On the Design of Quantum Gates and TPCP Maps Based on Supersymmetry with Noise Performance Analysis Based on the Hudson-Parthasarathy Quantum Stochastic Differential Equation

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July 23, 2024

1 Abstract

We use supersymmetry to enlarge the dimension of the Hilbert space on which the unitary evolution of the state of the quantum fields acts. We discuss how to control the unitary evolution or TPCP maps generated by the quantum evolution of the fields by controlling the vacuum expectations of other fields in the theory. This amounts to breaking supersymmetry using controlled vacuum expectation values of the other fields. The evolution of the wave functional or TPCP maps obtained by tracing out over other fields is based on the Feynman path integral formula for the fields. By using the methods of quantum stochastic filtering, we estimate the evolving state of the fields from non-demolition noise measurements and then design a family of TPCP maps evolving in time whose outputs match the estimated evolving state. In this way, we are able to simulate the evolution of the state of the quantum noisy fields. Direct matching of the designed TPCP map to output the evolving system state is not possible since there is no way by which we can determine the exact evolving state; we can only estimate it using non-demolition measurements. The family of designed TPCP maps can be based on using a simulated master equation with unknown parameters incorporated into the Hamiltonian and the other Lindblad operators, chosen so as to match the state outputted by the quantum filter.

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https://doi.org/10.32388/GPLGFN

2 Problem formulation and discussion

By assuming that the laws of nature are inherently supersymmetric in nature, i.e., invariant under Boson-Fermion exchange, we are able to introduce additional quantum fields into the action, leading to more degrees of freedom in the corresponding Hamiltonian, i.e., an enlargement of the dimension of the underlying Hilbert space obtained by truncation of the Boson and Fermion Fock spaces. This amounts to setting up a supersymmetric action and hence a supersymmetric Hamiltonian using Chiral superfields for describing matter and non-Abelian gauge superfields for describing the particles exchanged in the interaction of the matter superfields. The total Lagrangian density thus has the form $[\Phi^*.exp(t.V).\Phi]_D + Re[f(\Phi)]_F + Re[W_L^T\epsilon.W_L]_F$. Here, Φ is the left Chiral matter superfield whose components are the scalar field, the left Chiral Dirac field, and an auxiliary F-field. V is the non-Abelian gauge superfield whose components in the Wess-Zumino gauge are the Yang-Mills gauge field, the gaugino field, and an auxiliary D field. The first term in the above Lagrangian is the supersymmetric generalization of the standard Lagrangian of the Klein-Gordon scalar field plus the Lagrangian of the massless Dirac field plus the interaction between the Dirac field and the non-Abelian Yang-Mills gauge field. The second term is a superpotential term that, after eliminating the auxiliary F and D fields using Gaussian path integration, contains a term that corresponds to a mass term of the Dirac field whose mass depends on the vacuum expectation value of the scalar field via the superpotential, and the last term is the supersymmetric generalization of the Lagrangian density of the Yang-Mills non-Abelian gauge field. Now, the crucial step here is to introduce control parameters β into the superpotential $f(\Phi)$ and hence, eventhough these parameters can be varied, supersymmetry of the total Lagrangian will not be broken, and hence the laws of nature will remain preserved. By varying these parameters, we can therefore hope to control the unitary evolution of the wave functional of the fields $\phi, \psi, V_\mu^A, \lambda^A$ starting from $t = -\infty$ and going $uptot = +\infty$ in the scattering process. Here, $\phi, \psi, V_\mu^A, \lambda_A$ are respectively the scalar field, the Dirac field, the non-Abelian gauge field, and the gaugino field. We call ψ the Higgsino field, i.e., the Fermionic superpartner of the Bosonic Higgs field ϕ because under an infinitesimal supersymmetry transformation defined by the Salam-Strathdhee supervector field

$$\alpha^T L, L = \gamma^\mu\theta.\partial_\mu + \gamma^5\epsilon.\partial_\theta$$

Applied to the left chiral superfield Φ, it becomes clear that the variation of ϕ is proportional to ψ and that the variation of ψ is proportional to ϕ, plus the auxiliary F field. Likewise, the gaugino field λ^A is the fermionic superpartner of the bosonic gauge field V_μ^A, since these are exchanged under the action of the above supervector field. The variation of the parameters of the superpotential amounts, after eliminating the auxiliary fields, to controlling the mass of the Dirac field via the scalar field. There are two ways to design the quantum gate: as either a unitary map or a TPCP map, one based on Feynman's path integral method and the other based on Hamiltonian quantum mechanics using creation and annihilation operators. Here, we discuss both of these methods.

After eliminating the auxiliary F and D fields, the total Lagrangian density has the form

$$\mathcal{L}(\phi,\psi,\lambda,V) = (D_\mu\phi)^*(D^\mu\phi)+\psi^*\gamma^0(\gamma^\mu.(i\partial_\mu+gV_\mu^A t_A)-G(\phi))\psi-(1/4)F^{A\mu\nu}F_{\mu\nu}^A$$

$$+k(1).\lambda^*\gamma^0(\gamma^\mu(i\partial_\mu+gV_\mu^A ad(t_A))\lambda$$

$$+F_1(\psi,\lambda,\phi)+F_2(\phi)$$

, where in the second term, $G(\phi)$, the Dirac mass matrix is proportional to $f''(\phi)$ and where the second last term F_1 is trilinear in $(\psi,\psi^*),(\lambda,\lambda^*),(\phi,\phi^*)$. The last term F_2 is proportional to $|f'(\phi)|^2$. It comes mainly from the observation that $[\Phi^*\Phi]_4$ contains a term F^*F, while $Re[f(\Phi)]_2$ contains a term $Re(f'(\phi)F)$, so by eliminating F, setting the variation of the action w.r.t. it to zero, we get $F=-f'(\phi)^*/2$ and substituting this back into the same expressions, we get a contribution

$$F^*F+Re(f'(\phi)F)=-|f'(\phi)|^2/4$$

It easily follows that the effective potential of the supersymmetric action is $|f'(\phi)|^2/2$ so that for the vacuum to be supersymmetric, we require that $f'(\phi_0)=0$ where ϕ_0 is the vacuum expected value of the scalar field. Therefore, if we do not wish to break supersymmetry but simultaneously require control of the superpotential via the parameters β, we must then move on the p-dimensional surface $f'(\phi_0|\beta_1,...,\beta_p)=0$ in the parameter space. Usually, the superpotential is gauge invariant, i.e., $f'(\phi).t_A\phi=0$ forall ϕ where t_A runs over the gauge group generators. If d is the dimension of the gauge group, this equation imposes d constraints on the superpotential. Therefore, the effective number of constraints in the equation $f'(\phi_0|\beta)=0$ is $n-d$ where n is the number of ϕ-components. This means that the allowable dimension of the parameter manifold for varying β is $p-(n-d)$ for designing the optimally controlled gate.

In this expression,

$$D_\mu\phi=(\partial_\mu-igV_\mu^A t_A)\phi$$

Note that ψ transforms according to the vector representation of the gauge group while $\lambda=\lambda^A t_A$ transforms according to the adjoint representation of the gauge group. Now, from this Lagrangian, we can write down the equations of motion and obtain the free wave (unperturbed) solution components for the fields $\phi,\psi,\lambda^A,V_\mu^A$ in terms of Bosonic and Fermionic creation and annihilation operators. Specifically, the unperturbed equations are

$$\partial^\mu\partial_\mu\phi=0,\gamma^\mu\partial_\mu\psi=0,\gamma^\mu\partial_\mu\lambda^A=0,$$

$$(\delta_\rho^\mu\partial^\nu\partial_\nu-\partial^\mu\partial_\rho)V_\rho^A=0,$$

The solutions to these equations are plane waves with coefficients being Bosonic and Fermionic creation and annihilation operators in momentum space. We denote these unperturbed solutions by $\phi^0, \psi^0, \lambda^{0A}, V_\mu^{A0}$. These solutions represent the evolving observables in Dirac's interaction picture, wherein observables evolve according to the unperturbed Hamiltonian while states evolve according to the perturbation in the Hamiltonian, followed by a rotation using the adjoint representation of the unperturbed Hamiltonian.

3 Unitary gate and TPCP map design based on the Feynman path integral

We write down the total action functional for the chiral and gauge superfields interacting with each other as

$$S[\phi, \psi, V, \lambda] = S_{01}(\phi|\beta) + S_{02}(\psi) + S_{03}(V) + S_{04}(\lambda)$$

$$+S_{11}(\phi, V) + S_{12}(\psi, V) + S_{14}(\lambda, V) + S_{15}(\psi, \phi|\beta) + S_{16}(\phi, \psi, \lambda|\beta)$$

with obvious meanings for the various terms. The superpotential terms are present in S_{01}, and also in S_{15}, and S_{16}. These terms arise when we eliminate the auxiliary F and D by setting the variational derivative of the action w.r.t. these to zero. This is justified because the auxiliary fields enter into the Lagrangian linearly and quadratically, so the effective action obtained by path integrating the exponential of the total action w.r.t. these fields is a Gaussian functional integral and hence can be evaluated by setting these fields to their values at which the action is stationary w.r.t them.

Now, suppose that we have detected the superpartners λ and ϕ of the standard fields V_μ, ψ, namely the superpartners of the fields used in standard quantum electrodynanics. Then, we should, using path integrals, be able to write down the unitary evolution kernel for the three fields simply by path integrating $exp(iS)$ over all the fields from time $t = -\infty$ to $t = +\infty$ with specified spatial values of these fields at times $t = -\pm\infty$. On the other hand, if, as is the present case, we have not detected the superpartners λ, ϕ of the standard QED fields V_μ, ψ, then, by path integrating the evolution kernel in the adjoint domain over these superpartners, we would get a non-unitary TPCP map that transforms an initial pure or mixed state of the fields (ψ, V_μ^A) to a mixed state of the same fields. In both cases, we can control the β parameters to design either a unitary gate or a TPCP gate that is as close as possible w.r.t. some distance measure to a given unitary or TPCP gate. Of course, the TPCP gate acts in a lower dimensional Fock space as compared to the original unitary gate because it is obtained by path integrating, which amounts to partial tracing over the scalar and gaugino fields.

4 Fermionic Filter in the Formalism of John Gough et al.

Let $A(t)$ be the Bosonic annihilation process and $J(t)$ the Fermionic annihilation process. Let $\Lambda_A(t)$ be the associated Bosonic counting process and $\Lambda_J(t)$ the associated Fermionic counting process. The system Hilbert space $\langle$ is assumed to be $\mathbb{Z}_2$ graded:

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$$

Let P_0 denote the projection onto $\langle_0$ and $\mathfrak{h}_1$ the projection onto $\mathfrak{h}_1$. It is clear that for describing unitary evolution, Bosonic noise must be coupled to the even system operators while Fermionic noise must be coupled to odd system operators. Note that X is an even system operator iff

$$X(\mathfrak{h}_0) \subset \mathfrak{h}_0, X(\mathfrak{h}_1) \subset \mathfrak{h}_1,$$

and it is an odd system operator iff

$$X(\mathfrak{h}_0) \subset \mathfrak{h}_1, X(\mathfrak{h}_1) \subset \mathfrak{h}_0$$

Also note that $P_0 + P_1 = I$ and that if X is any system operator, then

$$X = X_0 + X_1$$

where X_0 and X_1 are respectively even and odd system operators. They are given by

$$X_0 = P_0 X P_0 + P_1 X P_1, X_1 = P_0 X P_1 + P_1 X P_0$$

Define the linear map τ on the space of system operators by the equation

$$\tau(X) = X_0 - X_1 = (P_0 - P_1)X(P_0 - P_1) = \theta.X.\theta$$

where

$$\theta = P_0 - P_1$$

5 $\mathbb{Z}_2$ graded tensor product between two $\mathbb{Z}_2$ graded Hilbert spaces

In order to describe the Fermionic filter, we must first learn how to add Fermionic quantum noise to the Hudson-Parthasarathy noisy Schrödinger evolution. To do so, we must note that the noise terms in the evolution must always be Bosonic because the unitary evolution operator is overall Bosonic. Thus, we must have even and odd system operators just as we have even and odd noise operators in the form of Bosonic and Fermionic noise, respectively. Thus, if we have an odd system operator M and Fermionic noise differential $dJ(t)$, then their tensor product $MdJ = M \otimes dJ$ is even and should satisfy the anticommutation relation

$$M \otimes dJ = -dJ \otimes M$$

In other words, we must use a graded tensor product between the system and noise Hilbert spaces. To this end, we consider two $\mathbb{Z}_2$ graded Hilbert spaces $\mathcal{H}_1 = \mathcal{H}_{1e} \oplus \mathcal{H}_{1o}$ and $\mathcal{H}_2 = \mathcal{H}_{2e} \otimes \mathcal{H}_{2o}$. Define $\theta_2 = P_{2e} - P_{2o}$ and for X_1, X_2 operators in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, we define their graded tensor product to be

$$X_1 \otimes_g X_2 = (X_1)_e \otimes X_2 + (X_1)_o \otimes \theta_2 X_2$$

To verify that this is indeed a graded tensor product of operators, we must verify that if X_2, Y_1 are operators in $\mathcal{H}_2$ and $\mathcal{H}_1$ of definite parity, then

$$(X_1 \otimes_g X_2)(Y_1 \otimes_g Y_2) = (-1)^{p(X_2)p(Y_1)} X_1 Y_1 \otimes_g X_2 Y_2$$

Indeed, we have

$$(X_1 \otimes_g X_2)(Y_1 \otimes_g Y_2)$$

$$((X_1)_e \otimes X_2 + (X_1)_o \otimes \theta_2 X_2).(((Y_1)_e \otimes Y_2 + (Y_1)_o \otimes \theta_2 Y_2)$$

and by expanding this and using

$$(X_1 Y_1)_e = (X_1)_e (Y_1)_e + (X_1)_o (Y_1)_o,$$

$$(X_1 Y_1)_o = (X_1)_e (Y_1)_o + (X_1)_o (Y_1)_e$$

and the fact that

$$\theta_2^2 = I_2, \theta_2 X_2 \theta_2 = (-1)^{p(X_2)} X_2,$$

$$X_2 \theta_2 = \theta_2^2 X_2 \theta_2 = (-1)^{p(X_2)} \theta_2 X_2$$

the required property can be easily verified.

6 Quantum Noise analysis of Schrodinger evolution equation for the supersymmetric fields

A, A^*, Λ_A are the Bosonic annihilation, creation, and number processes, J, J^*, Λ_J are the Fermionic annihilation, creation, and number processes. The HPS equation is then

$$dU(t) = (-(iH+P)dt + L_1 dA + L_2 dA^* + S_1 d\Lambda_A + M_1 dJ + M_2 dJ^* + S_2 d\Lambda_J)U(t)$$

where L_1, L_2, S_1, S_2 are even system operators while M_1, M_2 are odd system operators. H is the system Hamiltonian taking into account all the interactions. Let $c_V(k), c_V(k)^*$ denote the free non-interacting Bosonic annihilation and creation operators of the gauge field V_μ^A defined by the plane wave expansion of the solution to the linearized Yang- Mills equation for V_μ^A. These satisfy the Bosonic canonical commutation relations. Let $c_\lambda(k), c_\lambda(k)^*$ denote the free non-interacting Fermionic annihilation and creation operators of the gaugino field λ^A. These satisfy the canonical anticommutation relations. Let $c_\psi(k), c_\psi(k)^*$ denote the free non-interacting Fermionic annihilation and creation operators of the Fermionic field ψ. They also satisfy the canonical anticommutation relations. Finally, let $c_\phi(k), c_\phi(k)^*$ denote the annihilation and creation operators of the Bosonic scalar field ϕ. They also satisfy the canonical commutation relations. Taking into account interactions, the interaction Hamiltonian can, using standard plane wave Fourier analysis, be expressed as

$$H_{11}(\phi, V) + H_{12}(\psi, V) + H_{14}(\lambda, V) + H_{15}(\psi, \phi|\beta) + H_{16}(\phi, \psi, \lambda|\beta)$$
$$= H_{11}(c_\phi, c_\phi^*, c_V, c_V^*) + H_{12}(c_\psi, c_\psi^*, c_V, c_V^*)$$
$$+H_{14}(c_\lambda, c_\lambda^*, c_V, c_V^*)+$$
$$H_{15}(c_\psi, c_\psi^*, c_\phi, c_\phi^*|\beta)$$
$$+H_{16}(c_\phi, c_\phi^*, c_\psi, c_\psi^*, c_\lambda, c_\lambda^*|\beta)$$

where bilinearity is understood in $H_{11}, H_{12}, H_{14}, H_{15}$ and trilinearity in H_{16}. Note that H_{15} is quadratic in $(\psi, \bar{\psi})$ and linear in $f''(\phi)$, i.e., it is trilinear in $(\psi, \bar{\psi}, f''(\phi))$. It is precisely this term that gives mass to the Dirac field when the Higgs field acquires vacuum expectation values. The mass matrix of the Dirac field is therefore $f''(\phi) = f''(\phi|\beta)$, and this mass can be controlled by altering the superpotential parameters β. The H_{16} term is trilinear in ϕ, ψ, λ. It comes from the Lagrangian component $[\Phi^*.exp(t.V)\Phi]_D$, on noting that the $[.]_3$ term in V is proportional to λ with Φ contributing the $[.]_0$ term, namely ϕ, and Φ^* contributing the $[.]_1$ term, namely ψ, we get a $[.]_4$ term from the product of these three terms which is therefore trilinear in ϕ, ψ and λ. Of course, each term can accordingly be replaced by its conjugate. It should be noted that this trilinear term cannot be controlled since it does not depend on the superpotential. Its occurrence in physics is a purely supersymmetric effect which is not present in conventional quantum field theory, and it is hoped that by designing gates that take this supersymmetric effect into account, we can detect supersymmetry in nature and also exploit its presence in technology.

The non-interacting components of the Hamiltonian are derived from the following components of the action:

$$S_{01}(\phi|\beta) + S_{02}(\psi) + S_{03}(V) + S_{04}(\lambda)$$

In the absence of the superpotential term $f(\Phi)$, S_{01}, S_{02}, S_{04} yield the quadratic self terms, but S_{03} yields, apart from quadratic self terms, cubic and fourth degree self terms because it corresponds to the Yang-Mills gauge field V_μ^A. In the presence of the superpotential, the corresponding Hamiltonians can thus be expressed as

$$H_{01} = \sum_k \omega_\phi(k) c_\phi(k)^* c_\phi(k) + G_{01}(c_\phi, c_\phi^*|\beta)$$

where G_{01} is derived from the superpotential term $F_2 = -|f'(\phi)|^2$, which came by path integrating w.r.t. the auxiliary F field, resulting in replacing F by $f'(\phi)$ as explained earlier. Likewise,

$$H_{02}(\psi) = \sum_K \omega_\psi(k) c_\psi(k)^* c_\psi(k),$$

contains no control term,

$$H_{03}(V) = \sum_k \omega_V(k)c_V(k)^* c_V(k) + G_{03}(c_V, c_V^*)$$

also contains no control term, but G_{03} contains the cubic and fourth degree Yang-Mills gauge terms, i.e., cubic and fourth degree in c_V, c_V^* .

$$H_{04} = \sum_k \omega_\lambda(k)c_\lambda(k)^* c_\lambda(k)$$

also does not contain any control term, and neither does it contain any nonlinear self interactions.

7 Some general remarks on noise analysis and noise removal from the quantum gates using the Fermionic quantum filter

We set up the HPS equation for $U(t)$ with the system Hamiltonian being the sum of all the non-interacting components:

$$H = \sum_k \omega_V(k)c_V(k)^* c_V(k) + \sum_k \omega_\phi(k)c_\phi(k)^* c_\phi(k)$$

$$+ \sum_k \omega_\psi(k)c_\psi(k)^* c_\psi(k) + \sum_k \omega_\lambda(k)c_\lambda(k)^* c_\lambda(k)$$

The characteristic frequencies of oscillations $\omega_V(k), \omega_\phi(k), \omega_\psi(k), \omega_\lambda(k)$ of the Bosonic and Fermionic fields are obtained by solving the free field equations, namely the D'Alembert wave equations for the Bosonic components and Dirac's equation for the Fermionic components, with the appropriate boundary conditions on the fields at the cavity boundary. The Lindblad noise coupling operators of the cavity fields to the bath noise are obtained by superposing the system and noise fields, calculating the quadratic form of this superposed field corresponding to the second quantized Hamiltonian, and then extracting the cross terms from this Hamiltonian between the system and the noise fields. As an example, let $\psi(x)$ and $\psi_N(x)$ denote the system and noise wave operator fields corresponding to the Dirac particles. The energy operator corresponding to the superposition of these two fields is

$$\int_{cavity} (\psi(x) + \psi_N(x))^* ((\alpha, -i\nabla) + \beta m)(\psi(x) + \psi_N(x))d^3x$$

$\psi(x)$ is built out of the operators $c_\psi(k), c_\psi(k)^*$ linearly, while $\psi_N(x)$ is built out of the Fermionic annihilation and creation white noise processes $dJ_{\psi,N}(t)/dt, dJ_{\psi,N}(t)^*/dt$ again linearly. The cross term in the above energy gives the Lindblad term in the HPS as:

$$2Re \int_{cavity} \psi(x)^* H_D \psi_N(x) d^3x - - - (a)$$

where

$$H_D = (\alpha, -i\nabla) + \beta.m$$

is the first quantized Dirac Hamiltonian. The noise self term here is

$$\int \psi_N(x)^* H_D \psi_N(x) d^3x - -- (b)$$

(a) gives rise to the term $L_{\psi 1}(t)dJ(t) + L_{\psi 2}dJ(t)^*$ while (b) gives rise to the Fermion counting term: $S_\psi(t)d\Lambda_J(t)$.

The goal is to design a one-parameter family of $TPCPmapsT_t^d, t \geq 0$ acting on mixed system states so that the state $T_t^d(\rho_s(0))$ is as close as possible to the true system state $\rho_s(t)$ defined by

$$\rho_s(t) = Tr_2(U(t)(\rho_s(0) \otimes |\phi(u) >< \phi(u)|)U(t)^*)$$

where $U(t)$ satisfies the QSDE

$$dU(t) = (-(iH+P)dt + L_1 dA + L_2 dA^* + S_1 d\Lambda_A + M_1 dJ + M_2 dJ^* + S_2 d\Lambda_J)U(t)$$

We need to determine the evolution law of $\rho_s(t)$. To this end, let $X = X_e + X_o$ be a system operator and define $\tau(X) = X_e - X_o = \theta.X.\theta$ where $\theta = P_e - P_o$. Then define

$$j_t(X) = U(t)^* X U(t)$$

We get by application of quantum Ito's formula,

$$dj_t(X) = dU(t)^* X U(t) + U(t)^* X dU(t) + dU(t)^* X dU(t)$$

$$= j_t(\theta_0(x))dt + j_t(\theta_1(X))dA(t) + j_t(\theta_2(X))dA(t)^*$$

$$+ j_t(\theta_3(X))dJ(t) + j_t(\theta_4(X))dJ(t)^* + j_t(\theta_5(X))d\Lambda_A(t)$$

$$+ j_t(\theta_6(X))d\Lambda_J(t)$$

It should be noted that $\theta_k, k = 0, 1, 2, 5, 6$ are even maps on the space of system operators while $\theta_k, k = 3, 4$ are odd maps on the same space. Thus,

$$\tau(\theta_k(X)) = \theta_k(\tau(X)), k = 0, 1, 2, 5, 6,$$

$$\tau(\theta_k(X)) = -\theta_k(\tau(X)), k = 3, 4$$

We therefore obviously have from properties of the $\mathbb{Z}_2$ graded tensor product,

$$j_t(\theta_1(X))dA(t) = dA(t)j_t(\theta_1(X)),$$

$$j_t(\theta_2(X))dA(t)^* = dA(t)^* j_t(\theta_2(X)),$$

$$j_t(\theta_3(X))dJ(t) = -dJ(t)j_t(\theta_3(\tau(X))),$$

$$j_t(\theta_4(X))dJ(t)^* = -dJ(t)^* j_t(\theta_4(\tau(X))),$$

$$j_t(\theta_5(X))d\Lambda_A(t) = d\Lambda_A(t)j_t(\theta_5(X)),$$

$$j_t(\theta_6(X))d\Lambda_J(t) = d\Lambda_J(t)j_t(\theta_6(X)),$$

Note that if X is a system observable, then the parity of $j_t(X)$ is the same as that of X and that if W is a noise operator, then

$$XW = (X_e + X_o)(W_e + W_o) = W_eX_e + W_oX_e + W_eX_o - W_oX_o$$

$$= W_eX + W_o\tau(X) = \tau(W)X_o + WX_e$$

where by product, we mean graded tensor product on both the sides. Using these formulas, we can compute

$$dTr(j_t(X)(\rho_s(0) \otimes |\phi(u) >< \phi(u)|))$$

where $|\phi(u) >$ is the tensor product of a Bosonic coherent state and a Fermionic coherent state, i.e.,

$$|\phi(u) >= |\phi(u_B) > \otimes |\phi(u_F) >$$

This computation makes use of the formulas

$$dA(t)|\phi(u) >= u_B(t)dt|\phi(u) >,$$

$$dJ(t)|\phi(u) >= u_F(t)dt|\phi(u) >$$

and

$$Tr(dA(t)^*|\phi(u) >< \phi(u)|) < \phi(u)|dA(t)^*|\phi(u) >= \bar{u}_B(t)dt,$$

$$Tr(dJ(t)^*|\phi(u) >< \phi(u)|) < \phi(u)|dJ(t)^*|\phi(u) >= \bar{u}_J(t)dt,$$

By equating the above to

$$dTr(\rho_s(t)X)$$

where

$$\rho_s(t) = Tr_2(U(t)(\rho_s(0) \otimes |\phi(u) >< \phi(u)|)U(t)^*)$$

we can derive the master equation for the system state $\rho_s(t)$ in the form

$$d\rho_s(t)/dt = \chi(\rho_s(t), u_B(t), u_F(t)dt$$

where χ is a linear map on the space of operators in the system Hilbert space depending upon the Boson and Fermion sub-parameters $u_B(t), u_F(t)$ of the coherent parameter $u(t)$. Now we are in a position to formulate the quantum filter equations. We start with non-demolition noise

$$Y_o(t) = U(t)^*Y_i(t)U(t), Y_i(t) = c(1)A(t) + \bar{c}(1)A(t)^* + c(3)\Lambda_A(t) + c(4)\Lambda_J(t)$$

It is easy to prove that Y_o satisfies the non-demolition property using the fact that for $T \geq t$,

$$d(U(T)^*Y_i(t)U(T)) = 0$$

because of the unitarity condition on $U(T)$ and the fact that $Y_i(t)$ commutes with $dA(T), dA(T)^*, dJ(T), dJ(T)^*, d\Lambda_A(T), d\Lambda_J(T)$. Note that this would not be the case if we included a $J(t)$ or $J(t)^*$ term in $Y_i(t)$ since dJ(T) anticommutes with $J(t)$ etc. Then we define the measurement algebra up to time t as

$$\eta_o(t) = \sigma(Y_o(s) : s \leq t)$$

and construct the conditional expectation of an evolving system observable X:

$$\pi_t(X) = \mathbb{E}(j_t(X)|\eta_o(t))$$

satisfying the orthogonality principle (in the language of J.Gough et.al, the reference probability method)

$$\mathbb{E}[(j_t(X) - \pi_t(X))C(t)] = 0$$

where

$$dC(t) = f(t)C(t)dY_o(t), t \geq 0, C(0) = 1$$

Applying quantum Ito's formula to this and using the arbitrariness of the function $f(t)$ gives us the two equations

$$\mathbb{E}[(dj_t(X) - d\pi_t(X))|\eta_o(t)] = 0,$$

$$\mathbb{E}[(dj_t(X) - d\pi_t(X))dY_o(t)|\eta_o(t)]$$

$$+\mathbb{E}[(j_t(X) - \pi_t(X))dY_o(t)|\eta_o(t)] = 0$$

Using quantum Ito's formula to write

$$dY_o(t) = dY_i(t) + dU(t)^* dY_i(t)U(t) + U(t)^* dY_i(t)dU(t)$$

$$= j_t(Q_0)dt + j_t(Q_1)dA(t) + j_t(Q_2)dA(t)^* + j_t(Q_3)dJ(t) + j_t(Q_4)dJ(t)^*$$

$$+j_t(Q_5)d\Lambda_A(t) + j_t(Q_6)d\Lambda_J(t)$$

where Q_0, Q_1, Q_2, Q_5, Q_6 are even system operators and Q_3, Q_4 are odd system operators, and also expressing the filter differential as

$$d\pi_t(X) = F_t(X)dt + \sum_{k\geq 1} G_{k,t}(X)dY_o(t)^k$$

where $F_t(X), G_{k,t}(X)$ are elements of the Abelian algebra $\eta_o(t)$, we can apply the above equations of the orthogonality principle to calculate the filter coefficients $F_t(X), G_{k,t}(X)$ and hence derive the Fermionic filter. In the special case when only the Fermionic counting Λ_J is present in the measurement process or only Bosonic noise $cA + \bar{c}A^*$ is present, the relevant filter equations have been derived by J.Gough et.al.

8 A brief look at the simplest super-gravity action

The simplest super-gravity action contains just two fields: the graviton of spin two, which is a Boson, and its super-partner, the Gravitino of spin 3/2, which is a Fermion. Its action has local supersymmetry, apart from being invariant under local Lorentz transformations and diffeomorphism. To define it, we require first to cast the Einstein-Hilbert action for the gravitational field in terms of its curvature but expressed in spinorial language. Specifically, we start with a tetrad e_μ^n satisfying

$$\eta_{nm} e_\mu^n e_\nu^m = g_{\mu\nu}$$

and define the spinor connection ω_μ^{mn} so that the covariant derivative of the tetrad vanishes:

$$0 = D_\nu e_\mu^n = e_{\mu,\nu}^n - \Gamma_{\mu\nu}^\rho e_\rho^n$$

$$+\omega_\nu^{nm} e_{m\mu} = 0$$

Solving this equation, we get

$$\omega_\nu^{nm} = -e^{m\mu}(e_{\mu,\nu}^n - \Gamma_{\mu\nu}^\rho e_\rho^n)$$

$$= -e^{m\mu} e_{\mu:\nu}^n - - - (c)$$

The curvature of the spinor connection is given by

$$R_{\mu\nu}^{mn} = \omega_{\nu,\mu}^{mn} - \omega_{\mu,\nu}^{mn} + [\omega_\mu, \omega_\nu]^{mn}$$

Here, it should be noted that ω_μ^{mn} should be regarded as the element $\omega_\mu^{mn}\gamma_{mn}/4$ of the Lie algebra of the spinor representation of the Lorentz group where $\gamma_{mn} = [\gamma_m, \gamma_n]$. Note that $\gamma_{mn}/4, 0 \leq m < n \leq 3$ are the standard elements of the spinor representation of the Lie algebra of the Lorentz group and as such, they satisfy the Lorentz algebra commutation relations. The covariant derivative of the Dirac spinor field ψ w.r.t. the spinor connection is defined by

$$D_\mu \psi = (\partial_\mu + \omega_\mu^{mn}\gamma_{mn}/4)\psi$$

This covariant derivative satisfies, under the spinor representation of local Lorentz transformations, the standard transformation properties that the connection should satisfy under local gauge transformations:

$$S(g(x))D_\mu S(g(x))^{-1} = D'_\mu$$

where

$$D'_\mu = \partial_\mu + \omega_\mu^{'mn}\gamma_{mn}/4$$

where

$$\omega_\mu^{'mn}(x)\gamma_{mn}/4 = \omega_\mu^{mn} S(g(x))\gamma_{mn} S(g(x))^{-1}/4 + S(g(x))\partial_\mu S(g(x))^{-1}$$

where $g(x)$ is a local Lorentz transformation and $S(.)$ is the spinor representation of the Lorentz group. It should be noted that by using the standard anticommutation relations of the Dirac matrices and the fact that the matrices $\gamma_{mn}/4$ are generators of the spinor representation S of the Lorentz group, that for any element g of the Lorentz group,

$$S(g)\gamma_m S(g)^{-1} = g_m^n \gamma_m$$

and therefore,

$$S(g)\gamma_{mn} S(g)^{-1} = g_m^{m'} g_n^{n'} \gamma_{m'n'}$$

Alternately, if

$$g(x) = 1 + \theta^{mn}(x)\gamma_{mn}/4$$

is an infinitesimal local element of the spinor representation of the Lorentz group, then under this transformation, the spinor connection transforms as

$$\delta\omega_\mu^{mn}\gamma_{mn}/4 = (\omega_\mu^{'mn} - \omega_\mu^{mn})\gamma_{mn}/4 =$$

$$[\theta^{ab}\gamma_{ab}/4, \omega_\mu^{mn}\gamma_{mn}/4] - \partial_\mu\theta^{ab}\gamma_{ab}/4$$

By expressing the commutator $[\gamma_{ab}/4, \gamma_{mn}/4]$ once again as a linear combination of the $\gamma'_{cd}s/4$ using the standard commutation relations of the Lorentz Lie algebra, we can eliminate the gamma matrices in the above relation, thereby reducing it to a direct formula for $\delta\omega_\mu^{mn}$ in terms of ω_μ^{rs} and θ^{ab} and $\partial_\mu\theta^{ab}$. That this transformation law of the spinor connection required for the invariance of the Dirac equation in curved space-time under local Lorentz transformations can be counterverified by using the formula formula (c) for the connection in terms of the tetrad by making use of the transformation law of the tetrad under local Lorentz transformations:

$$e_\mu^n(x) \to g(x)_m^n e_\mu^m(x)$$

or equivalently, under infinitesimal local Lorentz transformations defined by

$$1 + \theta^{ab}(x)\epsilon_{ab},$$

$$\delta e_\mu^n(x) = \theta^{ab}(x)(\epsilon_{ab})_m^n e_\mu^m(x)$$

where

$$\epsilon(ab)_m^n = \delta_a^n \delta\eta_{bm} - \eta_{am}\delta_b^n$$

(Note that $\epsilon(ab), 0 \leq a < b \leq 3$ are the standard generators of the Lorentz Lie algebra). In the simplest super-gravity Lagrangian, we have a graviton with the Einstein-Hilbert Lagrangian $e.R_{\mu\nu}^{mn} e_m^\mu e_n^\nu$ where $e = det((e_\mu^n))$ and a gravitino χ, which is a Fermion but unlike the Dirac Fermion ψ, it transforms according to the adjoint representation of the spinor group. Specifically, the Lagrangian of the gravitino is

$$e\bar{\chi}_\rho.\gamma^{\rho\mu\nu} ad(D_\mu)\chi_\nu$$

where

$$\bar{\chi}^\mu = \chi^{\mu*}\gamma^0, \gamma^\nu(x) = \gamma^n e_n^\nu(x)$$

and $\gamma^{\rho\mu\nu}$ is obtained by complete antisymmetrization of $\gamma^\rho\gamma^\mu\gamma^\nu$.

It can be shown that the total supergravity Lagrangian, namely the sum of the Einstein-Hilbert action and that of the gravitino, is invariant under an appropriate local supersymmetry transformation of the graviton field e_μ^n and the gravitino field χ_μ provided that the spinor connection ω_μ^{mn} that appears in the covariant derivative $D_\nu) = (\partial_\nu + \omega_\nu^{mn}\gamma_{mn}/4)$ is defined by its equation of motion, i.e., that obtained by setting the variational derivative w.r.t. it of the supergravity action to zero. (It should be noted that the gravitino is a spin 3/2 particle which can be realized as the tensor product of a spin-one vector and a spin 1/2 spinor. Thus the gravitino has a vector index μ and a spinor index a. Note that unlike the Fermion in super-Yang-Mills theory, which transforms according to the adjoint representation of the spin group, the gravitino will transform according to the tensor product of the vector representation of the Lorentz group with the spinor representation of the Lorentz group). It is easily seen then that the resulting spinor connection will be a modification of the earlier purely Bosonic case, i.e., it will also contain a gravitino bilinear component apart from the graviton bilinear component defined in (c). In other words, the spinor connection in this simplest model of supergravity described by the Lagrangian density

$$L = e.R_{\mu\nu}^{mn} e_m^\mu e_n^\nu + K.e.\bar{\chi}_\mu\gamma^{\mu\rho\nu}(\partial_\rho\chi_\mu + \omega_\rho^{mn}(\gamma_{mn}/4)\chi_\nu) - - - (d)$$

will be given by setting its variational derivative w.r.t ω_μ^{mn} to zero, recalling that

$$R_{\mu\nu}^{mn} = \omega_{\nu,\mu}^{mn} - \omega_{\mu,\nu}^{mn} + [\omega_\mu, \omega_\nu]^{mn}$$

It is clear from the fact that if the gravitino term were absent, then the field equations would give the original answer, and hence, when the gravitino term is present so that the total Lagrangian becomes locally supersymmetric, then the spinor connection will have the form

$$\omega_\mu^{mn} = -e^{m\nu}e_{\nu:\mu}^n + K_0.\bar{\chi}_\nu\gamma_\mu^{\nu\rho}(\gamma^{mn}/4)\chi_\rho$$

$$= -e^{m\nu}e_{\nu:\mu}^n + K_0\chi_\nu^{pq*}\gamma^0(\gamma_{pq}/4)\gamma_\mu^{\nu\rho}(\gamma^{mn}/4)\chi_\rho - - - (e)$$

Note that we can represent the gravitino, which belongs and transforms according to the tensor product of the vector representation of the Lorentz group with the diffeomorphism group of space-time, as

$$\chi_\mu = \chi_\mu^a\gamma_a$$

Using the identity

$$S(g)\gamma_a S(g)^{-1} = g_a^b\gamma_b$$

where g is a Lorentz transformation of space-time and $S(.)$ is the spinor representation of the Lorentz group, it follows that

$$S(g)\chi_\mu S(g)^{-1} = \chi_\mu^a.g\gamma_a g^{-1}$$

$$= \chi^a_\mu g^b_a \gamma_b$$

It follows that the spinor index a in the graviton components χ^a_μ transforms according to the vector representation of the Lorentz group under the adjoint action, ie,

$$\chi^a_\mu \to g^b_a \chi^\mu_b$$

and under space-time diffeomorphisms,

$$\chi^a_\mu(x) \to \chi^a_\nu(x) \frac{\partial x^\nu}{\partial \bar{x}^\mu}$$

In the above expression, $\gamma^{\nu\rho}_\mu$ is obtained by complete antisymmetrization of $\gamma^\nu \gamma^\mu \gamma^\rho$ followed by lowering the space-time index μ using the metric $g_{\mu\nu} = \eta_{mn} e^m_\mu e^n_\nu$. Apparently, in this locally supersymmetric theory, there is no scope for adding any control parameters, but if supergravity is correct, then we would observe both gravitons and gravitinos, and by controlling the gravitino vacuum expectations so that it has a non-zero value, we would be able to change the dynamics of the gravitons. In this case, supersymmetry is broken because supersymmetry is broken whenever Fermions acquire non-zero vacuum expectation values. Let $< \chi^\mu >$ denote the vacuum expectation value of the gravitino. Then, the effective Lagrangian of the graviton is given by (d), where we replace χ^μ by its vacuum expected value $< \chi^{mu}(x) >$ and also $\omega^{mn}_\mu(x)$ by (e), with χ^μ in that expression also replaced by $< \chi^\mu(x) >$. The dynamics of the graviton can thus be controlled by controlling the vacuum expected value of the gravitino. This is a classical analysis. If we wish to do a purely quantum mechanical analysis, then we start with the supergravity action $S[e^n_\mu, \chi^a_\mu]$ and apply classical Gravitino fields $\chi^{0a}_\mu(x)$ and calculate the path integral over one-loop diagrams w.r.t. the quantum fluctuations in the gravitino field. In this way, we would be able to calculate the scattered state of the gravitons at time $t = +\infty$ starting from an initial state at $t = -infty$ in the presence of an external c-number gravitino field. This amounts to replacing χ^a_μ by $\chi^{0a}_\mu(x) + \delta\chi^a_\mu(x)$ in the action and carrying out the path integrals w.r.t. e^n_μ and $\delta\chi^a_\mu$. To one-loop order, we retain terms that are at the most quadratic in the $\delta\chi^a_\mu$ in the action and evaluate the path integral w.r.t it as a Fermionic Gaussian integral. This is analogous to the one loop calculation of the quantum corrections to the Yang-Mills matter and gauge field action performed in [Steven Weinberg, The Quantum Theory of Fields, Vol.II, Cambridge University Press].

This is the simplest model of super-symmetry breaking in supergravity by allowing the gravitino to acquire vacuum expected values. However, in conventional global supersymmetric theories of matter and gauge fields, not involving gravity, supersymmetry is broken when the $f'(\phi_0) \neq 0$ with ϕ_0 being the vacuum expected value of the scalar field. To connect this idea with the present theory of supersymmetry breaking when the Fermionic gravitino acquires vacuum expected values, we have to only note that in the former theory, we obtained by path integrating over the auxiliary fields F, D, the value $F = f'(\phi), D = K\phi^*\phi + \xi$ and according to the standard supersymmetry transformations, $\delta\psi$ is proportional to F in the left Chiral multiplet while $\delta\lambda$ is proportional to D in the gauge superfield $V^A(x,\theta)t_A$. Thus, vanishing of the vacuum expectations of the Fermion change $\delta\psi$ is equivalent, after path integrating over the auxiliary field F, to the vanishing of $f'(\phi_0)$ which is simply the condition that the vacuum has zero energy and is therefore supersymmetric. Likewise, vanishing of the Fermion change $\delta\lambda$ is, equivalent, after path integrating over the auxiliary field D, to the vanishing of $K\phi_0^*\phi_0 + \xi$ which amounts to saying that the vacuum expectation of the scalar field is a constant in space-time. It should be noted that when the vacuum has finite positive energy, then by applying a supersymmetry transformation to it using one of the supersymmetry generators Q, we can transform a Bosonic state to a Fermionic state of the same energy and vice versa. However, the condition for broken supersymmetry is that the vacuum has nonzero positive energy. Thus, broken super-symmetry means that the vacuum state has positive energy and hence that a Bosonic vacuum state can be paired with a Fermionic vacuum state and vice versa. Therefore, unbroken supersymmetry, which implies that the vacuum energy is zero, also means that the vacuum cannot be paired with any other state and is therefore invariant under the supersymmetry generators Q. A nice discussion of this account of unbroken and broken supersymmetry can be found in [Steven Weinberg, vol. III, Supersymmetry], where it is also mentioned that if F is the operator that has eigenvalue zero on a Bosonic state and eigenvalue one on a Fermionic state, so that $(-1)^F$ has eigenvalue 1 on Bosonic states and eigenvalue -1 on Fermionic states, then, the Witten index $Tr((-1)^F)$ will get contributions only from zero energy states, because positive energy states of Bosons and Fermions pair up, leading to an equal number of Bosonic and Fermionic states at any fixed positive energy. It follows that $Tr((-1)^F)$ is nonzero and equal to the number of Bosons minus the number of Fermions in the vacuum. In particular, supersymmetry is unbroken iff the vacuum has zero energy, iff Bosonic and Fermionic states in the vacuum cannot be perfectly paired. This amounts to saying that if supersymmetry is unbroken, then we can have a vacuum state that is either purely Bosonic without any Fermion to pair up with these Bosons or vice versa.

Now, based on this discussion of super-gravity and general global supersymmetry breaking, we can formulate a more complex super-gravity theory having, apart from the gravity and gravitino, matter and gauge superfields in addition, so that the total Lagrangian of gravity and these fields is the sum of a simple supergravity component discussed above, comprising the graviton e_μ^n, the gravitino χ_μ, chiral matter fields ϕ, ψ (i.e., Higgs and Higgsino), and gauge fields V_μ^A, λ^A (gauge and gaugino). This enlarged super-gravity Lagrangian is now invariant under local supersymmetry transformations, apart from being diffeomorphism invariant. The space-time derivatives that are used in defining the infinitesimal supersymmetry transformations are now replaced by covariant derivatives with the connection being the spinor connection built out of graviton bilinears and gravitino bilinears, as discussed above. The metric of gravity now naturally enters into the picture just as it enters into conventional (non-supersymmetric) theories of gravitation, such as the scalar field and gauge field interacting with gravity described by the Lagrangian,

$$L = c(1).R\sqrt{-g} + g^{\mu\nu}(x)D_\mu\phi(x)^* D_\nu(\phi(x))\sqrt{-g(x)}$$
$$-(1/4)F^{A\mu\nu}(x)F^A_{\mu\nu}(x)\sqrt{-g(x)}$$

where

$$D_\mu\phi(x) = (\partial_\mu + ie.V^A_\mu t_A)\phi(x),$$
$$ieF^A_{\mu\nu} = [D_\mu, D_\nu]$$

An even simpler non-supersymmetric situation in which we have just another field, the scalar field, will be based on the Lagrangian

$$L = R\sqrt{-g(x)} + g^{\mu\nu}(x)\sqrt{-g(x)}\partial_\mu\phi(x)\partial_\nu\phi(x)$$
$$-V(\phi)\sqrt{-g}$$

Here, we replace $\phi(x)$ by $\phi_0 + \delta\phi(x)$ and evaluate the path integral w.r.t $\delta\phi$ up to second order in $\delta\phi$ (i.e., up to one loop correction terms). Writing

$$Q(\phi,\chi) = \int g^{\mu\nu}\sqrt{-g}\partial_\mu\phi\partial_\nu\chi.d^4x$$

we get

$$\int exp(Q(\phi_0+\delta\phi, \phi_0+\delta\phi) - V(\phi_0+\delta\phi))D\delta\phi$$
$$= exp(Q(\phi_0,\phi_0) - \int V(\phi_0)\sqrt{-g}).\int exp(2Q(\phi_0,\delta\phi) + Q(\delta\phi,\delta\phi) - \int V'(\phi_0)\sqrt{-g}\delta\phi - (1/2)\int V''(\phi_0)\sqrt{-g}(\delta\phi)^2)$$
$$= exp(Q(\phi_0,\phi_0) - V(\phi_0)).\int exp(Q(\delta\phi,\delta\phi))$$
$$[1 - (1/2)\int V''(\phi_0)\sqrt{-g}(\delta\phi)^2 + (1/2)(\int V'(\phi_0)\sqrt{-g}\delta\phi)^2$$
$$+2(Q(\phi_0,\delta\phi)^2]D\delta\phi$$
$$= exp(Q(\phi_0,\phi_0) - \int V(\phi_0) + \delta S(\phi_0, g))$$

where $\delta S(\phi_0, g)$ is obtained by evaluating the zeroth and second moments of a Gaussian density functional. The effective gravitational action, after interacting with a quantum scalar field controlled by a classical control scalar field ϕ_0, is therefore given by

$$S_{eff}(g|\phi_0) =$$
$$\int R\sqrt{-g}d^4x + \delta S(\phi_0, g) + Q(\phi_0, \phi_0|g)$$

and, by controlling the classical field $\phi_0(x)$, we can therefore control the metric and also quantum gates based on the evolution of the wave function of the metric field. Remark $Q(\phi_0, \phi_0|g)$ is the same as $Q(\phi_0, \phi_0)$ with the dependence on the metric g being emphasized.

9 Simulation Results

Super Yang-Mills theory is the simplest supersymmetric field model having just one gauge boson and one gaugino field, which are super-partners of each other. This can be derived from the discussion at the beginning of this article by deleting the scalar Higgs field ϕ and its super-partner, the Dirac Higgsino field ψ. Thus, there are only two component fields, namely the gauge field V_μ^A and the gaugino field λ^A. This theory can be derived from the super-symmetric Lagrangian $[W_A^T \epsilon W_A]_2$ where W_A is the left chiral spinor superfield defined by

$$W_A = D_R^T \epsilon D_R exp(-t.V)(D_L exp(t.V))$$

with

$$V(x,\theta) = V^A(x,\theta)t_A$$

V^A being the gauge superfield. This Lagrangian contains a D^2 term which can be eliminated by setting it, by noting that path integration sets it to zero, its value at which the Lagrangian is stationary. More generally, we can add to this Lagrangian a supersymmetric term $\xi_A D^A$ linear in the D, and then path integration sets D^A to a constant.

Here, we simulate a quantum unitary gate and a TPCP map based on super Yang-Mills theory. The Lagrangian in this theory is

$$L = (-1/4)F^{A\mu\nu}F_{\mu\nu}^A + (\lambda^A)^T\gamma^5\epsilon\gamma^\mu[D_\mu, \lambda^A]$$

where

$$D_\mu = \partial_\mu + ig.V_\mu^A t_A$$

$$[t_A, t_B] = -iC(ABC)t_C$$

and

$$igF_{\mu\nu}^A = [D_\mu, D_\nu]^A$$

or equivalently,

$$F_{\mu\nu}^A = V_{\nu,\mu}^A - V_{\mu,\nu}^A + g.C(ABC)V_\mu^B V_\nu^C$$

It is easily seen that by adopting the gauge condition $V_0^A = 0$, with the canonical position fields as $Q_r^A V_r^A, r = 1,2,3$, so that the canonical momentum fields for the gauge part of the action become

$$P_r^A = \partial L/\partial V_{r,0}^A = F_{0r}^A$$

we can express the corresponding Hamiltonian by applying the Legendre transformation in the form $(1/2)F_{0r}^A F_{0r}^A + (1/4)F_{rs}^A F_{rs}^A$, or equivalently in abbreviated notation as

$$H_{gge} = C_1(rs)P_rP_s + C_2(rs)Q_rQ_s + C_3(rsk)Q_rQ_sQ_k + C_4(rskm)Q_rQ_sQ_kQ_m$$

Here, Q_r, P_r are abbreviated notations for the canonical position and momentum variables of the gauge part of the action. The Hamiltonian of the gaugino part of the action, on the other hand, is the sum of a free gaugino part

$i(\lambda^A)^T\gamma^5\epsilon\gamma^r\partial_r\lambda^A$ and an interaction part with the gauge potential $C(ABC)(\lambda^A)^T\gamma^5\epsilon\gamma^r V_r^B\lambda^C$. Denoting by q_r, p_r the canonical position and momentum variables of this gaugino part of the action, we can express the gaugino Hamiltonian along with its interaction with the gauge part as

$$H_{gino} = D_1(rs)p_r q_s + D_2(rsk)p_r q_k Q_s$$

Note that $Q_r's$ are built from V_r^A, $P_r's$ from $V_{r,0}^A$, q_r from λ^A, and finally, p_r from $(\lambda^A)^*\gamma^0 = (\lambda^A)^T\gamma^5\epsilon$ using the fact that the λ^A are Majorana Fermions. Now comes the crucial supersymmetry breaking argument: When the gaugino fields acquire vacuum expectation values, the second term in H_{gino} gets replaced by $D_2(rsk) < p_r >< q_k > Q_s$, which can be expressed as $a(s)Q_s$, where $a(s) = D_2(rsk) < p_r >< q_k >$ are parameters dependent upon the vacuum expectations of the gaugino field, and this amounts to the following effective Hamiltonian for the gauge field:

$$H_{gge,eff}(Q,P) = C_1(rs)P_rP_s + C_2(rs)Q_rQ_s + C_3(rsk)Q_rQ_sQ_k + C_4(rskm)Q_rQ_sQ_kQ_m + a(s)Q_s$$

We shall base our simulation studies on this model.

To simplify matters further, we shall assume that there is just one position variable for the gauge field, and therefore the controlled Hamiltonian can be expressed as

$$H(Q,P|a(t)) = (P^2+Q^2)/2 + c(1)Q^3 + c(2)Q^4 + b(t)Q$$

where we have allowed the parameter $b = b(t)$ to depend on time. This looks like the Hamiltonian of a one-dimensional harmonic oscillator with cubic and fourth-degree anharmonic terms plus a control electric field interaction term, assuming that the harmonic oscillator particle carries charge. Thus, the control parameter $b(t)$ can be interpreted as a time-varying control electric field. Introduce Using time-independent perturbation theory and the quantum theory of the harmonic oscillator, we can compute the approximate eigenfunctions $|u_n>, n \geq 0$ and corresponding energy eigenvalues $E_n, n \geq 0$ of the harmonic oscillator with anharmonic perturbations described by the Hamiltonian

$$H_0 = (P^2+Q^2)/2 + c(1)Q^3 + c(2)Q^4$$

Then, using time-dependent perturbation theory, we can compute the approximate evolution operator of the perturbed system as

$$U(T) \approx U_0(T) - i\int_0^T b(t)U_0(T-t)QU_0(t)dt$$

where

$$U_0(t) = exp(-itH_0) = \sum_n exp(-itE_n)|u_n><u_n|$$

Truncation to $N+1$ dimensions gives the $N+1 \times N+1$ dimensional unitary gate

$$U_N(T) = U_{0N}(T) - i\sum_{n,m=0}^N \int_0^T b(t)exp(-iE_n(T-t) - iE_m t) < u_n|QW|u_m > |u_n><u_m|$$

$$= \sum_{n,m=0}^{N} [\delta[n-m] - \int_0^T b(t) exp(i(E_n - E_m)t)dt) exp(-iE_nT) < u_n|Q|u_m >]|u_n >< u_m|$$

and we can control the function $b(t)$ or, better still, its truncated Fourier transform $\hat{b}_T(\omega) = \int_0^T b(t)exp(i\omega t)dt$ evaluated at $E_n - E_m$ so that the gate $U_N(T)$ having matrix elements

$$< u_n|U_N(T)|u_m >=$$

$$\delta[n-m] - \int_0^T b(t)exp(i(E_n - E_m)t)dt)exp(-iE_nT) < u_n|Q|u_m >$$

$(n, m = 0, 1, ..., N)$ is as close as possible to a desired $N + 1 \times N + 1$ unitary gate w.r.t. the Frobenius norm.

10 References

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