https://doi.org/10.32388/GJK3AU

Stochastic resonance and first passage time for excitable system exposed to underdamped medium

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Noise induced Brownian dynamics in underdamped medium is studied numerically to understand the firing time of excitable systems. By considering Brownian particles that move in underdamped medium, we study how the first arrival time behaves for different parameters of the model. We study the first arrival time for both single particle as well as the first arrival time of one particle out a system that has N particles. The present study helps to understand the intercellular calcium dynamics in cardiac tissue at the level of a single microdomain and at a tissue level (ensemble of microdomains). In the presence of time varying signal, we study how signal to noise ratio (SNR) depends on the model parameters. It is showed that the SNR exhibits a pronounced peak at a particular noise strength. The fact that the SNR is amplified as the number of micro domains (N) increase shows that the weak periodic signal plays a decisive role in controlling the noise induced dynamics of excitable systems which may also shed light on how to control the abnormal calcium release in a cardiac tissue.

PACS numbers: Valid PACS appear here

I. INTRODUCTION

Understanding the physics of thermally activated barrier crossing is vital since it has diverse physical applications and serves as a tool to understand stochastic paradigms. Particularly if one considers a Brownian particle moving in a viscous medium, assisted by the thermal background kicks, the particle crosses the potential barrier. The magnitude of its first passage time relies not only on the system parameters, such as the potential barrier height, but also on the initial and boundary conditions. Understanding of such noise induced thermally activated barrier crossing problem is vital to get a better understanding of most biological problems such as cardiac system [1–8]. In the past, considering temperature independent viscous friction, the dependence of the mean first passage time (equivalently the escape rate) on model parameters has been explored for various model systems [9]. However experiment shows that the viscous friction γ is indeed temperature dependent and it decreases as temperature increases. In this work we discuss the role of temperature on the viscous friction as well as on the first passage time by taking a viscous friction γ that decreases exponentially when the temperature T of the medium increases ($\gamma = Be^{-AT}$) as proposed originally by Reynolds [10].

Exposing excitable systems to time varying periodic forces may result in an intriguing dynamics where in this case the coordination of the noise with time varying force leads to the phenomenon of stochastic resonance (SR) [11, 12], provided that the noise induced hopping events synchronize with the signal. The phenomenon of stochastic resonance (SR) has obtained considerable interests because of its significant practical applications in a wide range of fields. SR depicts that systems enhance their performance as long as the thermal background noise is synchronized with time varying periodic signal. Since the innovative work of Benzi *et. al.* [11], the idea of stochastic resonance has been broadened and implemented in many model systems [13–21]. Recently the occurrence of stochastic resonance for a Brownian particle as well as for extended system such as polymer has been reported by us [22, 23]. Our analysis revealed that, due to the flexibility that can enhance crossing rate and change in chain conformations at the barrier, the power amplification exhibits an optimal value for optimal chain lengths and elastic constants as well as for optimal noise strengths. However most of these studies considered a viscous friction which is temperature independent. In this work considering temperature dependent viscous friction, we study how the signal to noise ratio (SNR) behaves as one varies the model parameters. We explore first the stochastic resonance (SR) of a single particle (equivalently of a single microdomain) and later we study the SR for many particle system (N microdomains) by considering both temperature dependent and independent viscous friction and compare the result.

The aim of this paper is to explore the crossing rate and stochastic resonance of a single as well as many Brownian particles in an underdamped medium. Although a generic model system is considered, the present study also helps to understand the dynamics of abnormal calcium cycle in a cardiac system since the thermally activated barrier crossing rate of a single particle mimics the dynamics of abnormal calcium flow at a single microdomain level while the dynamics for many particles case is related to the flow of calcium at tissue (many microdomains) level [24, 25].

To give a brief outline, in this work we first study the first passage time of a single particle for both temperature dependent and independent viscous friction cases. The numerical simulation results depict that the first passage time is smaller when γ is temperature dependent. In both cases the escape rate increases as the noise strength increases and decreases as the potential barrier increases. We then extend our study for N particle systems. The first passage time T_N that one particle (out of N particles) takes to cross the potential barrier can be studied via numerical simulation. It is found that T_N is considerably smaller when the viscous friction is temperature dependent. For both cases, T_N decreases as the noise strength increases and as the potential barrier steps down. In high barrier limit, $T_N = T_s/N$ where T_s is the first arrival time for a single particle. In general as the number of particles N increases, T_N decreases.

We then study our model system in the presence of time varying signal. In this case the interplay between noise and sinusoidal driving force in the bistable system may lead the system into stochastic resonance. Via numerical simulations, we study how the signal to noise ratio (SNR) behaves as a function of the model parameters. The SNR depicts a pronounced peak at particular noise strength T. The SNR is higher for temperature dependent γ case. In the presence of N particles, SNR is considerably amplified as N steps up showing the weak periodic signal plays a vital role in controlling the noise induced dynamics of excitable systems.

The rest of the paper is organized as follows. In section II, we present the model. In section III, by considering both temperature dependent and independent viscous friction cases, we explore the dependence for first arrival time on model parameters for both single as well as many particle system. The role of sinusoidal driving force on enhancing the mobility of the particle is studied in IV. Section V deals with Summary and Conclusion.

II. THE MODEL

We consider a Brownian particle that moves in underdamped medium. The particle is exposed to a piecewise linear potential with a reflective boundary on the left edge and an absorbing boundary on the right. The potential is given by

$$U(x) = \begin{cases} -U_0\left(\frac{2x}{L_0}+1\right), & \text{if} \quad x < -L_0; \\ U_0\left(\frac{2x}{L_0}+1\right), & \text{if} \quad -L_0 \le x \le 0; \\ U_0\left(-\frac{2x}{L_0}+1\right), & \text{if} \quad 0 \le x \le L_0 \\ U_0\left(\frac{2x}{L_0}-1\right), & \text{if} \quad x > L_0. \end{cases} \tag{1}$$

where U_0 and $2L_0$ denote the height and width of the barrier, respectively. The potential exhibits its maximum value U_0 at $x = 0$ and its minima at $x = -L_0$ and $x = L_0$. The ratchet potential is coupled with a uniform temperature T as shown in Fig. 1.

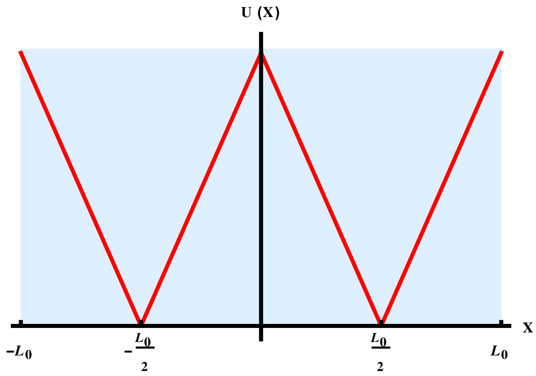


FIG. 1: Schematic diagram for a Brownian particle in a piece-wise linear potential.

For a Brownian particle that is arranged to undergo a random walk in an underdamped medium, the dynamics of the particle is governed by the Langevin equation

$$m\frac{d^2x}{dt^2} = -\frac{dU(x)}{dx} - \gamma\frac{dx}{dt} + \sqrt{2k_B\gamma T}\xi(t) \tag{2}$$

where m is the mass of the particle and k_B is the Boltzmann's constant. The viscous friction γ is assume to have an exponential temperature dependence as

$$\gamma \; = Be^{-AT}, \text{ if } \quad -L_0 \le x \le L_0. \tag{3}$$

Here A and B are constants that characterize the system, and for brevity we will work on units where m and k_B are unity. The background thermal noise $\xi(t)$ is assumed to be Gaussian and has no correlation; *i. e.* $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$.

As discussed in the work [24, 25], the timing of spontaneous excitations in a cardiac tissue can be studied via master equation. For long one dimensional cable (where N is large enough), the voltage fluctuations can be approximated by Gaussian statistics. In other words, if the number of ensembles N is large enough, the calcium dynamics can be well approximated by the corresponding Fokker-Planck type of equation where the effective potential is very complicated and model dependent as discussed in our previous work [24]. Hence the piece-wise linear potential presented in this work only qualitatively addresses the cardiac problem. Moreover, since different excitable systems have different effective potential, the simplified potential presented in this work helps to understand these model systems at least qualitatively.

Please note that the dynamics of such a system can be realized experimentally. One makes negatively charged particle, then put the particle within positively and negatively charged fluidic channel to mimic the piecewise linear potential. The fluidic channel is subjected to an external periodic force (AC field). Since the particle is negatively charged, it encounters a difficulty of crossing through the negatively charged part of the channel. Assisted by the thermal background kicks along with its conformational change, the particle ultimately overcomes the barrier. The presence of time varying force, may further enhance the rate of crossing.

To simplify the numerical simulation we rescale position, time, temperature and barrier potential and work in terms of dimensionless quantities; length $\bar{x} = x/L_0$, time $\bar{t} = t/\beta$, barrier height $\bar{U}_0 = U_0/T$, and temperature $\beta = BL_0^2/T$. For brevity we drop all the bars from all equations from here on.

III. THE FIRST ARRIVAL TIME

In this section, the dependency of the first arrival time on the different model parameters is explored via numerical simulations by integrating the Langevin equation (2) (employing Brownian dynamics simulation). In the simulation, a Brownian particle is initially situated in one of the potential wells. Then the trajectories for the particle is simulated by considering different time steps Δt and time length t_{max}. In order to ensure the numerical accuracy 10^9 ensemble averages have been obtained.

Before exploring the first arrival time of one particle out of N particles, we first numerically evaluate the first passage time distribution both for a single particle system and many particle systems. This gives us a qualitative clue on how the first passage time behaves because the first passage time is given by $T = \int_0^\infty t'P(t')dt'$ where $P(t')$ is the first time distribution.

The first passage time distribution $P_N(t)$ as a function of t is shown in Fig. 2 for fixed values of $U_0 = 2.0$ and $L_0 = 1.0$. Figure 2(a) shows the distributions for temperature independent γ case ($A = 0$) and while Fig. 2(b) shows temperature dependent γ case ($A = 1.0$). The simulation results show that when the number of particles N in the system increases, the peak of the distribution decreases, revealing that the firing time for one particle (out of the N particles) decreases when N increases.

Note that in the high barrier limit, the first passage time distribution $P(t)$ is given by

$$P_s(t) = \frac{e^{\frac{-t}{T_s}}}{T_s} \tag{4}$$

where T_s is the first passage time for a single particle. For a system that has N particles, the distribution of first passage time for one of the particles to cross the barrier is given as

$$P_N(t) = \frac{e^{\frac{-t}{T_N}}}{T_N}. \tag{5}$$

The first arrival time T_N, *i. e.* the time for one of the particles first to cross the potential barrier, is given by

$$T_N = \frac{T_s}{N}. \tag{6}$$

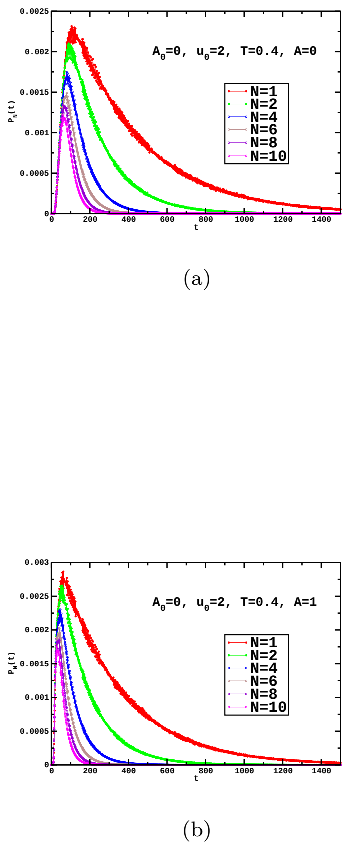
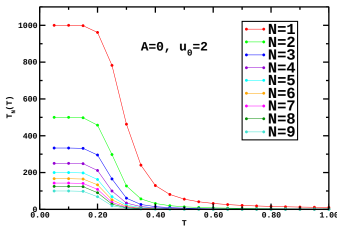


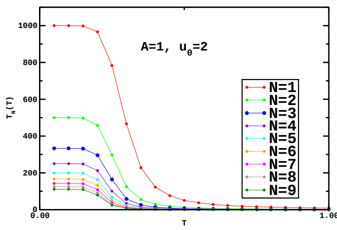
FIG. 2: (Color online) The first passage time distribution $P_N(t)$ as a function of t for different values of N. We use parameter values of $U_0 = 2.0$ and $L_0 = 1.0$. Figures (a) and (b) show the distributions for constant γ ($A = 0$) and temperature dependent γ ($A = 1$) cases, respectively.

As we discussed before, exploring noise induced thermally activated barrier crossing is vital to get a better understanding on how the escape rate or equivalently the first arrival time depends on the different model parameters. If one considers a Brownian particle moving in underdamped medium assisted by the background thermal kicks, the particle crosses the potential barrier after some time. The magnitude of its first arrival time relies not only on the system parameters, such as the potential barrier height, but also on the initial and boundary conditions.

Most previous studies considered temperature independent viscous friction. However, experiment shows that viscous friction γ is indeed temperature dependent and it decreases as temperature increases**??**. Using con-



(a)



(b)

FIG. 3: Color online) The mean first passage time T_N as a function of temperature T for the parameter values of $U_0 = 2.0$ and $L_0 = 1.0$ for different N values. Figs. 3(a) and 3(b) are plotted by considering constant and variable γ cases, respectively.

stant and exponential dependence of γ on temperature, we have plotted the first passage time $T_N(T)$ as a function of temperature T in Figs. (3a) and (3b). The first passage time decreases as N and the strength of the temperature step up in both cases. In the figures the parameters are fixed as $U_0 = 2.0$ and $L_0 = 1.0$. The the mean first passage time is lower in the variable γ case. This can be more appreciated if one plots the ratio of the first passage time between cases where $A = 1$ and $A = 0$ ($\bar{T}_N$) as shown in Fig. 4. In the figure, the ratio is flat at low temperature and increases very rapidly at higher temperature T. This is plausible since the diffusion constant $D = T/\gamma = k_B T e^T$ is also valid when viscous friction to be temperature dependent showing that the effect of temperature on the particles' mobility is twofold. First,

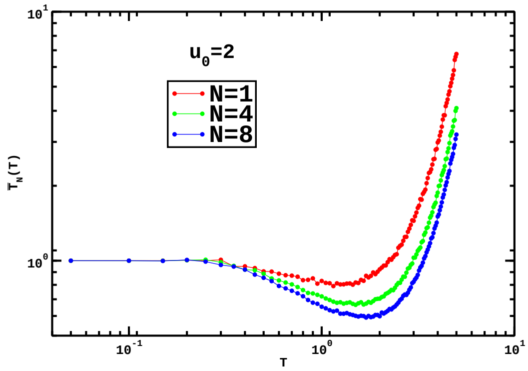


FIG. 4: The ratio of first passage time between the temperature dependent γ ($A = 1$) and constant γ ($A = 0$) cases.

it directly assists the particles to surmount the potential barrier; *i. e.* particles jump the potential barrier at the expenses of the thermal kicks. Secondly, when temperature increases viscous friction gets attenuated and hence diffusibility of the particle increases. Since most biological systems operate at physiological temperature, our study suggests that macromolecules in such systems are transported fast when molecules exposed at temperature dependent γ.

Furthermore, when N increases the first arrival time decreases. This suggests that in cardiac tissue, when the number of microdomains increases the chance for cardiac tissue to release calcium abnormally increases. This is because spontaneous calcium release at microdomain level triggers diffusion of calcium in the neighboring domains and hence causes calcium release at tissue level. Hence increasing the number of microdomains increases the chance for one of the domain to fire calcium steps up, resulting in reduced mean arrival time.

IV. STOCHASTIC RESONANCE

In the past, various studies have shown that exposing excitable systems to time varying periodic signals may result in coordination between the noise and the signal to leads the phenomenon of stochastic resonance (SR) [11, 12]. This coordination brings SR provided that the noise induced hopping events synchronize with the time varying periodic signal. To quantify this resonance in the presence of varying γ, in this section we study how the signal noise ratio (SNR) behaves as function of different model parameters.

In the presence of a time varying periodic signal $A_0 \cos(\Omega t)$, the Langevin equation that governs the dynamics of the system is written as

$$m\frac{d^2x}{dt^2} = -\frac{dU(x)}{dx} - \gamma\frac{dx}{dt} + A_0\cos(\Omega t) + \sqrt{2k_B\gamma T}\xi(t). \tag{7}$$

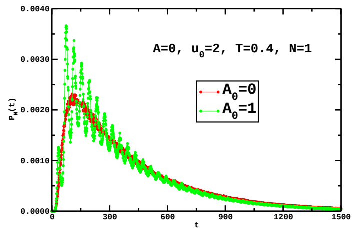
where A_0 and Ω are the amplitude and angular frequency of the external signal, respectively. We have numerically simulated eq. 7 for small barrier height and explore the dependence of the first passage time distribution $P_N(t)$ in the presence of time varying signal $(A_0\cos(\Omega t))$.

The first passage time distribution function $P_N(t)$ as a function of time is plotted in Fig. 5 for fixed values of $U_0 = 2.0$ and $T = 0.4$ considering a constant γ $(A = 0)$. Figs. 5a, 5b and 5c show the change in the distribution for $N = 1$, $N = 6$ and $N = 10$, respectively. As a comparison, we plot the first time distributions both in the presence of the external signal $A_0 = 1.0$ (green solid line) and in the absence of signal $A_0 = 0.0$ (red solid line). As it can be seen clearly, the presence of external time varying signal cases multiple resonances. .

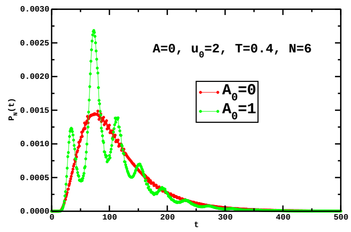
The resonance profile can be better observed by looking at the ratio between the first passage time distribution functions in Fig. 6, namely by plotting the ratio of the green curve to the red one. Fig. 6a shows the case for $N = 1$ and it clearly shows the resonance profile. It turns out that the ratio of the first time distribution is independent of the number of particles as shown in Fig. 6b where we plot the ratio for different N values varying from 1 to 10.

In the limit of small barrier height, we also plot the first passage time distribution in Fig. 7. In Figs. 7a, 7b and 7c, the number of particles is fixed as $N = 1$, $N = 6$ and $N = 10$, respectively. The figures show that the resonance is more pronounced when γ is temperature dependent (green line) than temperature independent γ (red line).

Via numerical simulations, we further study how the SNR behaves as a function of the determinate model parameters by introducing additional dimensionless parameter: $\bar{A}_0 = A_0L_0/U_0$, and for brevity we drop the bar hereafter. Fig. 8a depicts the plot for the SNR as a function of T for the parameter values of $A_0 = 0.1$, $A = 0$ and $U_0 = 2.0$. The parameter N is varied from $N = 1$ to $N = 10$ for a variable γ case. The SNR exhibits monotonous noise strength dependence revealing a peak at an optimal noise strength T_{opt}. Our analysis shows that $T_{opt} = U_0/3$ showing that T_{opt} increases as U_0 decreases. In Fig. 8b, the SNR as a function of T is plotted for the parameter values of $U_0 = 2.0$ and $L_0 = 1.0$ for a constant γ case and $A_0 = 0.1$. As shown in the figures, once again the SNR exhibits a peak at



(a)



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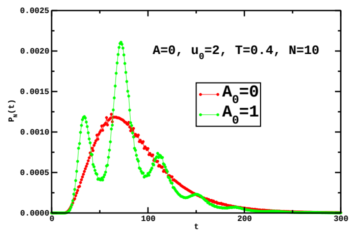
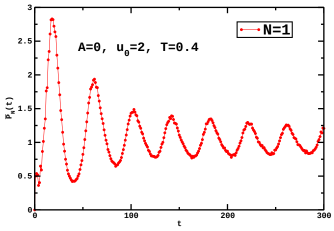
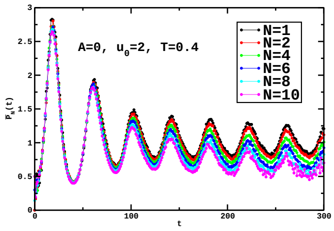


(c)

FIG. 5: (Color online) The first passage time distribution $P_N(t)$ as a function of time for the parameter values of $U_0 = 2.0$, $\Omega = 0.4$, and $A_0 = 0.1$. The red and green lines show the case where external signal is turned on and off, respectively. In Figs. 5(a), 5(b) and 5(c), N is fixed at $N = 1$, $N = 6$ and $N = 10$, respectively.



(a)



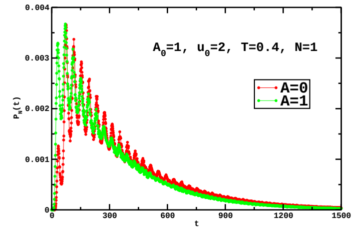
(b)

FIG. 6: (Color online)) The ratio of the first passage time distribution functions $\bar{P}_N(t)$ when the signal amplitude is $A_0 = 0.1$ and frequency is $\Omega = 0.4$. In Fig. 6(a), the particle number is fixed as $N = 1$. In Fig. 6(b), the resonance profile is independent on N.

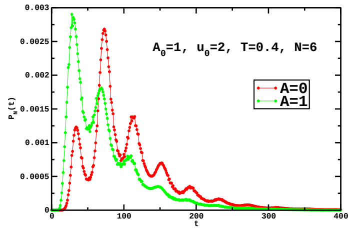
an optimal noise strength T_{opt}. Furthermore, as N increases, T_{opt} decreases and the SNR increases with N. This numerical result clearly indicates that the abnormal calcium firing rate increases in the presence of weak external signals since the first passage time considerably decreases in the presence of signals. Thus our study is vital in developing antiarrhythmic strategy.

V. SUMMARY AND CONCLUSION

In this work first we study the first passage time of a single particle both for temperature dependent and independent viscous friction cases. The the simulation results depict that the first passage time is considerably smaller



(a)



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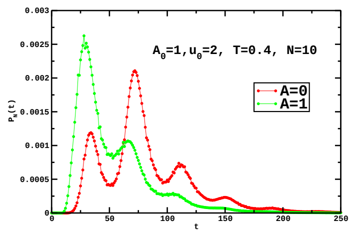
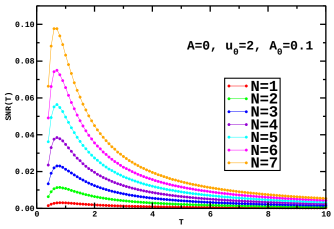
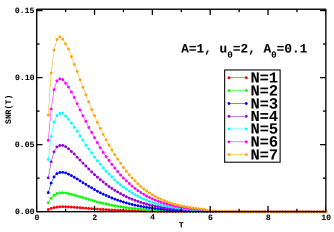


(c)

FIG. 7: (Color online) The first passage time distribution $P_N(t)$ as a function of time in small barrier limit. Figs. (a), (b) and (c) are plotted by considering one, six and ten particles cases respectively. The amplitude and frequency of the signal are set at $A_0 = 0.1$ and $\Omega = 0.4$ respectively.



(a)



(b)

FIG. 8: (Color online) The SNR as a function of T for the parameter values of $A_0 = 0.1$, and $U_0 = 2.0$. Figures 8(a) and 8(b) are plotted for a constant γ ($A = 0$) and variable γ ($A = 1$) cases, respectively.) respectively.

when γ is temperature dependent. In both cases the escape rate increases as the noise strength increases and decreases as the potential barrier increases. We then extend our study for N particle systems. The first passage time T_N for one particle out of N particles to cross the potential barrier can be studied via numerical simulation for any cases. It is found that T_N is considerably smaller when the viscous friction is temperature dependent. For both cases, T_N decreases as the noise strength increases and as the potential barrier steps down. In high barrier limit, $T_N = T_s/N$ where T_s is the first passage time for a single particle. In general as the number of particles N increases, T_N decreases. .

We also study our model in the presence of time varying periodic signal. In this case the interplay between noise and sinusoidal driving force in the bistable system may lead the system into stochastic resonance. Via numerical simulations we study how the signal to noise ratio (SNR) behaves as a function of the model parameters. The SNR depicts a pronounced peak at particular noise strength T_{opt}. The SNR is higher when γ is temperature dependent. For many particles system, SNR is considerably amplified as the number of particles N steps up showing the weak periodic signal plays a vital role in controlling the noise induced dynamics of excitable systems.

In conclusion, in this work we presented a very important model which helps to understand the dynamics of excitable systems such as neural and cardiovascular systems. Thus, this numerical study is crucial not only for the fundamental understanding of excitable systems, but also in providing a basic paradigm in to understand thermally activated transport features in various biological systems.

Acknowledgment.— We would like to thank Mulugeta Bekele for the interesting discussions we had. MA would like to thank Mulu Zebene for the constant encouragement. SFD's research was supported in part by the Intramural Research Program of the National Institute of Health, National Library of Medicine.

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