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SIMPLE MODULES AS INVARIANT SPACES OF HECKE ALGEBRAS

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ABSTRACT: It is shown that all simple modules of the symmetric groups S_n and the finite groups of Lie type can be realized as invariant spaces of some Hecke algebras with respect to certain multiplicative characters. We ask whether this phenomenon is true for all finite groups.

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INTRODUCTION

We present and discuss a phenomenon on the structure of the simple modules for two (important and large) classes of finite groups; namely the symmetric groups and the finite groups of Lie type. The simple modules of those classes of groups turn out to be invariant spaces for certain Hecke algebras of the finite groups with respect to certain multiplicative characters of the Hecke algebra. This gives an indication and raise a question whether the simple modules of all finite groups can be realized this way.

PRELEMINARIES

Let A be a finite dimensional unitary k-algebra with a unit element 1_A over a field k. A left A-module Y is an Abelian group $(Y,+)$ with a left A-action $A \times Y \to Y$; $(a, y) \longmapsto ay$ of A on Y. Every A-module Y can be regarded as a k-module (vector space over k) as follows: $\lambda y \coloneqq (\lambda. 1_A)y$; $\forall \lambda \in k, y \in Y$. We write $T \leq_A Y$ when T is an A-submodule of Y. For two A-modules X, Y, write (X, Y) for the A-homomorphism space $Hom_A(X, Y)$, $End_A(Y)$ for the A-endomorphism algebra of Y and $r(X, Y) = \{f \in (X, Y) | \, fg \in J(End_A(Y); \, \forall g \in (Y, X)\}$ for the **radical subspace** of (X, Y), where $J(End_A(Y)$ is the Jacobson radical of the algebra $End_A(Y)$. Clearly, (X, Y) can naturally be regarded as a left $End_A(Y)$-module via the action defined by the composition $\circ$ of A-maps as follows: $\alpha\beta \coloneqq \alpha \circ \beta$; $\forall \alpha \in End_A(Y), \beta \in (X, Y)$, and also $r(X, Y) \leq_{End_A(Y)} (X, Y)$. If $\psi: A \to k^\times$ is a multiplicative character of A (i.e. a one-dimensional representation of A), then the subspace $I_\psi(Y) = \{y \in Y | \, ay = \psi(a)y; \, \forall a \in A\}$ of Y is called **the invariant subspace of Y relative to ψ** (or **the (A, ψ)-invariant space of Y**).

PERMUTAION MODULES - HECKE ALGEBRA

Let G be a finite group. We take $A = kG$; the group algebra of G over k. There is a well-known equivalence of categories between the category of matrix representations of G over k and the category of kG-modules so studying kG-modules is equivalent to studying matrix representations. If $H \leq G$ then the induced kG-module $Y = Ind_H^G(k_H) = kG \otimes_{kH} k_H$ of the trivial kH-module k_H (i.e. the abelian group $(k, +)$ of k together with the trivial kH-action; i.e. $a\lambda \coloneqq \lambda, \forall a \in kH, \lambda \in k$) is known to be a permutation kG-module isomorphic to the left ideal $kG[H]$ generated by the group algebra element $[H] = \sum_{h \in H} h \in kG$ regarded as left kG-module. The endomorphism algebra $End_{kG}(Y)$ =$\boldsymbol{H}(G, H)$ is called the **Hecke algebra of $\boldsymbol{G}$ by $\boldsymbol{H}$**. If T is a transversal for the (H, H)-double cosets of G then $\boldsymbol{H}(G, H)$ has a k-basis $\{a_t | t \in T\}$ indexed by the elements of T, where $a_t \in \boldsymbol{H}(G,H)$ is defined by: $a_t([H]) = \sum_{x \in HxH} x$, and, as we saw previously, Y has a natural structure of $\boldsymbol{H}(G,H)$-module. Let $Y = \sum_{1\le i\le r}^{\oplus} m_i Y_i$ be a decomposition of Y into a direct sum of non-isomorphic indecomposable kG-modules where $m_i \in \mathbb{N}$ is the multiplicity of Y_i in that decomposition. Then, (Y_i, Y) is a projective indecomposable $End_{kG}(Y)$-module and the quotient module $S_i := (Y_i, Y)/r(Y_i, Y)$ is simple $End_{kG}(Y)$-module. The **Brauer-Fitting correspondence**

$$Y_i \longleftrightarrow S_i := (Y_i, Y)/r(Y_i, Y) \ (1)$$

defines a bijection between the set $Inds(Y)$ of all indecomposable direct kG-summands of Y and the set $Irr(\boldsymbol{H}(G,H))$ of all simple $\boldsymbol{H}(G,H)$-modules (see [PL], 1.4). If $\psi: \boldsymbol{H}(G,H) \to k^{\times}$ is a one-dimensional (hence obviously simple) representation of $\boldsymbol{H}(G,H)$, then the corresponding indecomposable direct summand Y_ψ of Y appears with multiplicity 1. Green [JG1] defined an $\boldsymbol{H}(G,H)$-**invariant subspace** $I_\psi(Y) = \{y \in Y | ay = \psi(a)y; \forall a \in \boldsymbol{H}(G,H)\}$ of Y with respect to $\boldsymbol{\psi}$ [$(\boldsymbol{H}(G,H), \boldsymbol{\psi})$**-invariant subspace**] and also proved that

$$I_\psi(Y) \le_{kG} Y_\psi \ (2)$$

We shall show below that every simple kG-module for the finite groups G under consideration is contained in a certain $(\boldsymbol{H}(G,H), \boldsymbol{\psi})$-invariant space and as such it is an invariant space for some Hecke algebra.

A MOTIVATION EXAMPLE (THE TRIVIAL kG-MODULE)

If G is a finite group and $H \le G$ then the Hecke algebra $\boldsymbol{H}(G,H)$ has a multiplicative character IND (from index) given by: $IND(a_t) = |H : H \cap {}^{t}H|$; ${}^{t}H = tHt^{-1}$. The corresponding direct kG-summand Y_{IND} of $Y = Ind_H^G(k_H)$ is the known **Alperin-Scott module** $\boldsymbol{S}(G,H)$ which is, by definition, the unique kG-summand of Y having the trivial kG-module k_G in its **head** $Hd(Y)$ (the maximal completely reducible quotient kG-module) and its **socle** $Soc(Y)$ (the maximal completely reducible kG-submodule) (cf. [JG2],§4). Since $I_{IND}(Y) \le_{kG} Y_{IND}$ (by (2)), and $k_G \le_{kG} Socl(Y_{IND})$, (as $Y_{IND} = \boldsymbol{S}(G,H)$), it follows that $k_G \le_{kG} I_{IND}(Y)$. Therefore, we have

LEMMA1: If G is any finite group and $H \le G$, then the trivial kG-module k_G is an $(\boldsymbol{H}(G,H), IND)$-invariant space for the Hecke algebra $\boldsymbol{H}(G,H)$. ∎

SIMPLE kS_n-MODULES

Let $G = S_n$; the symmetric group on *n* letters. It is known that the isomorphism classes of simple kS_n-modules are indexed by partitions of *n* [GJ]. If $\lambda = (\lambda_1, \lambda_2, \dots)$ is a partition of n, we follow [GJ] in writing D^λ for the corresponding simple kS_n-module. Let $H = \Re_\lambda$ ($K = \boldsymbol{C}_\lambda$) be the row (column) stabilizer subgroup for the partition λ and take $Y = kG[H]$. Green[JG1] proved the existence of a multiplicative character $\psi_\lambda: \boldsymbol{H}(G,H) \to k^{\times}$ for the **Hecke (Schur)** algebra $\boldsymbol{H}(G,H)$ given by $\psi_\lambda(a_t) = \sum_{\alpha \in HtH \cap K} \varepsilon(\alpha)$, where ε is **the sign character** of S_n defined by: $\varepsilon(\alpha) = (-1)^{\ell(\alpha)}$ and $\ell(\alpha)$ is the **length** of α (the minimal number of the simple generators of S_n of which α is a product). It was also proved in [JG1] that $S^\lambda \le_{kG} I_{\psi_\lambda}(Y)$, where S^λ is the **Specht module** of type λ [GJ]. The corresponding component Y_{ψ_λ} of Y is the **Young module** Y^λ (see [GJ]). It follows that $S^\lambda \le_{kG} I_{\psi_\lambda}(Y) \le_{kG} Y^\lambda$. As $D^\lambda \le_{kG} S^\lambda \le_{kG} I_{\psi_\lambda}(Y)$, we then have

THEOREM1: The simple kS_n-module D^λ is an $(\boldsymbol{H}(G,H), \psi_\lambda)$-invariant space for the Hecke (Schur) algebra $\boldsymbol{H}(G,H)$. ∎

Note that when $chark = 0$, we have $S^\lambda \cong I_{\psi_\lambda}(Y) \cong Y^\lambda$.

THE SIMPLE MODULES FOR THE FINITE GROUPS OF LIE TYPE

Let $G = (G, B, N, R, W)$ be a finite group with BN-pair. Let $Y = kG[B]$, then the **(Iwahori) Hecke algebra** $\boldsymbol{H}(G, B)$ has a k-basis $\{a_w | w \in W\}$ indexed by the elements of **the Weyl group** W of G. $\boldsymbol{H}(G, B)$ has a 1-dimensional representation $\psi: \boldsymbol{H}(G, B) \to k^{\times}$ given by $a_w \mapsto (-1)^{\ell(w)}$; where $\ell(w)$ is the length of $w \in W$. Write St_G for **the Steinberg representation** of G, then, according to [AK1], $St_G \leq_{kG} I_{\psi}(Y) \cong H_{n-1}(\Delta) \leq_{kG} Y_{\psi}$, where Δ is the **Tit's simplicial complex** associated with G; this holds in any characteristic of k. Therefore, we have

THEOREM2([AK2]): $St_G \cong I_{\psi}(Y) = \bigcap_{\boldsymbol{w \in R}} Ker(a_w + \boldsymbol{1})$ is an $(\boldsymbol{H}(G, B), \psi)$-invariant space for the Iwahori Hecke algebra $\boldsymbol{H}(G, B)$. ■

Now, if $G = (G, B = UH, N, R, W)$ has a split BN-pair (**e.g. finite groups of Lie type**) of characteristic $p = chark$, and if we take $Y = kG[U]$, then; as the elements of N index the (U, U)-double cosets of G, we have $H(G, U) = End_{kG}(Y) = \prec a_n | n \in N \succ$. The permutation kG-module $Y = kG[U]$ plays an essential role in studying modular representation theory of G in the natural characteristic. In fact, according to [SA], $Y = kG[U]$ is known to have a multiplicity-free decomposition $Y = \sum_{(\chi,J)}^{\oplus} Y(\chi, J)$ as a sum of indecomposable kG-modules, where the sum is taken over all admissible pairs (χ, J). Each indecomposable summand $Y(\chi, J)$ is known to have a simple head and a simple socle (denoted by $M(\chi, J)$) and those heads (and socles) give all simple kG-modules $\{M(\chi, J) | (\chi, J)\}$ [C1]. If $\psi(\chi, J): \boldsymbol{H}(G, U) \to k^{\times}$ is the multiplicative character of the Hecke algebra $H(G, U)$ which corresponds to the indecomposable kG-summand $Y(\chi, J)$ of Y, then, as $M(\chi, J)$ is the socle of $Y(\chi, J)$, (2) implies that $M(\chi, J) \leq_{kG} I_{\psi(\chi,J)}(Y) \leq_{kG} Y(\chi, J)$, hence we have

THEOREM3: If $G = (G, B = UH, N, R, W)$ is a finite group of Lie type of characteristic p and k is a field having the same characteristic. Then, the simple kG-module $M(\chi, J)$ of G is an $(\boldsymbol{H}(G, U), \psi(\chi, J))$-invariant space for the Hecke algebra $\boldsymbol{H}(G, U)$. ■

Motivated by the cases discussed above, we may raise the following question

<u>**QUESTION**</u>: Can every simple module for any finite group G be realized as an invariant space for some Hecke algebra $\boldsymbol{H}(G, H)$; $H \leq G$?.

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