overall structure of these natural sequences allows for the recursive search for attached primes: knowing one prime enables the generation of another, which can then generate yet another, and so forth. Finally, a brief overview of an alternative approach based on *generalized twin* Mersenne numbers further expands the scope of this prime-number framework. Several illustrative examples and tables, along with various results, are also included. Two conjectures that encompass all this prime-generating potential are also proposed.

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1. Introduction

Since its initial description by Mersenne, a 17th-century French Minim friar, the OEIS A001348 [1] natural-number sequence has been used across various domains and studied in depth in many sources over the years, see, for example, reference [2]. The historical development of Mersenne numbers and an extended bibliography can be found in the general review provided in reference [3].

As prime number generation is important in various branches of number theory (see, for example, [4][5][6] [7]), Mersenne prime numbers, although they constitute a small cardinality set, are well studied, among other aspects, for their cryptographic relevance [8][9].

Another aspect of Mersenne numbers, which has generated a large body of literature, concerns with their generalization.

For example, looking at historical [1][10][11][12][13][14][15] and recent literature [16][17][18][19][20], one can note that various attempts to obtain generalized Mersenne sequences have taken many paths, but none, like the one proposed by the present approach, has been found in the printed lore of Mersenne numbers.

Another point we may emphasize is the description of the simple, general proposed vector algorithm for prime generation, which transforms the initial Mersenne numbers [21] by replacing the even prime 2 with any odd prime, and the unit subtraction with a possibly limited set of even numbers.

This contribution to the description of generalized Mersenne numbers and the possible generation of sequences bearing primes, as in the original Mersenne sequence, will also be illustrated by several examples to demonstrate the utility of the planned prime generators inspired by the structure of the original Mersenne numbers.

The present study is organized as follows. First, the vector structure of sets of natural numbers is described. Then, the generalized Mersenne numbers are defined. Next, an example of generalized Mersenne numbers is provided. Subsequently, the binary form of Mersenne numbers is compared with their generalized counterparts. A reflection on a new generalized Mersenne example is included, and the extension of the generalized Mersenne numbers to the class of twin generalized Mersenne numbers is introduced. Several additional examples are presented as the study progresses. Then, evidence is presented to show that the recursive nature of the generalized Mersenne number sequences allows one to derive new primes from known primes. The following section presents and discusses two tables of relevant results that illustrate the overall structure of generalized Mersenne numbers and their role in generating primes. Finally, a table of generalized twin Mersenne primes is provided. In addition to two general conjectures, the manuscript concludes with potential applications, future developments, and a bibliography.

2. Vector description of natural number sets and the Mersenne numbers

One can use infinite–dimensional row vectors in Dirac's bra notation (chosen in this form for typographic reasons) to represent sets of natural numbers, as proposed earlier in the literature; see, for example, [22] and the references therein.

Then, adopting this notation, and the symbol $\mathbf{N}$ for the natural numbers set, Mersenne numbers can be defined by the vector expression:

$$
\left\langle \mathbf{M}_{(2,1)} \right| = \left\langle 2^{\mathbf{N}} \right| - \langle \mathbf{1} | =
$$
$$
\left(1, 3, 7, \ldots, \left[2^{I} - 1\right], \ldots \right) \Leftarrow \forall I \in \mathbf{N} = \{1, 2, 3, \ldots, I, \ldots\} \equiv \langle \mathbf{N} | = (1, 2, 3, \ldots). \tag{1}
$$

Note that this definition uses general natural–number powers, not only primes, for example, the vector: $\left\langle 2^{\mathbf{N}} \right| = (2, 4, 8, \ldots, 2^{n}, \ldots)$, corresponds to the sequence of the *natural powers* of 2, the natural even prime. Also, the *unity* vector: $\langle \mathbf{1} | = (1, 1, 1, \ldots, 1, \ldots)$ has been introduced to describe Mersenne numbers as in the equation .

At that point, it seems logical that when describing the odd primes set, one writes, using the natural numbers Q as prime generators:

$$
\forall P = 2Q + 1 \in \mathbf{P} = \{3, 5, 7, 11, 13, \ldots\} \subset \mathbf{N} \equiv
$$
$$
\langle \mathbf{P} | = (3, 5, 7, 11, 13, \ldots) \rightarrow Q = \frac{1}{2}[P - 1] \tag{2}
$$

that is, the prime generator vector can be written in vector form as a homothecy of the prime numbers vector, previously origin shifted by the unity vector:

$$
\langle \mathbf{Q} | = \frac{1}{2} \left(\langle \mathbf{P} | - \langle \mathbf{1} | \right). \tag{3}
$$

The Mersenne number sequence contained in the vector $\left\langle \mathbf{M}_{(\mathbf{2,1})} \right|$ of the equation is described in Sloane's OEIS entry A000225.

3. The generalized Mersenne numbers

Afterwards, one can define a generalized Mersenne number sequence vector $\left\langle \mathbf{M}_{(\mathbf{P;K})} \right|$ with the following algorithm:

$$
\forall P \in \mathbf{P} : \left\langle \mathbf{M}_{(P;K)} \right| = \left\langle P^{\mathbf{N}} \right| - 2K \langle \mathbf{1} | \Leftarrow K = (1, 2, \ldots, Q). \tag{4}
$$

Because the set of K numbers must be multiplied by 2, it is preferable to construct the homothecy of the unity vector as:

$$2\langle\mathbf{1}| \equiv \langle\mathbf{2}| = (2, 2, 2, \ldots, 2, \ldots),\tag{5}$$

such that it can be used to build the infinite sequences of *generalized* Mersenne numbers like:

$$\forall P \in \mathbf{P} : \langle\mathbf{M}_{(P;K)}| = \langle P^{\mathbf{N}}| - K\langle\mathbf{2}| \Leftarrow K = (1, 2, \ldots, Q).\tag{6}$$

Thus, one can consider the generalized Mersenne number sequences as a vector whose components are the natural powers of every prime number, origin-shifted by natural homotheties of the vector $\langle\mathbf{2}|$.

3.1. Some remarks on the definition of generalized Mersenne numbers.

In the algorithm associated with the equation , the prime number that generates the prime powers vector might be called the *base prime*, and the homothetic vector $K\langle\mathbf{2}|$ can be referred to as the *origin-shift vector*. Logically, one can also refer to the natural number K as the *origin-shift parameter*.

The reason for using the origin-shifting homothetic vector $K\langle\mathbf{2}|$, is double. On the one hand, has the role of subtracting a constant less than the prime number from the prime-power vector as in classical Mersenne numbers. On the other hand, the maximal value of the origin-shift parameter $1 \leq K \leq Q$ ensures that the first element of the generalized Mersenne vector $\langle\mathbf{M}_{(\mathbf{P};\mathbf{K})}|$ does not appear negative.

However, the restriction on the values of the origin-shift parameter, not allowing it to become larger than the base prime, can be relaxed when the base prime powers are shifted, so the origin-shifting parameter can be allowed to be $K \geq Q$ without taking into account the first element of the generalized Mersenne vector. Some examples will be presented later on.

Moreover, considering the increasing nature of the power base prime vector's elements, one can admit that for each value of the cumulative exponents of the base prime, one can also define a maximal value of the origin-shift parameter that can be subtracted.

Then, this could modify the equation , but it would make it diverge from the simplicity associated with the definition of classical Mersenne numbers, which one can also insist is present in the definition of generalized Mersenne numbers. The equation that takes into account this generalization can be written as:

$$\forall P \in \mathbf{P} : \langle\mathbf{M}_{(P;K_{\mathbf{N}})}| = \langle P^{\mathbf{N}}| - 2\langle K_{\mathbf{N}}| \Leftarrow \forall I \in \mathbf{N} : K_I = (1, 2, \ldots, Q_I),\tag{4}$$

but it will not be used in what follows for ease of description. Though some examples will be given in the section that shows tables of computed generalized Mersenne numbers.

4. Mersenne twins and generalized Mersenne twins

Also, one can be less restrictive when defining the Mersenne numbers by substituting the subtraction in the equation with an addition, that is:

$$\left\langle \mathbf{T}_{(2,1)} \right| = \left\langle 2^{\mathbf{N}} \right| + \left\langle \mathbf{1} \right| = \left(3, 5, 9, \ldots, \left[2^I + 1 \right], \ldots \right) \Leftarrow \forall I \in \mathbf{N}, \tag{8}$$

one can call these numbers Mersenne twins, simply because one can easily grasp that the numbers of both sequences are related by the simple vector equation:

$$\left\langle \mathbf{T}_{(2,1)} \right| = \left\langle \mathbf{M}_{(2,1)} \right| + \left\langle \mathbf{2} \right|. \tag{9}$$

The sequence contained within the twin Mersenne vector $\left\langle \mathbf{T}_{(\mathbf{2},\mathbf{1})} \right|$ is described under the entry A000051 in the Sloane OEIS.

Therefore, in the same trend, one can define generalized Mersenne twins using the addition instead of the subtraction in the equation , then writing now:

$$\forall P \in \mathbf{P} : \left\langle \mathbf{T}_{(P;K)} \right| = \left\langle P^{\mathbf{N}} \right| + K \left\langle \mathbf{2} \right| \Leftarrow \forall K \in \mathbf{N}. \tag{10}$$

Where the restrictions already discussed on the origin–shift parameter K do not apply here, since such a parameter can take any natural value.

As with classical Mersenne numbers and their twins, one can write a relationship between the generalized counterparts, taking into account the restrictions on the origin–shift parameter:

$$\left\langle \mathbf{T}_{(P,K)} \right| = \left\langle \mathbf{M}_{(P,K)} \right| + K \left\langle \mathbf{4} \right| \Leftarrow \left\langle \mathbf{4} \right| = 4 \left\langle \mathbf{1} \right|. \tag{11}$$

Although the twins of Mersenne numerical structures are also appealing as prime number generators, in order to make this study not too complicated and extensive, only some examples of generalized twin Mersenne numbers will be presented throughout the text, and some at the end of this discussion in the tables section. The reason is to keep this text as close as possible to the structure of classical Mersenne numbers.

5. A first example of generalized Mersenne numbers

The handiest example is the first candidate of a generalized Mersenne numbers sequence based on the odd prime 3:

$$\left\langle \mathbf{M}_{(3;1)} \right| = \left\langle \mathbf{3^N} \right| - \left\langle \mathbf{2} \right| = (1, \mathit{7}, 25, \mathit{79}, \mathit{241}, \mathit{727}, 2185, 6559, \mathit{19681}, \dots), \tag{12}$$

where one can see that five primes appear (in italic) within the first nine vector positions.

Such a sequence is catalogued in the Sloane OEIS entry A058481.

This kind of generalized Mersenne numbers list: $\left\langle \mathbf{M}_{(3;1)} \right|$ seems even more prolific in terms of primes than the Mersenne numbers themselves, which in the present notation, has been written as the vector: $\left\langle \mathbf{M}_{(2;1)} \right|$.

To obtain five primes in the typical Mersenne vector sequence, see, for example, [3], one needs to pick them from the first 13 members of the list.

The corresponding twin vector is stimulating because it presents a similar content of prime numbers (in italic):

$$\left\langle \mathbf{T}_{(3,1)} \right| = \left\langle \mathbf{3^N} \right| + \left\langle \mathbf{2} \right| = (\mathit{5}, \mathit{11}, \mathit{29}, \mathit{83}, 245, 731, 2189, \mathit{6553}, 19635, \dots) \tag{13}$$

interestingly enough, the generalized Mersenne twin vector presents the same number of primes in the first 9 elements as the original Mersenne–like counterpart. This sequence also appears as the entry A168607 in the Sloane OEIS.

6. Binary form of Mersenne numbers and the base representation of their generalized companions

The binary form of each element of the Mersenne sequence is the binary unity vector of dimension equal to the exponent in the number, that is:

$$\forall I \in \mathbf{N} : \left(2^I - 1\right) \leftrightarrow \left\langle \mathbf{1}_I^B \right|, \tag{14}$$

where the superfix B means that the 1's in the unity vector are bits $\left\{1^B\right\}$, and the suffix I refers to the bit vector dimension.

It is interesting to conclude this comparison of the Mersenne sequence $\left\langle \mathbf{M}_{(2;1)} \right|$ with the sequence $\left\langle \mathbf{M}_{(3;1)} \right|$ in the equation , whose elements in base 3 can be expressed as sequences like: $(1, 2, 2, 2, \dots 2, \dots, 2)$, making the comparison quite remarkable. In fact, this means that:

$$\left\langle \mathbf{M}_{(3;1)} \right| + \left\langle \mathbf{1} \right| \leftrightarrow \left\langle \mathbf{2}^{\{3\}} \right| = (2, 2, 2, \dots, 2, \dots 2)^{\{3\}}, \tag{15}$$

where the superfix $\{3\}$ means the numerical base used.

However, in general, the representation of the generalized Mersenne numbers of the type: $\langle \mathbf{M}_{(P;1)} |$, can be expressed in the base P, in a manner that even includes the classical Mersenne representation. That is:

$$\langle \mathbf{M}_{(P;1)} | = \langle P^{\mathbf{N}} | - \langle \mathbf{2} | = (1, R, R, R, \ldots, R, \ldots) \Leftarrow R = P - 1, \tag{16}$$

7. A reflection on another numerical example

As a result of these definitions, an infinite number of generalized Mersenne vectors could be proposed and studied using the construction algorithm given by the equation .

For instance, choosing now the prime 7, it is obtained (prime numbers in italic) the sequence $\{\langle \mathbf{M}_{(7,K)} | | K = 1, 3 \}$:

$$\langle \mathbf{M}_{(7;1)} | = \langle 7^{\mathbf{N}} | - \langle \mathbf{2} | = (\mathit{5}, \mathit{47}, 341, \mathit{2399}, 16805, 117647, \ldots) \tag{17}$$

$$\langle \mathbf{M}_{(7;2)} | = \langle 7^{\mathbf{N}} | - 2 \langle \mathbf{2} | = (\mathit{3}, 45, 339, 2397, 16803, 117645, \ldots) \tag{18}$$

$$\langle \mathbf{M}_{(7;3)} | = \langle 7^{\mathbf{N}} | - 3 \langle \mathbf{2} | = (1, \mathit{43}, \mathit{337}, 2395, 16801, \mathit{117643}, \ldots), \tag{19}$$

none of these three sequences can be found in the Sloane OEIS entries, to the present authors' knowledge.

Such a lack of presence in the OEIS entries is an interesting result, suggesting that the generalized Mersenne sequences described here have not been sufficiently noticed or recognized as interesting as the sequential sets, constructed from infinite sets, as described in the equation .

Thus, one might conclude they have not been considered as important as the Mersenne sequence itself or as the $\langle \mathbf{M}_{(3;1)} |$ sequence.

8. A brief exploration of several examples

Furthermore, these findings show that the generalized Mersenne numbers produce infinite prime sequences that follow patterns similar to those of the original Mersenne sequence.

While one can search for any sequence of generalized Mersenne numbers, it will be adequate to consider, as an additional example, the following sequences: $\{\langle \mathbf{M}_{(13,K)} | | K = 1, 6 \}$, and test the appearance of primes in them.

The obtained results are (primes in italic):

$$\langle \mathbf{M}_{(13,1)} | = (\mathit{11}, \mathit{167}, 2195, \mathit{28559}, \mathit{371291}, 4826807, \ldots) \tag{20}$$

$$\langle \mathbf{M}_{(13,2)} | = (9, 165, 2193, 28557, 371289, 4826805, \ldots) \tag{21}$$

$$\left\langle \mathbf{M}_{(13,3)} \right| = (\mathit{7},\mathit{163},2191,28555,371287,4826803,\ldots) \tag{22}$$

$$\left\langle \mathbf{M}_{(13,4)} \right| = (\mathit{5},161,2189,28553,371285,4826801,\ldots) \tag{23}$$

$$\left\langle \mathbf{M}_{(13,5)} \right| = (\mathit{3},159,2187,28551,371283,4826799,\ldots) \tag{24}$$

$$\left\langle \mathbf{M}_{(13,6)} \right| = (1,\mathit{157},2185,\mathit{28549},\mathit{371281},\mathit{4826797},\ldots) \tag{25}$$

The first and last sequences appear to yield a large number of primes compared with the intermediate ones, which seem to contain only trivial primes.

The last sequence is particularly remarkable, as it yields a large prime triplet. Due to this, it is worth continuing with some even larger powers of 13 in both sequences.

However, the results are disappointing as no primes follow the initial ones:

$$\left\langle \mathbf{M}_{(13,1)} \right| = (\ldots,62748515,815730719,10604499371,\ldots) \tag{26}$$

$$\left\langle \mathbf{M}_{(13,6)} \right| = (\ldots,62748505,815730709,10604499361,\ldots). \tag{27}$$

Given the results and , and using the fact that every sequence of generalized Mersenne numbers can be obtained from the previous one by subtracting the vector $\langle \mathbf{2} |$, it is easy to see that the intermediate sequences do not contain primes in these positions either.

Thus, it seems that the appearance of primes in generalized Mersenne number sequences follows an aleatory allure, similar to how the Mersenne primes behave. This will be made visible further at the end of this work, where extended results will be presented.

This behavior of the prime yielding can be tested again with another base prime number, 61, generating the vectors $\left\{ \left\langle \mathbf{M}_{(61,K)} \right| \big| K = 1, 30 \ \right\}$, which starts with the first sequence:

$$\left\langle \mathbf{M}_{(61,1)} \right| = (\mathit{59},\mathit{3179},226979,13845839,844596299,51520374359,\ldots), \tag{28}$$

an intermediate one:

$$\left\langle \mathbf{M}_{(61,15)} \right| = (\mathit{31},\mathit{3691},226951,13845811,844596271,51520374331,\ldots), \tag{29}$$

and the last one:

$$\left\langle \mathbf{M}_{(61,30)} \right| = (1,3661,226921,13845781,\mathit{844596241},51520374301,\ldots). \tag{30}$$

Constituting an additional simplified sample that can generate even a larger prime in each row.

9. A generated prime used to start new sequences of generalized Mersenne numbers

Finally, one of the larger previously generated primes, see the red element in the equation , will be used to start a new sequence, like:

$$\left\langle \mathbf{M}_{(844596241,1)} \right| = \begin{pmatrix} 844596239, 713342810311330079, \\ 60248665613332542612282551 9, \\ 50885796502286624973306163933546655 9, \dots \end{pmatrix} \tag{31}$$

also, one has tried:

$$\left\langle \mathbf{M}_{(844596241,32)} \right| = \begin{pmatrix} 844596177, 713342810311330017, \\ 60248665613332542612282545 7, \\ 50885796502286624973306163933546649 7, \dots \end{pmatrix} \tag{32}$$

and at random it has been used:

$$\left\langle \mathbf{M}_{(844596241,172839)} \right| = \begin{pmatrix} 844250563, 713342810310984403, \\ 60248665613332542612247984 3, \\ 50885796502286624973306163933512088 3, \dots \end{pmatrix} . \tag{33}$$

The results suggest that large primes may generate sequences in which primes appear with increasing difficulty, as found in the Mersenne sequence.

However, one needs to be aware that the sequences , , and are indeed infinite, and at least one prime could appear somewhere as one moves forward.

To provide an example of such a possibility, the following generalized Mersenne number has been computed, which has serendipitously provided a 99–digit prime number (in italic):

$$844596241^{11} - 2 * 7777777777 =$$
$$\textit{1560057429985655831898543916060836101891611082922} \leftrightarrow \tag{34}$$
$$\textit{7787191211923025465170870355628760040051006713108 7}$$

The sign $\leftrightarrow$ means that the digits continue in the next line.

10. Primes to start new prime search sequences via generalized Mersenne numbers

Yet, nothing opposes the possibility that an even larger prime number than the one found in the classical Mersenne sequence $\left\langle \mathbf{M}_{(2,1)} \right|$, does appear in some $\left\langle \mathbf{M}_{(P,K)} \right|$ particular sequence, where the choice of the

generating prime P powers can be any prime number, even the largest known Mersenne prime, which is the largest prime generated ever [23].

As it was already commented, one must be aware that the sequences $\left\langle \mathbf{M}_{(P,K)} \right|$ are connected by a recurrence; because of this, one can write:

$$\left\langle \mathbf{M}_{(P,K+1)} \right| = \left\langle \mathbf{M}_{(P,K)} \right| - \left\langle \mathbf{2} \right|. \tag{35}$$

One can also ask what happens when any prime contained in $\left\langle \mathbf{M}_{(2,1)} \right|$ is chosen as a base prime to start a sequence, which could be written, using the superfix π to indicate that the number in brackets is prime, as:

$$\left\langle \mathbf{M}_{\left(\left[2^I-1\right],K\right)} \right| \rightarrow \left[2^I - 1\right]^\pi - 2K = \left(\sum_{\lambda=0}^{\pi} \binom{\pi}{\pi-\lambda} (-1)^\lambda 2^{I(\pi-\lambda)} \right) - 2K. \tag{36}$$

The same can be developed using a generalized Mersenne prime as a generating prime, then one can write:

$$\left\langle \mathbf{M}_{\left(\left[P^I-2K\right],L\right)} \right| \rightarrow \left[P^I - 2K\right]^\pi - 2L = \left(\sum_{\lambda=0}^{\pi} \binom{\pi}{\pi-\lambda} (-1)^\lambda P^{I(\pi-\lambda)} (2K)^\lambda \right) - 2L \tag{37}$$

Therefore, no extra complexity appears in using generalized Mersenne primes as elements for new generating sequences of generalized Mersenne numbers.

11. The recursiveness of generalized Mersenne numbers and a conjecture

While the classical Mersenne numbers cannot be recursively iterated within the classical Mersenne sequence, their generalized odd prime counterparts can.

Starting from any odd prime P_0, one can employ the appropriate set of prime powers $\{N_0; N_1; \ldots; N_{M-1}; \ldots\}$ and origin shift parameters $\{K_0; K_1; \ldots; K_{M-1}; \ldots\}$:

$$\exists P_1 \in \mathbf{P} : P_1 = P_0^{N_0} - 2K_0 \Rightarrow$$
$$\exists P_2 \in \mathbf{P} : P_2 = P_1^{N_1} - 2K_1 \Rightarrow$$
$$\ldots$$
$$\exists P_M \in \mathbf{P} : P_M = P_{M-1}^{N_{M-1}} - 2K_{M-1} \tag{38}$$

From the iterative sequence , a conjecture can be enunciated:

Conjecture 1. "*If a prime number P_0 is known, a larger prime can be found using the algorithm:* $\exists\,\{N,K\} \in \mathbf{N} : P_1 = P_0^N - 2K \rightarrow P_1 \in \mathbf{P}$".

Such conjecture has, in fact, stimulated the trial–and–error search for the prime in the equation (34).

12. Extensive calculations using the algorithm for assorted prime numbers

To provide deeper insight into the search for prime numbers in the generalized Mersenne fashion, Table 1 presents findings from the prime number sequences associated with the base prime set: $\{3, 7, 31, 127, 8191\}$.

The number of primes generated by the proposed algorithm is shown in the table below.

Table 1. The first terms of the generated power sequences from generalized Mersenne numbers: $M_{[P,K]} = \left\{ \left(P^N - 2K \right) | N = 1, 8 \right\}$, for various prime base numbers P and values of the origin–shift $K \leq Q$. K parameters in the first column. Found primes in red.

$P = 3 \qquad Q = 1$

$N/$

	1	2	3	4	5	6	7	8
1	1	7	25	79	241	727	2185	6559

$P = 7 \qquad Q = 3$

$N/$

	1	2	3	4	5	6	7	8
1	5	47	341	2399	16805	117647	823541	5764799
2	3	45	339	2397	16803	117645	823539	5764797
3	1	43	337	2395	16801	117643	823537	5764795

$P = 31$ $Q = 15$

N/

	1	2	3	4	5	6	7	8
1	29	959	29789	923519	28629149	887503679		
2	27	957	29787	923517	28629147	887503677	27512614107	852891037437 852891037447 (K= 4)
3	25	955	29785	923515	28629145	887503675	27512614105	852891037435
4	23	953	29783	923513	28629143	887503673	27512614103	852891037433
5	1	951	29781	923511	28629141	887503671	27512614101	852891037431
6	19	949	29779	923509	28629139	887503669	27512614099	852891037429
7	17	947	29777	923507	28629137	887503667	27512614097	852891037427
8	15	945	29775	923505	28629135	887503665	27512614095	852891037425
9	13	943	29773	923503	28629133	887503663	27512614093	852891037423
10	11	941	29771	923501	28629131	887503661	27512614091	852891037421
11	9	939	29769	923499	28629129	887503659	27512614089	852891037419
12	7	937	29767	923497	28629127	887503657	27512614087	852891037417
13	5	935	29765	923495	28629125	887503655	27512614085	852891037415
14	3	933	29763	923493	28629123	887503653	27512614083	852891037413
15	1	931	29761	923491	28629121	887503651	27512614081	852891037411

$P = 127 \qquad Q = 63$

$N/$

	1	2	3	4	5	6	7	8
1	125	16127	2048381	260144639	33038369405	4195872914687	532875860165501	67675234241018879
2	123	16125	2048379	260144637	33038369403	4195872914685	532875860165499	67675234241018877
3	121	16123	2048377	260144635	33038369401	4195872914683	532875860165497	67675234241018875
4	119	16121	2048375	260144633	33038369399	4195872914681	532875860165495	67675234241018873
5	117	16119	2048373	260144631	33038369397	4195872914679	532875860165493	67675234241018871
6	115	16117	2048371	260144629	33038369395	4195872914677	532875860165491	67675234241018869
7	113	16115	2048369	260144627	33038369393	4195872914675	532875860165489	67675234241018867
8	111	16113	2048367	260144625	33038369391	4195872914673	532875860165487	67675234241018865
9	109	16111	2048365	260144623	33038369389	4195872914671	532875860165485	67675234241018863
10	107	16109	2048363	260144621	33038369387	4195872914669	532875860165483	67675234241018861
11	105	16107	2048361	260144619	33038369385	4195872914667	532875860165481	67675234241018859
12	103	16105	2048359	260144617	33038369383	4195872914665	532875860165479	67675234241018857
13	101	16103	2048357	260144615	33038369381	4195872914663	532875860165477	67675234241018855
14	99	16101	2048355	260144613	33038369379	4195872914661	532875860165475	67675234241018853
15	97	16099	2048353	260144611	33038369377	4195872914659	532875860165473	67675234241018851
16	95	16097	2048351	260144609	33038369375	4195872914657	532875860165471	67675234241018849 67675234241018843 (K=19)

$P = 8191 \quad Q = 4095$

$N/$

	1	2	3	4	5	6	7	8
1	8189	67092479	549554511869	4501401006735359	36870975646169341949	302010161517773079920639	2473765232992079297629970429	20262611023438121526887087800319
2	81877	67092477	549554511867	4501401006735357	6870975646169341947	302010161517773079920637	2473765232992079297629970427	20262611023438121526887087800317
3	8185	67092475	549554511865	4501401006735355	36870975646169341945	302010161517773079920635	2473765232992079297629970425	20262611023438121526887087800315
4	8183	67092473	549554511863	4501401006735353	36870975646169341943	302010161517773079920633	2473765232992079297629970423	20262611023438121526887087800313
5	8181	67092471	549554511861	4501401006735351	36870975646169341941	302010161517773079920631	2473765232992079297629970421	20262611023438121526887087800311
6	8179	67092469	549554511859	4501401006735349	36870975646169341939	302010161517773079920629	2473765232992079297629970419	20262611023438121526887087800309
7	8177	67092467	549554511857	4501401006735347	36870975646169341937	302010161517773079920627	2473765232992079297629970417	20262611023438121526887087800307
8	8175	67092465	549554511855	4501401006735345	36870975646169341935	302010161517773079920625	2473765232992079297629970415	20262611023438121526887087800305
9	8173	67092463	549554511853	4501401006735343	36870975646169341933	302010161517773079920623	2473765232992079297629970413	20262611023438121526887087800303
10	8171	67092461	549554511851	4501401006735341	36870975646169341931	302010161517773079920621	2473765232992079297629970411	20262611023438121526887087800301
11	8169	67092459	549554511849	4501401006735339	36870975646169341929	302010161517773079920619	2473765232992079297629970409	20262611023438121526887087800299
12	8167	67092457	549554511847	4501401006735337	36870975646169341927	302010161517773079920617	2473765232992079297629970407	20262611023438121526887087800297

13	8165	67092455	549554511845	4501401006735335	36870975646169341925	302010161517773079920615	2473765232992079297629970405	20262611023438121526887087800295
14	8163	67092453	549554511843	4501401006735333	36870975646169341923	302010161517773079920613	2473765232992079297629970403	20262611023438121526887087800293
15	8161	67092451	549554511841	4501401006735331	36870975646169341921	302010161517773079920611	2473765232992079297629970401	20262611023438121526887087800291
16	8159	67092449	549554511839	4501401006735329	36870975646169341919	302010161517773079920609	2473765232992079297629970399	20262611023438121526887087800289

13. Extensive calculations using the algorithm without restrictions for assorted prime numbers

Next, Table 2 presents the results of prime generation when the restriction of the origin–shift parameter of the algorithm is not considered. The same set of base primes is employed, and therefore, the first column of the unit powers is left blank.

One can note the abundant number of primes that can be generated in this manner.

Table 2. The first terms of the generated power sequences from generalized Mersenne numbers: $M_{[P,K]} = \left\{ \left(P^N - 2K \right) | N = 2, 8 \right\}$, for various prime base numbers P and values of the origin–shift $K > Q$. K parameters in the first column. Found primes in red.

$$P=3 \quad Q=1$$

N/	2	3	4	5	6	7	8
2		5	23	77	239	725	2183	6557
3		3	21	75	237	723	2181	6555
4		1	19	73	235	721	2179	6553
5			17	71	233	719	2177	6551
6			15	69	231	717	2175	6549
7			13	67	229	715	2173	6547
8			11	65	227	713	2171	6545
9			9	63	225	711	2169	6543
10			7	61	223	709	2167	6541
11			5	59	221	707	2165	6539
12			3	57	219	705	2163	6537
13			1	55	217	703	2161	6535
14				53	215	701	2159	6533
15				51	213	699	2157	6531
16				49	211	697	2155	6529

$$P=7 \quad Q=3$$

N/	2	3	4	5	6	7	8
4		41	335	2393	16799	117641	823535	5764793
5		39	333	2391	16797	117639	823533	5764791
6		37	331	2389	16795	117637	823531	5764789
7		35	329	2387	16793	117635	823529	5764787
8		33	327	2385	16791	117633	823527	5764785
9		31	325	2383	16789	117631	823525	5764783
10		29	323	2381	16787	117629	823523	5764781
11		27	321	2379	16785	117627	823521	5764779
12		25	319	2377	16783	117625	823519	5764777
13		23	317	2375	16781	117623	823517	5764775
14		21	315	2373	16779	117621	823515	5764773
15		19	313	2371	16777	117619	823513	5764771
16		17	311	2369	16775	117617	823511	5764769

$$P = 31 \quad Q = 15$$

N/	2	3	4	5	6	7	8
16		929	29759	923489	28629119	887503649	27512614079	852891037409

14. Examples of generalized Mersenne number recurrence

Table 3 provides several examples of possible recurrences among prime numbers, as given by the equation . Among the interesting observations, one can deduce from this table that the same prime number can be used to generate several primes, provided an appropriate set of base prime power and origin-shift parameter: $\{N, K\}$.

P_0	N	K	P_1
1009	8	11	10743092865916626547798743
6449	3	7	268211336847
65521	2	14	4293001469
65521	3	11	281281747415783
65521	4	5	18429861372428076491
65521	5	26	1207542946982859999111653
65521	6	111	79119421429263970001791209343
65521	7	50	5183983611466804578487361812817141
102217	3	2	1067995423452317
102217	5	6	1115875269783974638985086 9
102217	4	39	1091672881990250779 99
102217	6	35	114061422451508535673138505003 9
286749949	3	2	235781674663661750101173 53
386749969	3	20	5784833488634880250022024 9
486749969	3	1	115323495512799090800220211
1946999993	2	30	3790808972742000109
1946999993	3	30	7380705043393011286208999717
4293001429	2	3	18429861269396042047
144924114737	3	43	304384104157860031466092878811363 9
1036488922571	2	3	10743092866123924332500 47
1064645023921	2	5	1133469026959746662214251
1064645023921	2	23	1133469026959746662214287
4175614324837	2	2	17435754989783955355076573
32434291280957	2	6	1051983250897963271910835861

P_0	N	K	P_1
1067995423452281	2	3	1140614224515017005024284102967
2872641517626821	2	29	8252069288793325346047818566099

Table 3. Generalized Mersenne primes generated by the recursion $P_1 = P_0^N - 2K$ for adequate natural numbers of power *N* and origin–shift parameters *K*.

15. Examples of recurrence using generalized twin Mersenne primes

Table 4 shows the recursive possibilities of the generalized twin Mersenne primes. One can see that the generalized twin Mersenne primes recurrence yields primes in the same manner as with the generalized Mersenne primes.

One can grasp that this procedure is as valid as the one based on the generalized Mersenne prime generation, as larger primes can be obtained from already known primes.

Similar observations to those of the previous Table 3 can be repeated here. That is: the same base prime can be transformed into different primes by choosing the adequate power and origin–shift parameter.

P_0	N	K	P_1
1009	8	11	1074309286591662654798743
6449	3	7	268211336847
65521	2	14	4293001469
65521	3	11	281281747415783
65521	4	5	18429861372428076491
65521	5	26	1207542946982859999111653
65521	6	111	79119421429263970001791209343
65521	7	50	5183983611466804578487361812817141
102217	3	2	1067995423452317
102217	5	6	1115875269783974638985069
102217	4	39	1091672881990025077999
102217	6	35	1140614224515085356731385050039
286749949	3	2	235781674663661750101117353
386749969	3	20	578483348863488025002202249
486749969	3	1	1153234955127990908000220211
1946999993	2	30	3790808972742000109
1946999993	3	30	738070504339301128620899717
4293001429	2	3	18429861269396042047
144924114737	3	43	30438410415786003146609287881136 39
1036488922571	2	3	1074309286612392433250047
1064645023921	2	5	1133469026959746662214251
1064645023921	2	23	1133469026959746662214287
4175614324837	2	2	17435754989783955355076573
32434291280957	2	6	10519832508979632719108358 61

$\mathbf{P_0}$	**N**	**K**	$\mathbf{P_1}$
1067995423452281	2	3	1140614224515017005024284102967
2872641517626821	2	29	8252069288793325346047818566099

Table 4. Generalized twin Mersenne primes generated by the recursion: $P_1 = P_0^N + 2K$, for adequate natural numbers of power *N* and origin-shift parameters *K*.

Based on Table 4, Conjecture 1 can be restated as an even more general version, such as:

Conjecture 2. "*If a prime number P_0 is known, a larger prime can be found using the algorithm:* $\exists\,\{N, K\} \in \mathbf{N} : P_1 = P_0^N \mp 2K \rightarrow P_1 \in \mathbf{P}$".

16. Conclusions

A well-defined extension of the Mersenne number sequence and its ability to include large prime numbers have been introduced. Several alternative nuances have also been explored, leading to the concept of generalized Mersenne twins.

Several examples of prime generation in the generalized Mersenne numbers have been examined, starting with selected exploratory cases and later creating a large table showcasing various sequences of generalized Mersenne numbers.

Among the interesting topics developed here, one can highlight the calculation of a 99-digit prime number.

A conjecture and its extension regarding prime generation have been proposed.

An appealing result of this study is that the infinite generalized Mersenne sequences described here form a broad, unlimited extension of the Mersenne number sequence.

It allows the calculation, in principle, of an infinite number of large prime numbers, comparable in size or even larger than those listed in Sloan's OEIS entry A001348.

Additionally, the ease with which prime numbers, regardless of their origin, can be iterated to produce larger primes has been recognized as a feature of generalized Mersenne primes and their twins.

Furthermore, generalized or classical Mersenne prime numbers of any size can be repeatedly iterated to generate even larger primes.

The relationship between generalized Mersenne numbers and their twins is interesting, as evidenced by Tables 3 and 4. In both tables, large prime numbers are used as bases. The following comments apply to large base primes, such as those of the mentioned tables.

Although it is difficult to assess whether a base prime can generate several new primes with appropriate powers and origin shifts, some certainly do. Also, if a generated prime and its twin can become both primes, that will constitute a generalization of twin prime existence. Even if only one of the original or the twin is prime, usually the non-prime decomposes into a product of large primes. Even if both the original and the twin are non-primes, they are usually the product of several large primes.

In this way, the largest Mersenne prime recently calculated: $2^{136279841} - 1$ (October 12, 2024) [23], can be seen as a simple step toward the goal of discovering even larger prime numbers. It is a matter of using these large numbers as base primes and determining the appropriate power and origin shift that yields a new prime.

Acknowledgements

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Declarations

Potential competing interests: No potential competing interests to declare.