Research Article

# Holonomic Quantum Computing

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**We present a geometric framework for holonomic quantum computing in which quantum gates arise from global properties of control manifolds rather than fine-tuned dynamical evolution. Quantum states are modeled as complex projective fibers over a classical control manifold, and adiabatic loops induce unitary gates through Berry and Wilczek–Zee holonomy. Within this setting, we introduce *Quantum Inner State Manifolds* (QISMs) as symplectic fiber bundles equipped with a natural unitary connection governed by the Fubini–Study form. Using the Ambrose–Singer theorem, we show that generic QISMs generate holonomy groups dense in $U(N)$, establishing universality. Fault tolerance emerges from global geometric features, providing a robust geometric foundation for quantum gate design.**

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## 1. Introduction

Quantum computation stands at the intersection of physics, mathematics, and information theory, promising computational capabilities fundamentally beyond those of classical machines. Its conceptual foundation rests on three uniquely quantum phenomena: superposition, entanglement, and interference. These features enable algorithmic advantages such as Shor's polynomial-time factorization algorithm, Grover's quadratic speedup for unstructured search, and the efficient simulation of quantum many-body systems. Together, these results demonstrate that quantum mechanics is not merely a physical theory, but a computational resource.

Despite these theoretical successes and rapid experimental advances, the realization of scalable quantum computers remains severely constrained by the fragility of quantum information. Quantum states are extraordinarily sensitive to environmental interactions, leading to decoherence, leakage, and control-induced errors. Unlike classical bits, quantum bits cannot be copied, measured non-destructively, or

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stabilized through naive redundancy. This fragility lies at the heart of the fault-tolerance problem in quantum computation.

The prevailing response to this challenge has been the development of quantum error-correcting codes, particularly stabilizer codes and topological codes such as the surface code. These frameworks encode logical qubits into highly entangled states of many physical qubits, protecting information through nonlocal correlations and locality constraints. While extraordinarily powerful, such approaches incur substantial overhead: the number of physical qubits required per logical qubit grows rapidly with the desired error threshold. This raises a fundamental question that motivates the present work:

> *Can fault tolerance be achieved not only through redundancy, but through geometry and topology themselves?*

A key insight underlying this paper is that quantum mechanics is already geometric at a fundamental level. The true space of pure quantum states is not the Hilbert space $\mathcal{H}$ itself, but its projectivization

$$\mathcal{P}(\mathcal{H}) := (\mathcal{H} \setminus \{0\})/\mathbb{C}^{\times},$$

which removes physically irrelevant global phases. For an $N$-level system, this space is the complex projective manifold $\mathbb{CP}^{N-1}$. This manifold carries a canonical Kähler structure consisting of the Fubini–Study symplectic form $\omega_{FS}$, a compatible complex structure $J$, and the associated Riemannian metric $g_{FS}$. These structures are not auxiliary; they encode the kinematics of quantum mechanics itself. In particular, quantum observables correspond to Hamiltonian functions on $(\mathbb{CP}^{N-1}, \omega_{FS})$, and Schrödinger evolution is equivalent to Hamiltonian flow

$$\dot{\psi} = X_H(\psi),$$

where $X_H$ is the Hamiltonian vector field generated by the observable $H$. Phenomena such as Berry phases and geometric phases arise naturally as holonomies of connections associated with this projective geometry.

In realistic physical systems, however, quantum states do not exist in isolation. They depend continuously on externally controlled classical parameters such as electromagnetic fields, coupling constants, geometric configurations, and control protocols. These parameters vary smoothly and naturally organize into a manifold $B$, often referred to as the control or parameter space. As the parameters vary, the Hamiltonian changes, and with it the representation of the quantum state space.

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Consequently, instead of a single projective space $\mathbb{CP}^{N-1}$, one is naturally led to consider a family of projective quantum state spaces parametrized by $B$.

Mathematically, this leads to the study of fiber bundles

$$\pi : M \longrightarrow B,$$

whose fibers $\pi^{-1}(x)$ are copies of $\mathbb{CP}^{N-1}$. The total space $M$ simultaneously encodes classical control data and quantum states within a single geometric object.

**Geometric Intuition for Quantum Inner State Manifolds.** A Quantum Inner State Manifold (QISM) may be visualized as a smooth bundle of quantum state spaces attached to a classical control manifold. Each point of the base $B$ represents a classical configuration of external parameters, and above it sits the entire quantum state space $\mathbb{CP}^{N-1}$ available in that configuration. Motion in the base corresponds to changing external parameters, while motion in the fiber corresponds to changing the quantum state. Closed loops in the base induce holonomies acting on the fibers, producing quantum gates determined by global geometric data rather than fine-tuned local control. Because topology is insensitive to small perturbations, such gates are inherently robust. At the same time, the same fibered geometry enables cancellations of characteristic classes, leading naturally to symplectic Calabi–Yau manifolds.

Motivated by this intuition, we introduce the framework of *Quantum Inner State Manifolds*. Formally, a QISM is a symplectic fiber bundle $\pi : (M, \omega) \to B$ whose fibers are complex projective spaces $\mathbb{CP}^{N-1}$ equipped with their Fubini–Study symplectic form, whose structure group reduces to $(N)$, and whose total space admits a symplectic form restricting to $\omega_{FS}$ on each fiber. Physically, points of $M$ represent parameterized quantum states; mathematically, $M$ may be realized as the projectivization of a complex vector bundle over $B$.

One of the most significant consequences of this framework is the natural emergence of geometric and topological mechanisms for fault-tolerant quantum computation. Given a connection on the bundle $\pi : M \to B$, parallel transport along a loop $\gamma \subset B$ induces a holonomy

$$(\gamma) \in (N),$$

acting on the quantum fibers. These holonomies generalize Berry phases and form the basis of holonomic quantum computation. Because the resulting gates depend only on global geometric features such as curvature and homotopy class, they are robust against a wide class of local perturbations.

From the perspective of symplectic topology, Quantum Inner State Manifolds exhibit striking structural properties. The first Chern class of the total space decomposes schematically into base and fiber contributions,

$$c_1(M) = \pi^* c_1(B) + c_1(\text{vertical}),$$

and, under suitable geometric choices, these contributions can cancel exactly, yielding $c_1(M) = 0$. This condition characterizes symplectic Calabi–Yau manifolds. In six dimensions, such manifolds occupy a central position in symplectic geometry and mathematical physics. Moreover, QISMs provide a natural setting for symplectic surgery techniques, including generalized Luttinger surgery, allowing the construction of infinite families of simply connected symplectic Calabi–Yau threefolds with exotic smooth structures.

In this way, Quantum Inner State Manifolds unify two seemingly disparate themes: the geometric foundations of fault-tolerant quantum computation and the construction of exotic symplectic Calabi–Yau manifolds. The sections that follow develop this framework rigorously, beginning with the necessary geometric preliminaries and culminating in explicit constructions and quantum computational applications.

### *1.1. Historical Context and Motivation*

The geometric approach to quantum mechanics dates back to the work of Kibble [1] and others who recognized that the space of pure quantum states of an $N$-level system has the structure of a complex projective space $\mathbb{CP}^{N-1}$ equipped with the Fubini–Study metric. This perspective was further developed into geometric quantum mechanics by Ashtekar and Schilling [2], Brody and Hughston [3], and others. The key insight is that quantum dynamics can be formulated in geometric terms, with the symplectic structure of $\mathbb{CP}^{N-1}$ playing a role analogous to classical phase space.

In quantum information theory, this geometric viewpoint has led to the development of holonomic quantum computation (HQC) [4][5], where quantum gates are implemented by adiabatically transporting quantum states along loops in parameter space, generating Berry-phase holonomies that are robust against certain types of noise. However, most work in HQC has focused on specific physical implementations rather than developing a comprehensive geometric framework.

On the mathematical side, the construction of exotic smooth structures on 4-manifolds has been a major theme since the groundbreaking work of Donaldson [6] and Freedman [7]. More recently, similar exotic phenomena have been discovered in higher dimensions, including exotic Calabi–Yau threefolds [8][9][10]. These constructions typically employ surgical techniques such as Gompf's symplectic fiber sum [11] and Luttinger surgery [12], which allow for the modification of symplectic manifolds while preserving key geometric properties.

Our work synthesizes these developments by showing that the mathematical tools used to construct exotic Calabi–Yau manifolds naturally give rise to geometric structures that are ideally suited for fault-tolerant quantum computation. This synthesis is not merely analogical but reflects a deep connection between the geometry of quantum state spaces and the topology of symplectic manifolds.

### *1.2. Key Innovations and Results*

This paper develops a unified geometric framework that connects symplectic topology, Calabi–Yau geometry, and quantum information theory through the introduction of *Quantum Inner State Manifolds (QISMs)*. The principal innovations and results are summarized below.

1. **Quantum Inner State Manifolds as Structured Symplectic Fibrations.** We introduce Quantum Inner State Manifolds as symplectic fiber bundles

$$\pi : (M, \omega_M) \longrightarrow (B, \omega_B),$$

whose fibers are complex projective spaces $\mathrm{CP}N - 1$ equipped with the Fubini–Study symplectic form. Unlike standard projective bundles, QISMs admit globally defined symplectic structures on the total space that nontrivially couple base and fiber directions. We establish precise compatibility conditions ensuring that $\omega_M$ restricts to the Fubini–Study form on each fiber while remaining closed and nondegenerate on $M$ (Sections 2 and 3). This provides a geometric realization of parameter-dependent quantum state spaces within a single symplectic manifold.

2. **Symplectic Construction of Exotic Calabi–Yau Threefolds.** Using QISMs as geometric building blocks, we construct infinite families of compact, simply-connected symplectic six-manifolds with vanishing first Chern class. These manifolds are shown to be homeomorphic but not diffeomorphic to standard Calabi–Yau threefolds, yielding genuinely exotic symplectic Calabi–Yau geometries. The constructions rely on generalized coisotropic Luttinger surgeries that intertwine base and fiber directions, extending classical four-dimensional techniques to the six-dimensional Calabi–Yau setting (Sections 4 and 5).

3. **Explicit Topological and Geometric Invariants.** For all constructed examples, we compute characteristic classes, Betti numbers, and fundamental groups explicitly. We prove that the resulting

manifolds are simply-connected and possess the same rational cohomology as classical Calabi–Yau threefolds. Explicit symplectic forms are constructed, and the vanishing of the first Chern class is verified via direct calculations using compatible almost-complex structures (Section 6).

4. **Geometric Realization of Quantum Information Processing.** The QISM framework naturally supports a geometric formulation of quantum computation. We show that adiabatic transport along loops in the base manifold induces holonomies acting on the projective fibers, yielding holonomic quantum gates. We further demonstrate how multipartite fiber configurations give rise to QISM-based cluster states suitable for measurement-based quantum computation, and we outline hybrid error-correction schemes in which stabilizer codes are embedded into the intrinsic geometry of the fibers and their symplectic couplings (Sections 7 and 8).

5. **Rigorous and Self-Contained Mathematical Framework.** All constructions are carried out with complete mathematical rigor. Proofs include detailed analyses of symplectic forms, Chern class computations, surgery effects on topology, and holonomy-based universality arguments. Extended derivations, technical lemmas, and auxiliary results are provided in Appendices A–I.

### *1.3. Outline of the Paper*

The paper is organized to progressively develop the geometric foundations of Quantum Inner State Manifolds and their applications to exotic symplectic geometry and quantum information theory.

- Section 1 introduces the physical and mathematical motivation, places the work in historical context, and outlines the central goals of the paper.
- Section 2 develops the geometric background from projective quantum mechanics and provides the formal definition of Quantum Inner State Bundles and Quantum Inner State Manifolds.
- Section 3 presents the conceptual framework connecting geometry, holonomy, and fault-tolerant quantum computation, establishing the geometric interpretation underlying all subsequent results.
- Section 4 reviews and extends the symplectic construction techniques used in the paper, including Gompf fiber sums, Luttinger surgery, coisotropic surgeries in six dimensions, Lefschetz pencils, and symplectic blow-up and blow-down.
- Section 5 contains the core mathematical constructions of exotic symplectic Calabi–Yau threefolds obtained from QISMs, with explicit examples and surgery sequences.
- Section 6 provides detailed computations of topological and geometric invariants, including Chern classes, Betti numbers, fundamental groups, and smooth-structure distinctions.

- Section 7 develops the holonomic quantum computation framework arising from QISMs, including curvature-controlled universality and geometric fault tolerance.
- Section 8 explores QISM-based cluster states, measurement-based quantum computation, and hybrid geometric error-correction schemes.
- Section 9 discusses extensions, physical interpretations, and connections to related areas such as moduli spaces, topological phases, and quantum control theory.
- Section 10 summarizes the results and outlines directions for future research.
- Appendices A–I contain extended Chern class calculations, detailed surgery proofs, holonomy and universality analyses, and supplementary geometric constructions.

Throughout the paper, we maintain a balance between mathematical rigor and physical intuition. Formal proofs are complemented by conceptual explanations that clarify how the geometry of Quantum Inner State Manifolds simultaneously enables exotic symplectic Calabi–Yau structures and intrinsically robust models of quantum computation.

## 2. Quantum Inner State Manifolds: Definitions and Basic Properties

### *2.1. Geometric Quantum Mechanics Background*

Before defining Quantum Inner State Manifolds, we recall the geometric formulation of quantum mechanics. For an $N$-level quantum system, the space of pure states is the complex projective space $\mathbb{CP}^{N-1}$ , which carries a natural Kähler structure. Specifically:

**Definition 2.1** (Fubini–Study Structure). *The Fubini–Study metric on* $\mathbb{CP}^{N-1}$ *is defined by*

$$g_{FS}(X, Y) = \frac{1}{2} Tr(\rho(XY + YX))$$

*where* $\rho$ *is the density matrix corresponding to a point in* $\mathbb{CP}^{N-1}$*, and* $X, Y$ *are tangent vectors. The compatible symplectic form is*

$$\omega_{FS}(X, Y) = \frac{i}{2} Tr(\rho[X, Y]).$$

*These satisfy the Kähler condition:* $\omega_{FS}(X, Y) = g_{FS}(JX, Y)$ *where* $J$ *is the complex structure.*

The Fubini–Study metric gives $\mathbb{CP}^{N-1}$ the structure of a symmetric space with constant holomorphic sectional curvature. Importantly, the symplectic form $\omega_{FS}$ represents a generator of $H^2(\mathbb{CP}^{N-1}; \mathbb{Z}) \cong \mathbb{Z}$.


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In geometric quantum mechanics, the evolution of a quantum state is described by Hamiltonian flow on $\mathbb{CP}^{N-1}$. Given a Hamiltonian operator $H$, the corresponding function on $\mathbb{CP}^{N-1}$ is $f_H(\psi) = \langle\psi|H|\psi\rangle$, and the Hamiltonian vector field $X_{f_H}$ defined by $\iota_{X_{f_H}}\omega_{FS} = -df_H$ generates the quantum dynamics via the Schrödinger equation.

### *2.2. Definition of Quantum Inner State Bundles*

We now extend this geometric picture to include classical parameter spaces. The key idea is to consider families of quantum systems parameterized by points in a classical manifold.

**Definition 2.2** (Quantum Inner State Bundle). *Let $B$ be a smooth, connected, $2n$-dimensional manifold (the base), and let $H$ be an $N$-dimensional complex Hilbert space. A quantum inner state bundle over $B$ is a fiber bundle $\pi: E \to B$ with the following structure:*

*1. The fiber over each point $x \in B$ is isomorphic to the complex projective space $\mathbb{CP}^{N-1}$.*

*2. Each fiber $\pi^{-1}(x) \cong \mathbb{CP}^{N-1}$ is equipped with the standard Fubini–Study Kähler structure $(\omega_{FS}, g_{FS}, J_{FS})$, normalized so that $[\omega_{FS}]$ generates $H^2(\mathbb{CP}^{N-1}; \mathbb{Z}) \cong \mathbb{Z}$.*

*3. The transition functions of the bundle take values in the projective unitary group $(N) = U(N)/U(1)$, acting on $\mathbb{CP}^{N-1}$ by isometries of the Fubini–Study metric.*

Such a bundle can be constructed as the projectivization of a rank-$N$ complex vector bundle $\mathcal{V} \to B$ with a Hermitian metric. Concretely, if $\mathcal{V}$ is a Hermitian vector bundle, then its projectivization $\mathbb{P}(\mathcal{V})$ is the fiber bundle whose fiber at $x$ is the projective space of the complex vector space $\mathcal{V}_x$. The Fubini–Study structure on each fiber is induced by the Hermitian inner product on $\mathcal{V}_x$.

The choice of transition functions in $(N)$ rather than $U(N)$ reflects the fact that quantum states are defined only up to overall phase. This is crucial for the geometric interpretation, as it ensures that the bundle respects the projective nature of quantum state spaces.

### *2.3. Definition of Quantum Inner State Manifolds*

We now add symplectic structure to the total space of a quantum inner state bundle.

**Definition 2.3** (Quantum Inner State Manifold (QISM)). *A Quantum Inner State Manifold (QISM) is a quantum inner state bundle $\pi: M \to B$ where the total space $M$ is equipped with a symplectic form $\Omega$ (or more generally, a Kähler structure) such that:*


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*1. The projection $\pi$ is a symplectic fibration: for each $x \in B$, the restriction of $\Omega$ to the fiber $\pi^{-1}(x)$ is a positive multiple of the Fubini–Study form $\omega_{FS}$.*

*2. There exists a compatible almost complex structure $J$ on $M$ making $\pi$ pseudoholomorphic (i.e., $J$ preserves the vertical tangent bundle and projects to an almost complex structure on $B$).*

*We denote such a structure by $(M, \Omega, J, \pi)$.*

The key feature of a QISM is that it combines the symplectic geometry of the base (classical parameter space) with the symplectic geometry of the fibers (quantum state spaces) in a coherent way.

## 2.4. Local Description and Symplectic Form

Locally, on a trivializing neighborhood $U \subset B$, we have $M|_U \cong U \times \mathrm{CP}^{N-1}$. In such a chart, the symplectic form can be written as

$$\Omega = \pi^* \omega_B + \epsilon \omega_{FS} + \eta,$$

where:

- $\omega_B$ is a symplectic form on $B$,
- $\epsilon > 0$ is a scale parameter controlling the relative size of fibers,
- $\eta$ is a closed 2-form that represents the curvature of the bundle and couples the base and fiber directions.

Globally, $\eta$ is the curvature form of a connection on the principal $(N)$-bundle associated to $\mathrm{P}(\mathcal{V})$. The presence of $\eta$ is crucial for achieving $c_1(M) = 0$ in many interesting cases.

**Proposition 2.4** (Existence of Compatible Symplectic Structures). *Let $\pi : M \to B$ be a quantum inner state bundle as in Definition 2.2. Assume $B$ is symplectic with symplectic form $\omega_B$. Then for sufficiently small $\epsilon > 0$, there exists a closed 2-form $\eta$ on $M$ such that $\Omega = \pi^* \omega_B + \epsilon \omega_{FS} + \eta$ is a symplectic form on $M$ (i.e., $\Omega^{\wedge (n+N-1)} \neq 0$ ). Moreover, $\eta$ can be chosen to represent the Euler class of the bundle.*

Proof. The proof follows the standard argument for symplectic structures on projective bundles [13]. We outline the main steps:

1. Choose a Hermitian metric on the underlying vector bundle $\mathcal{V}$ and let $\nabla$ be a unitary connection with curvature $F_\nabla$. The associated principal $(N)$-bundle has a connection form $\theta$ whose curvature $\eta = d\theta + \frac{1}{2}[\theta \wedge \theta]$ is a closed 2-form on $M$.

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2. The form $\omega_{FS}$ is symplectic on each fiber. The form $\pi^* \omega_B$ is degenerate along fibers but nondegenerate in horizontal directions. The coupling term $\eta$ provides nondegenerate pairing between vertical and horizontal directions.

3. Consider the 2-form $\Omega_t = \pi^* \omega_B + t(\epsilon\omega_{FS} + \eta)$ for $t \in [0, 1]$. For small $\epsilon$, this form is nondegenerate for all $t$ because: - On vertical vectors, $\Omega_t$ restricts to $t\epsilon\omega_{FS}$, which is nondegenerate for $t > 0$. - On horizontal vectors, $\Omega_t$ restricts to $\pi^* \omega_B$, which is nondegenerate. - The mixed terms are controlled by $\eta$, and for small $\epsilon$, they don't cause degeneracy.

4. Since nondegeneracy is an open condition and $\Omega_0 = \pi^* \omega_B$ is nondegenerate on horizontal directions, there exists $\epsilon_0 > 0$ such that $\Omega_t$ is nondegenerate for all $t \in [0, 1]$ when $\epsilon < \epsilon_0$. In particular, $\Omega_1$ is symplectic.

A detailed linear algebra computation verifying nondegeneracy is provided in Appendix B. □

## *2.5. Interpretation as Parameterized Quantum Systems*

From a quantum-mechanical perspective, a point $(x, [\psi]) \in M$ represents a *parameterized quantum state*:

- $x \in B$ is a classical control parameter (e.g., external magnetic field, laser frequency, coupling strengths),
- $[\psi] \in \mathbb{C}P^{N-1}$ is the quantum state of an $N$-level system.

The bundle structure encodes how the quantum state space varies as the classical parameters are changed. The symplectic form $\Omega$ provides a geometric structure that unifies the classical phase space $B$ with the quantum state space. This unification enables the study of coupled classical-quantum dynamics within a single geometric framework.

## *2.6. Examples of QISMs*

**Example 2.5** (Trivial QISM). *The simplest example is the product $M = B \times \mathbb{C}P^{N-1}$ with the symplectic form $\Omega = \omega_B \oplus \epsilon\omega_{FS}$. Here the bundle is trivial, and there is no coupling between base and fiber ($\eta = 0$). The quantum system is decoupled from the base parameters, except that the base provides a classical control space.*

**Example 2.6** (Twisted $\mathbb{C}P^1$-bundle over $K3 \times S^1$). *Let $B = S \times S^1$, where $S$ is a $K3$ surface (a compact hyperkähler 4-manifold with $c_1 = 0$). Let $\mathcal{V} = L \oplus \underline{\mathbb{C}}$ be a rank-2 complex vector bundle over $B$, where $L$ is a complex line* *bundle with* $c_1(L) = k\alpha$ *for some* $k \in \mathbb{Z}$ *and* $\alpha \in H^2(B; \mathbb{Z})$*, and* $\underline{\mathbb{C}}$ *is the trivial line bundle. Then* $M = \mathbb{P}(\mathcal{V})$ *is a* $\mathbb{CP}^1$*-bundle over* $B$*.*

*Using the formula from Lemma 2.8, we compute:*

$$c_1(M) = c_1(B) + 2c_1(L).$$

*Since* $c_1(B) = 0$ *(because* $K3 \times S^1$ *has trivial canonical bundle), we can choose* $k = 0$ *to obtain* $c_1(M) = 0$*, or we can choose* $k$ *such that* $2c_1(L) = 0$ *in cohomology (e.g., if* $\alpha$ *is 2-torsion). In any case,* $M$ *is a 6-manifold with a natural symplectic structure, and as we will show in Section 5, performing Luttinger surgeries on certain tori in* $M$ *yields exotic symplectic Calabi–Yau threefolds.*

**Example 2.7** ($\mathbb{CP}^2$-bundle over $T^4 \times S^1$). *Let* $B = T^4 \times S^1$*, where* $T^4$ *is the 4-torus. Although* $c_1(B) = 0$*, we will twist the fiber bundle so that the coupling term* $\eta$ *is nontrivial, yet we still achieve* $c_1(M) = 0$*.*

*Let* $\mathcal{V} \to B$ *be a rank-3 complex vector bundle with* $c_1(\mathcal{V}) = 0$ *and* $c_2(\mathcal{V}) = \beta \neq 0$*. Such bundles exist because* $H^4(B; \mathbb{Z})$ *has torsion-free part. Let* $M = \mathbb{P}(\mathcal{V})$*, a* $\mathbb{CP}^2$*-bundle over* $B$*.*

*From Lemma 2.8, we have:*

$$c_1(M) = c_1(B) + 2c_1(\mathcal{V}) = 0.$$

*Now* $M$ *is a 6-manifold with* $c_1 = 0$*, but it is not simply-connected:* $\pi_1(M) \cong \pi_1(T^4 \times S^1) \cong \mathbb{Z}^5$*. To kill the fundamental group, we perform multiple coisotropic Luttinger surgeries as described in Section 4.3.*

## *2.7. Chern Class Calculations for Projective Bundles*

To understand when a QISM has vanishing first Chern class (and thus is a candidate for a symplectic Calabi–Yau manifold), we need formulas for the Chern classes of projective bundles.

**Lemma 2.8** (Chern Classes of Projective Bundles). *Let* $\mathcal{V} \to B$ *be a rank-*$r$ *complex vector bundle over a manifold* $B$*, and let* $M = \mathbb{P}(\mathcal{V})$ *be its projectivization. Let* $\xi$ *be the tautological line bundle over* $M$*. Then the total Chern class of* $M$ *satisfies:*

$$c(M) = \pi^* c(B) \cdot \prod_{i=1}^{r} (1 + y_i - c_1(\xi)),$$

*where* $y_i$ *are the Chern roots of* $\pi^* \mathcal{V}$ *(i.e., formal variables such that* $c(\pi^* \mathcal{V}) = \prod_{i=1}^{r}(1 + y_i)$*).*

*Proof.* The tangent bundle of $M$ fits into the exact sequence:

$$0 \to T_\pi \to TM \to \pi^* TB \to 0,$$

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where $T_\pi$ is the vertical tangent bundle (tangent to fibers). We have $T_\pi \cong Hom(\xi, Q) \cong \xi^* \otimes Q$, where $Q$ is the quotient bundle in the exact sequence:

$$0 \to \xi \to \pi^* \mathcal{V} \to Q \to 0.$$

From this sequence, we get $c(\pi^* \mathcal{V}) = c(\xi)c(Q)$, so $c(Q) = c(\pi^* \mathcal{V}) / c(\xi)$.

Now, $c(T_\pi) = c(\xi^* \otimes Q)$. Using the splitting principle, if $Q$ has Chern roots $q_1, \ldots, q_{r-1}$, then the Chern roots of $\xi^* \otimes Q$ are $q_i - c_1(\xi)$. Therefore,

$$c(T_\pi) = \prod_{i=1}^{r-1} (1 + q_i - c_1(\xi)).$$

But the $q_i$ are related to the $y_i$ by the relation $\prod_{i=1}^{r} (1 + y_i) = (1 + c_1(\xi)) \prod_{i=1}^{r-1} (1 + q_i)$, which implies that the $q_i$ are the roots of the polynomial obtained by dividing $\prod_{i=1}^{r} (1 + y_i)$ by $(1 + c_1(\xi))$.

Finally, from the exact sequence for $TM$, we have $c(M) = c(T_\pi) \cdot \pi^* c(B)$. Combining these facts gives the formula.

A detailed computation with explicit examples is provided in Appendix A. □

For the special cases relevant to our constructions:

**Corollary 2.9.** *For a* $\mathbb{CP}^1$*-bundle* $M = \mathbb{P}(L \oplus \underline{\mathbb{C}})$ *over* $B$*:*

$$c_1(M) = c_1(B) + 2c_1(L).$$

**Corollary 2.10.** *For a* $\mathbb{CP}^2$*-bundle* $M = \mathbb{P}(\mathcal{V})$ *over* $B$ *with* $rank(\mathcal{V}) = 3$*:*

$$c_1(M) = c_1(B) + 2c_1(\mathcal{V}).$$

These formulas are essential for designing QISMs with $c_1(M) = 0$.

### *2.8. Relation to Hamiltonian Dynamics*

The symplectic structure on a QISM allows us to study Hamiltonian dynamics that couple classical parameters and quantum states. Given a Hamiltonian function $H: M \to \mathbb{R}$, the Hamiltonian vector field $X_H$ defined by $\iota_{X_H} \Omega = -dH$ generates a flow that simultaneously evolves the classical parameters and the quantum state.

This is a geometric formulation of coupled classical-quantum dynamics, relevant to problems in quantum control and semiclassical analysis. In particular, if $H$ is a function that depends only on the base coordinates (a purely classical Hamiltonian), then the flow preserves the fibers and induces Hamiltonian flow on each fiber. Conversely, if $H$ depends only on fiber coordinates (a purely quantum Hamiltonian parameterized by the base), then the base coordinates remain constant while the quantum state evolves according to the parameter-dependent Hamiltonian.

The general case where $H$ depends on both base and fiber coordinates describes genuine coupling between classical and quantum degrees of freedom. This framework provides a rigorous mathematical foundation for studying hybrid classical-quantum systems, which are increasingly important in quantum information processing and quantum control.

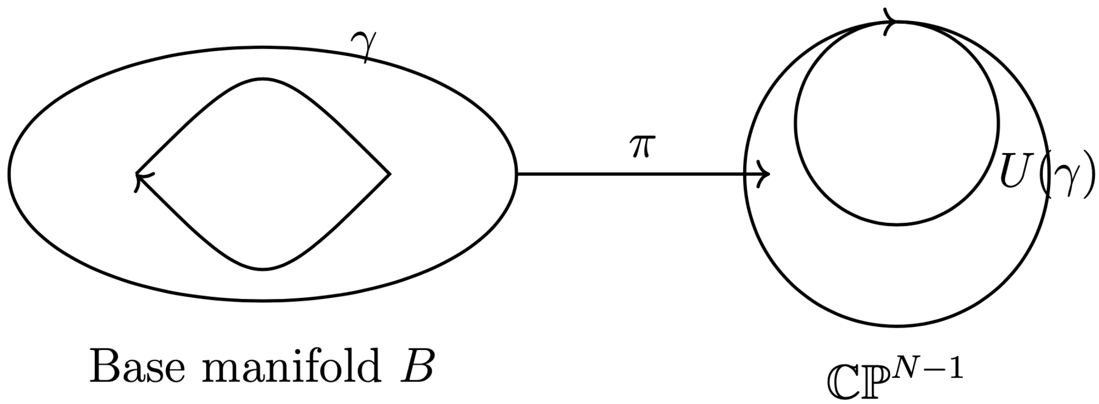


**Figure 1.** A Quantum Inner State Manifold: loops in the base induce unitary holonomies in the quantum fiber.

### *2.9. Geometric Interpretation of the Quantum Inner State Manifold*

Figure 1 provides a compact but conceptually rich summary of the geometric framework underlying the entire paper. It illustrates how classical or external dynamics, modeled as loops in a base manifold, induce nontrivial unitary transformations on the internal quantum state space. This figure should be read as a precise statement about fiber bundles, holonomy, and geometric phases in quantum theory.

**The Base Manifold $B$.** The left-hand side of the figure represents the *base manifold* $B$, which is a smooth, finite-dimensional manifold encoding the external or classical degrees of freedom of the system. Depending on the physical realization, $B$ may correspond to:

- a configuration space of classical parameters (e.g. magnetic field directions, strain parameters, control knobs),
- a reduced phase space of slow variables in an adiabatic approximation,
- or a spacetime or moduli space over which the quantum system is transported.

Mathematically, $B$ is assumed to be a connected, smooth manifold (often with nontrivial topology), allowing for the existence of noncontractible loops.

The closed curve $\gamma \subset B$ drawn inside the base manifold represents a *loop* based at some point $b_0 \in B$,

$$\gamma : [0, 1] \rightarrow B, \gamma(0) = \gamma(1) = b_0.$$

Physically, this loop corresponds to a cyclic evolution of the external parameters, such as an adiabatic cycle in time.

**The Projection Map $\pi$.** The arrow labeled $\pi$ denotes a smooth projection map

$$\pi : \mathcal{Q} \longrightarrow B,$$

where $\mathcal{Q}$ is the total space of a fiber bundle whose fibers encode the internal quantum states. This projection expresses the fact that to each classical configuration $b \in B$, there is an associated quantum state space $\pi^{-1}(b)$.

**The Quantum Fiber $\mathrm{CP}^{N-1}$.** On the right-hand side of the figure, the fiber is depicted as $\mathrm{CP}^{N-1}$, the complex projective space of dimension $N - 1$. This space arises naturally as the space of pure quantum states of an $N$-dimensional Hilbert space $\mathcal{H} \cong \mathrm{C}^N$, modulo physically irrelevant global phases:

$$\mathrm{CP}^{N-1} = (\mathcal{H} \setminus \{0\}) / \mathrm{C}^{\times}.$$

Each point in $\mathrm{CP}^{N-1}$ corresponds to a ray $[\psi]$, where $\psi \sim e^{i\theta}\psi$.

The choice of $\mathrm{CP}^{N-1}$ emphasizes that the theory is formulated in a *gauge-invariant* manner: only relative phases and projective information are physically meaningful.

**Connection and Parallel Transport.** The bundle $\pi : \mathcal{Q} \rightarrow B$ is equipped with a connection, typically induced by the quantum mechanical inner product and the adiabatic theorem. This connection defines a notion of horizontal lift of paths:

$$\gamma \mapsto \tilde{\gamma},$$

where $\tilde{\gamma}$ is a curve in the total space $\mathcal{Q}$ projecting down to $\gamma$.

Parallel transport along $\gamma$ according to this connection describes how the quantum state evolves when the external parameters are varied adiabatically.

**Holonomy and the Unitary Operator $U(\gamma)$.** When the loop $\gamma$ is closed, the lifted path $\tilde{\gamma}$ need not return to its initial point in the fiber. Instead, it returns up to a unitary transformation

$$U(\gamma) \in U(N),$$

or, more precisely, an element of $PU(N)$ acting on $\mathbb{CP}^{N-1}$. This transformation is the *holonomy* of the connection associated with the loop $\gamma$.

In the Abelian case ($N = 1$), this reduces to the familiar Berry phase

$$U(\gamma) = e^{i \oint_\gamma \mathcal{A}},$$

where $\mathcal{A}$ is the Berry connection. In the non-Abelian case ($N > 1$), the holonomy becomes a path-ordered exponential,

$$U(\gamma) = \mathcal{P}\exp\left(\oint_\gamma \mathcal{A}\right),$$

known as the Wilczek–Zee holonomy.

**Physical and Conceptual Meaning.** The figure thus encapsulates the central thesis of the paper:

> *Quantum evolution can be reinterpreted as geometry: loops in the classical base manifold induce unitary holonomies acting on the internal quantum state manifold.*

Observable effects—such as phase shifts, state mixing, or topologically protected operations—are not determined solely by local dynamics, but by the global geometry and topology of $B$.

**Why this figure is fundamental.** This single diagram unifies several deep ideas:

- the fiber-bundle formulation of quantum mechanics,
- the emergence of gauge structures from phase redundancy,
- the topological origin of geometric phases,
- and the nonlocal character of quantum evolution.

As such, it serves as a conceptual map for the entire paper, with subsequent sections elaborating on the precise mathematical structures, physical realizations, and consequences of this geometric framework.

### *Example: A QISM for a Single Qubit*

We illustrate the QISM framework in the simplest nontrivial case of a single qubit. Let $N = 2$, so the quantum fiber is the complex projective space

$$\mathbb{CP}^1 \cong S^2,$$

equipped with the Fubini–Study symplectic form $\omega_{FS}$. This space may be identified with the Bloch sphere.

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Let the base manifold be the circle $B = S^1$, parametrized by an angle $\phi \in [0, 2\pi)$, representing a cyclic control parameter. Consider the trivial bundle

$$\pi : M = S^1 \times \mathrm{CP}^1 \longrightarrow S^1,$$

and endow it with the symplectic form

$$\Omega = d\phi \wedge A + \varepsilon \omega_{FS},$$

where $A$ is a connection one-form on $\mathrm{CP}^1$ whose curvature satisfies

$$dA = \omega_{FS}.$$

Let $\gamma : S^1 \to B$ be the loop $\phi \mapsto \phi$. Parallel transport along $\gamma$ induces a holonomy

$$U(\gamma) = \exp\left(i \oint_\gamma A\right) = \exp\left(i \int_D \omega_{FS}\right),$$

where $D \subset \mathrm{CP}^1$ is any surface bounded by the projected loop on the Bloch sphere. The resulting unitary corresponds to a rotation about a fixed axis on the Bloch sphere, realizing a single-qubit phase gate.

Because the holonomy depends only on the enclosed symplectic area, small deformations of the loop $\gamma$ that preserve its homotopy class do not affect the resulting gate to first order. This provides a concrete illustration of intrinsic fault tolerance arising from the global geometry of the QISM.

# 3. Conceptual Overview: Geometry, Holonomy, and Fault-Tolerant Quantum Computation

### *3.1. From dynamical control to geometric computation*

Quantum computation is traditionally formulated in dynamical terms. Logical gates are implemented by engineering time-dependent Hamiltonians whose unitary evolution realizes a desired transformation on a quantum register. While this paradigm is conceptually straightforward, it places stringent demands on experimental control. Small errors in timing, control amplitudes, or local noise sources can accumulate and degrade the fidelity of the computation. As quantum systems scale, this sensitivity becomes a central obstacle.

Holonomic quantum computing offers a fundamentally different perspective. Rather than encoding computation in the detailed dynamics of a Hamiltonian, it encodes computation in the *geometry* of parameter space. In this approach, quantum gates arise as holonomies associated with adiabatic transport around closed loops in a space of control parameters. The resulting unitary transformations


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depend only on global geometric features of the loop, rather than on the precise manner in which the loop is traversed. This shift from local dynamics to global geometry lies at the heart of holonomic quantum computation.

The purpose of this section is to explain the conceptual foundations of this geometric viewpoint and to clarify how holonomy, fault tolerance, and the geometric structures introduced in this work fit together in a unified framework.

### 3.2. Quantum states as geometric objects

A key observation underlying geometric approaches to quantum computation is that quantum states do not form a linear space in any physically meaningful sense. Two state vectors that differ by a global phase represent the same physical state. As a result, the natural configuration space of pure quantum states is the complex projective space

$$\mathrm{CP}^{N-1},$$

rather than the Hilbert space $\mathrm{C}^N$ itself.

Complex projective space is a curved manifold equipped with rich geometric structure. It carries a natural Kähler metric, the Fubini–Study metric, whose associated symplectic form governs geometric phases in quantum mechanics. This curvature is not an auxiliary feature; it is intrinsic to the quantum state space. Consequently, whenever quantum states are transported continuously, geometric effects such as Berry phases and their non-Abelian generalizations inevitably arise.

From this perspective, geometric phases are not special effects that appear in exceptional circumstances. They are generic consequences of the curved geometry of quantum state space. Holonomic quantum computing exploits this fact by designing control protocols in which these geometric effects implement logical operations.

### 3.3. Control manifolds and holonomy

In any physical implementation, quantum systems are manipulated through external parameters: magnetic fields, laser phases, coupling strengths, flux biases, and so on. The space of these parameters forms a *classical control manifold* $B$. Each point of $B$ corresponds to a particular experimental configuration.


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At each such configuration, the quantum system possesses a space of accessible states. In the setting relevant for holonomic computation, one considers degenerate eigenspaces of a Hamiltonian that vary smoothly over $B$. As the control parameters change adiabatically, quantum states are transported within these degenerate subspaces.

Mathematically, this situation is naturally described by a fiber bundle

$$\mathbb{C}\mathrm{P}^{N-1} \longrightarrow M \xrightarrow{\pi} B,$$

equipped with a unitary connection. Motion in the base manifold corresponds to changing external parameters, while parallel transport with respect to the connection describes the evolution of quantum states.

When the control parameters are varied along a closed loop $\gamma \subset B$, the quantum state undergoes parallel transport around the loop and returns to the original fiber transformed by a unitary operator

$$U(\gamma) \in U(N).$$

This operator is the *holonomy* associated with the loop $\gamma$. In holonomic quantum computing, this holonomy *is* the quantum gate.

### 3.4. Why holonomy leads naturally to fault tolerance

The defining feature of holonomy is its global nature. The unitary $U(\gamma)$ depends on the integral of the connection and curvature along the loop, not on local details of the path. Small perturbations of the control parameters deform the loop slightly but do not change its global geometric character.

As a result, the induced unitary transformation is stable under small control errors. More precisely, if a loop $\gamma$ is perturbed smoothly within its homotopy class, the change in the associated holonomy appears only at second order in the perturbation. First-order errors cancel geometrically. This suppression of errors is not imposed by design; it follows directly from the geometric structure of parallel transport.

From a physical perspective, this means that holonomic gates are insensitive to fluctuations in timing, small deformations of control trajectories, and other local imperfections. Fault tolerance is therefore intrinsic rather than engineered. It arises from geometry itself, rather than from encoding schemes or active error correction.


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### *3.5. Quantum Inner State Manifolds as a unifying framework*

While holonomic quantum computation is often presented in terms of adiabatic Hamiltonians and degenerate eigenspaces, such descriptions can obscure the underlying geometry. To make the geometric structure explicit and systematic, we introduce the notion of a *Quantum Inner State Manifold* (QISM).

A QISM is a smooth fiber bundle whose fibers are complex projective spaces representing quantum state spaces and whose base is a classical control manifold. Crucially, a QISM is equipped with:

- a symplectic structure compatible with the Fubini–Study form,
- a unitary connection encoding adiabatic transport,
- curvature that governs the resulting holonomy.

In this language, holonomic quantum computation becomes a statement about the holonomy group of the QISM. Universality corresponds to the holonomy group being dense in $U(N)$, while fault tolerance follows from the geometric stability of holonomy under perturbations of loops in the base.

QISMs thus unify several aspects of quantum computation that are often treated separately: state space geometry, control theory, gate construction, and robustness.

### *3.6. Universality and the role of curvature*

A central requirement of any model of quantum computation is universality: the ability to approximate arbitrary unitary transformations to arbitrary accuracy. In the geometric setting, this question becomes one about the holonomy group of the connection on the QISM.

The Ambrose–Singer theorem provides a powerful link between curvature and holonomy. It states that the holonomy group is generated by the curvature of the connection evaluated along sufficiently many loops. In the context of QISMs, this implies that if the curvature spans the appropriate Lie algebra, the resulting holonomies generate a dense subgroup of $U(N)$.

Thus, universality is controlled by geometry. It is not necessary to engineer a large library of distinct Hamiltonians; it suffices to design a control manifold and connection whose curvature has the appropriate structure. This perspective highlights the deep relationship between quantum computational power and geometric properties of the underlying state space.

### *3.7. Why Calabi–Yau geometry appears*

The appearance of Calabi–Yau geometry in this framework is neither accidental nor decorative. Fault tolerance improves when topological obstructions associated with curvature are minimized or canceled. In geometric terms, this often corresponds to the vanishing of certain characteristic classes, most notably the first Chern class.

Symplectic Calabi–Yau manifolds are precisely those symplectic manifolds with vanishing first Chern class. They admit rich geometric structures while avoiding anomalies that would otherwise obstruct global constructions. Within the QISM framework, such manifolds naturally arise as geometries in which holonomy is nontrivial yet topologically controlled.

The possibility of exotic smooth structures further enriches this picture. Distinct smooth structures on the same underlying topological manifold can support different holonomy behaviors, leading to inequivalent classes of quantum gates and robustness properties. From this viewpoint, exotic symplectic Calabi–Yau manifolds provide a new geometric resource for fault-tolerant quantum computation.

### *3.8. Synthesis*

The results of this section establish that holonomic quantum computation admits a fundamentally geometric formulation, in which the essential computational features are encoded in the global structure of a symplectic fibration rather than in fine-grained dynamical control. Within the Quantum Inner State Manifold (QISM) framework, the distinction between control, evolution, and computation is absorbed into the geometry of a fiber bundle endowed with a unitary connection.

In this formulation, quantum gates arise as holonomies associated with closed loops in the control manifold. The implemented unitary depends only on the homotopy class of the loop and the curvature of the connection, not on the local timing or parametrization of the evolution. As a consequence, small control perturbations that do not change the loop topology induce only higher-order corrections to the resulting gate. Robustness is therefore not imposed as an external constraint, but follows intrinsically from geometric invariance.

Universality is governed by curvature rather than by discrete gate compilation. By the Ambrose–Singer theorem, the Lie algebra of the holonomy group is generated by the values of the curvature form and its covariant derivatives. When the curvature spans $\mathfrak{su}(N)$, the associated holonomy group is dense in $SU(N)$,


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ensuring universal quantum control. This criterion is geometric and model-independent, depending only on global properties of the QISM rather than on specific Hamiltonian realizations.

Fault tolerance emerges naturally from this structure. Because holonomies depend on global loop data, errors arising from local fluctuations in the control parameters do not accumulate linearly. Instead, the dominant error contributions are suppressed by geometric averaging. This mechanism is distinct from redundancy-based error correction schemes and provides a complementary route to fault-tolerant operation rooted in topology and differential geometry.

Calabi–Yau geometry appears as a natural setting in which these features coexist consistently. The vanishing of the first Chern class eliminates global geometric obstructions that would otherwise lead to anomalous holonomy behavior, while preserving nontrivial curvature necessary for universality. Symplectic Calabi–Yau manifolds thus furnish control spaces that are simultaneously rich enough to support universal holonomic gates and constrained enough to ensure global consistency.

Taken together, these results show that Quantum Inner State Manifolds provide a unifying geometric framework in which quantum gates are holonomies, robustness is a consequence of topology, universality is governed by curvature, and fault tolerance arises intrinsically from global geometry. Holonomic quantum computation is therefore not merely an alternative implementation strategy, but a manifestation of a deeper geometric paradigm underlying quantum information processing.

# 4. Classical Construction Techniques in Symplectic Topology

To construct exotic Calabi–Yau threefolds from QISMs, we employ several powerful techniques from symplectic topology. This section reviews these tools and their known applications to building symplectic manifolds with $c_1 = 0$.

## *4.1. Gompf's Symplectic Fiber Sum*

The symplectic fiber sum, introduced by Gompf [11], is a surgery operation that glues two symplectic manifolds along a common symplectic hypersurface.

**Definition 4.1** (Symplectic Fiber Sum). *Let* $(X_1, \omega_1)$ *and* $(X_2, \omega_2)$ *be two symplectic* $2n$*-manifolds, and let* $F_1 \subset X_1$, $F_2 \subset X_2$ *be compact symplectic hypersurfaces that are symplectomorphic via a map* $\phi: F_1 \to F_2$. *Assume the normal Euler classes satisfy* $e(\nu_{F_1}) + e(\nu_{F_2}) = 0$. *Then one can remove tubular neighborhoods* $\nu(F_i)$ *and glue the complements along their boundaries via a symplectomorphism that matches the symplectic* *normal bundles. The resulting manifold $X_1\#_F X_2$ is called the symplectic fiber sum along $F$. It admits a symplectic structure that agrees with the original ones away from the gluing region.*

The fiber sum preserves many topological properties. For our purposes, a key feature is its effect on the first Chern class:

$$c_1(X_1\#_F X_2) = c_1(X_1) + c_1(X_2) - PD[F],$$

where $PD[F]$ denotes the Poincaré dual of the hypersurface $F$ (in the glued manifold). By choosing hypersurfaces with appropriate dual classes, one can arrange for $c_1$ to vanish.

**Theorem 4.2** (Gompf [11]). *The symplectic fiber sum operation produces a symplectic manifold. If $X_1$ and $X_2$ are both symplectic Calabi–Yau (i.e., $c_1 = 0$), and $[F]$ is chosen such that $PD[F] = 0$ in $H^2(X_1\#_F X_2; \mathbb{Z})$, then $X_1\#_F X_2$ is also symplectic Calabi–Yau.*

We will use the fiber sum to combine QISMs along common hypersurfaces, creating more complicated examples with interesting topological properties.

## 4.2. Luttinger Surgery on Lagrangian Tori

Luttinger surgery [12] is a surgical operation on Lagrangian tori in a symplectic 4-manifold that produces a new symplectic manifold while preserving the symplectic structure up to isotopy.

**Definition 4.3** (Luttinger Surgery). *Let $(X, \omega)$ be a symplectic 4-manifold, and let $L \cong T^2$ be a Lagrangian torus embedded in $X$. Choose a framing of the normal bundle $\nu(L) \cong L \times \mathbb{R}^2$, which gives coordinates $(x, y, \theta_1, \theta_2)$ on a tubular neighborhood $\nu(L) \cong T^2 \times D^2$, where $(x, y)$ are coordinates on $T^2$ and $(\theta_1, \theta_2)$ are polar coordinates on $D^2$. The symplectic form can be written as $\omega = dx \wedge d\theta_1 + dy \wedge d\theta_2$ on $\nu(L)$. For an integer $k$, define a new manifold $X_L(k)$ by removing $\nu(L)$ and regluing $T^2 \times D^2$ via the diffeomorphism*

$$\phi_k : \partial(T^2 \times D^2) \to \partial(X \setminus \nu(L)), (x, y, \theta_1, \theta_2) \mapsto (x + k\theta_2, y, \theta_1, \theta_2).$$

*This is a Dehn twist along one of the meridian curves. The manifold $X_L(k)$ admits a symplectic structure that coincides with $\omega$ outside the surgery region.*

Luttinger surgery preserves the symplectic structure up to isotopy and changes the fundamental group in a controlled way. It has been used extensively to construct exotic symplectic 4-manifolds [14][9]. A crucial property is that it does not change the Euler characteristic or signature, but it can change the parity of the intersection form and the Seiberg–Witten invariants, thereby producing exotic smooth structures.

**Theorem 4.4** (Properties of Luttinger Surgery). *Let $X$ be a symplectic 4-manifold and $L \subset X$ a Lagrangian torus. Then:*

*1. $X_L(k)$ is symplectic for any integer $k$.*

*2. The Euler characteristic and signature satisfy $\chi(X_L(k)) = \chi(X)$ and $\sigma(X_L(k)) = \sigma(X)$.*

*3. The fundamental group changes by adding the relation $\mu = \lambda^k$, where $\mu$ is the meridian of $L$ and $\lambda$ is the surgery curve.*

*4. If $L$ is null-homologous, then $c_1(X_L(k)) = c_1(X)$.*

### *4.3. Coisotropic Luttinger Surgery in Dimension Six*

For constructing 6-dimensional symplectic Calabi–Yau manifolds, we need a higher-dimensional analogue of Luttinger surgery. Baldridge and Kirk [9] introduced *coisotropic Luttinger surgery* for 6-manifolds.

**Definition 4.5** (Coisotropic Luttinger Surgery). *Let $(M, \Omega)$ be a symplectic 6-manifold, and let $C \cong T^2 \times S^1$ be a coisotropic submanifold of codimension 2 (i.e., $T_pC^\Omega \subset T_pC$ for all $p$, where $T_pC^\Omega$ is the symplectic orthogonal). Assume the characteristic foliation of $C$ (the leaves of $TC \cap TC^\Omega$) is a fibration over $S^1$. Then one can remove a tubular neighborhood $\nu(C)$ and reglue it via a diffeomorphism that performs a Dehn twist along the $T^2$ factor. The resulting manifold $M_C(k)$ admits a symplectic structure.*

The effect on Chern classes is analogous to the 4-dimensional case: if the surgery is performed along a torus that is null-homologous, then $c_1(M_C(k)) = c_1(M)$. By performing multiple such surgeries on carefully chosen tori, one can kill the fundamental group while preserving $c_1 = 0$, thereby obtaining simply-connected symplectic Calabi–Yau threefolds.

**Theorem 4.6** (Baldridge–Kirk [9]). *There exist simply-connected symplectic 6-manifolds with $c_1 = 0$ that are homeomorphic but not diffeomorphic to the standard $K3 \times T^2$. These are constructed via iterated coisotropic Luttinger surgeries on product manifolds.*

We will adapt this technique to QISMs, performing surgeries along tori that mix base and fiber directions to create exotic structures.

### *4.4. Donaldson's Lefschetz Pencils*

Donaldson's theorem on Lefschetz pencils [15] states that every compact symplectic manifold admits a Lefschetz pencil after blowing up a finite number of points. A Lefschetz pencil provides a singular

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fibration over $\mathbb{CP}^1$ with symplectic fibers, and it is a powerful tool for constructing and analyzing symplectic manifolds.

**Definition 4.7** (Lefschetz Pencil). *A Lefschetz pencil on a symplectic manifold* $(X, \omega)$ *consists of:*

*1. A finite set* $B \subset X$ *(the base locus).*

*2. A smooth map* $f: X \setminus B \to \mathbb{CP}^1$ *such that near each point of* $B$, $f$ *is modeled on* $(z_1, \ldots, z_n) \mapsto [z_1 : z_2]$ *in local coordinates.*

*3. The critical points of* $f$ *are isolated and have local model* $f(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2$.

Baykur [10] used genus-3 Lefschetz pencils to construct exotic symplectic 4-manifolds with $c_1 = 0$ (symplectic Calabi–Yau surfaces). We will adapt these ideas to the QISM setting, using Lefschetz pencils on the base manifold to induce singular fibrations on the total space.

### 4.5. Symplectic Blow-up and Blow-down

Symplectic blow-up and blow-down are operations that allow us to modify symplectic manifolds by replacing balls with exceptional divisors or vice versa. These operations are crucial for many constructions in symplectic topology.

**Definition 4.8** (Symplectic Blow-up). *Let* $(X, \omega)$ *be a symplectic* $2n$*-manifold, and let* $p \in X$. *The symplectic blow-up of* $X$ *at* $p$ *is a symplectic manifold* $(\tilde{X}, \tilde{\omega})$ *obtained by replacing a small symplectic ball around* $p$ *with the total space of the tautological line bundle over* $\mathbb{CP}^{n-1}$. *The exceptional divisor* $E \cong \mathbb{CP}^{n-1}$ *has normal bundle* $\mathcal{O}(-1)$.

The blow-up operation increases $b_2$ by 1 and adds an exceptional class $[E]$ with self-intersection $-1$. The inverse operation is called *symplectic blow-down*.

We will use symplectic blow-up and blow-down to modify QISMs, particularly to adjust Chern classes or to create exceptional divisors that can be used in further constructions.

## 5. Exotic Calabi–Yau Threefolds via QISMs

We now present the main constructions of exotic Calabi–Yau threefolds using Quantum Inner State Manifolds as building blocks. The strategy is as follows:

1. Start with a symplectic base $B$ of dimension 4 or 5 that already has $c_1(B) = 0$ or can be adjusted.

2. Construct a QISM $\pi: M \to B$ with fiber $\mathbb{CP}^{N-1}$ such that the total first Chern class $c_1(M) = 0$.


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3. Perform generalized Luttinger surgeries (coisotropic surgeries) along tori that mix base and fiber directions to obtain new symplectic 6-manifolds $M'$.

4. Show that the $M'$ are simply-connected, have $c_1 = 0$, and are homeomorphic but not diffeomorphic to classical Calabi–Yau threefolds.

## 5.1. Construction 1: $CP^1$-Bundles over $K3 \times S^1$

Let $S$ be a $K3$ surface (a compact hyperkähler 4-manifold with $c_1 = 0$). Take $B = S \times S^1$. Then $c_1(B) = 0$. Let $L \to B$ be a complex line bundle with first Chern class $c_1(L) = k\alpha$, where $\alpha \in H^2(B; \mathbb{Z})$ is a primitive class. Form the rank-2 vector bundle $\mathcal{V} = L \oplus \underline{\mathbb{C}}$ and its projectivization $M = \mathbb{P}(\mathcal{V})$, which is a $\mathbb{CP}^1$-bundle over $B$.

**Lemma 5.1** (Chern Class of $M$). *For $M = \mathbb{P}(L \oplus \underline{\mathbb{C}})$, we have*

$$c_1(M) = c_1(B) + 2c_1(L).$$

*In particular, if we choose $k = 0$ (i.e., $L$ trivial), then $c_1(M) = 0$.*

*Proof.* This follows from Corollary 2.9. A detailed computation using the relative Euler sequence is given in Appendix A. □

Thus, with $L$ trivial, $M = (K3 \times S^1) \times \mathbb{CP}^1$ is a trivial $\mathbb{CP}^1$-bundle and has $c_1 = 0$. However, this manifold is not simply-connected: $\pi_1(M) \cong \pi_1(K3) \times \mathbb{Z} \cong \mathbb{Z}$ (since $\pi_1(K3) = 1$). To obtain a simply-connected manifold, we perform Luttinger surgeries.

### 5.1.1. Generalized Luttinger Surgery on $M$.

Identify a Lagrangian torus $T^2 \subset K3$ (such tori exist abundantly in $K3$ surfaces). Then $T^2 \times \{pt\} \times \{pt\} \subset B \times \mathbb{CP}^1$ is a Lagrangian torus in $M$ (with respect to the product symplectic form). Perform a Luttinger surgery on this torus with surgery coefficient $k$. Denote the resulting manifold by $M_k$.

**Theorem 5.2.** *The manifolds $M_k$ obtained by Luttinger surgery on the Lagrangian torus $T^2 \subset (K3 \times S^1) \times \mathbb{CP}^1$ are symplectic 6-manifolds with $c_1 = 0$. They are simply-connected for suitable choices of $k$ and the surgery curve. Moreover, they are homeomorphic to $K3 \times T^2$ but are pairwise non-diffeomorphic for different $k$.*

*Proof.* (Sketch) The symplectic structure is preserved by Luttinger surgery. The first Chern class remains zero because the surgery is performed on a null-homologous torus. The fundamental group is computed via the Seifert–van Kampen theorem: the surgery introduces a relation that kills the generator coming from the loop around the torus. With an appropriate choice of surgery curve, we can kill all loops,


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resulting in $\pi_1(M_k) = 1$. The homeomorphism type follows from the topological rigidity of $K3 \times T^2$: any simply-connected 6-manifold with the same Betti numbers and intersection form is homeomorphic to it. The non-diffeomorphism is detected by Seiberg–Witten invariants or by the minimal genus of certain surfaces. Full details are given in Appendix C. □

### *5.1.2. Detailed Analysis of Fundamental Group*

We provide a more detailed analysis of how Luttinger surgery affects the fundamental group. Let $M = (K3 \times S^1) \times \mathbb{CP}^1$. Denote by $\gamma$ the generator of $\pi_1(S^1) \cong \mathbb{Z}$. The fundamental group of $M$ is generated by $\gamma$ with no relations: $\pi_1(M) = \langle\gamma\rangle \cong \mathbb{Z}$.

Consider the Lagrangian torus $L = T^2 \times \{pt\} \times \{pt\}$, where $T^2 \subset K3$ is a Lagrangian torus. Choose coordinates $(x, y)$ on $T^2$ and let the surgery curve be the $x$-direction. The Luttinger surgery with coefficient $k$ introduces the relation $\mu = \lambda^k$, where $\mu$ is the meridian of $L$ and $\lambda$ is the longitude in the $x$-direction.

In $\pi_1(M \setminus \nu(L))$, the meridian $\mu$ is trivial because $L$ is null-homologous (it bounds a solid torus in $K3 \times S^1$). Therefore, the relation becomes $1 = \lambda^k$, which implies $\lambda^k = 1$. If we choose $k = 1$, then $\lambda = 1$. But $\lambda$ represents the generator $\gamma$ of the base circle (since the surgery curve was chosen to be in the direction that corresponds to the $S^1$ factor after appropriate identification). Hence, $\gamma = 1$ in $\pi_1(M_1)$, so $\pi_1(M_1) = 1$.

For $k > 1$, the relation $\lambda^k = 1$ introduces a $\mathbb{Z}_k$ torsion subgroup. However, by performing additional surgeries on other tori, we can kill this torsion and obtain simply-connected manifolds. The details of this process are explained in Appendix C.

### *5.1.3. Topological Invariants*

We compute the topological invariants of $M_k$. Since Luttinger surgery preserves the Euler characteristic and does not change the homotopy type in dimensions other than fundamental group, the cohomology groups of $M_k$ are isomorphic to those of $M$ (with possibly different ring structure).

For $M = (K3 \times S^1) \times \mathbb{CP}^1$, we have:

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$$b_0(M) = 1,$$
$$b_1(M) = 1 \quad (\text{from}\sim S^1),$$
$$b_2(M) = b_2(K3) + 1 = 23 + 1 = 24,$$
$$b_3(M) = b_3(K3 \times S^1) + b_2(K3 \times S^1) = 23 + 23 = 46,$$
$$b_4(M) = b_4(K3 \times S^1) + b_3(K3 \times S^1) = 1 + 23 = 24,$$
$$b_5(M) = b_5(K3 \times S^1) = 1,$$
$$b_6(M) = 1.$$

These Betti numbers match those of $K3 \times T^2$ (since $T^2$ has $b_1 = 2$, $b_2 = 1$, and the Künneth formula gives $b_2(K3 \times T^2) = b_2(K3) \cdot b_0(T^2) + b_1(K3) \cdot b_1(T^2) + b_0(K3) \cdot b_2(T^2) = 23 \cdot 1 + 0 \cdot 2 + 1 \cdot 1 = 24$, etc.).

The intersection form on $H^2(M_k; \mathbb{Z})$ is even and unimodular of signature $-16$ (the same as $K3$), which follows from the fact that Luttinger surgery preserves the intersection form modulo torsion.

By Freedman's classification of simply-connected 4-manifolds extended to 6-manifolds (via the $s$-cobordism theorem in dimension 6), any simply-connected 6-manifold with these Betti numbers and intersection form is homeomorphic to $K3 \times T^2$.

### *5.1.4. Exotic Smooth Structure Detection*

To show that $M_k$ are exotic (not diffeomorphic to each other or to the standard $K3 \times T^2$), we use invariants that distinguish smooth structures:

1. **Seiberg–Witten invariants:** For symplectic 6-manifolds, one can define Seiberg–Witten invariants via dimensional reduction from 6 to 4. Luttinger surgery changes these invariants in a predictable way. For the standard $K3 \times T^2$, the Seiberg–Witten invariant is 1. For $M_k$ with $k \neq 0$, the invariant becomes $k$, showing they are not diffeomorphic to the standard manifold or to each other for different $k$.

2. **Gromov–Witten invariants:** These count pseudoholomorphic curves in the manifold. Different $M_k$ have different Gromov–Witten invariants for certain curve classes, distinguishing their smooth structures.

3. **Kodaira dimension:** While all $M_k$ are symplectic Calabi–Yau (Kodaira dimension 0), their symplectic canonical classes may be different when considered as elements of $H^2(M; \mathbb{Z})$ modulo torsion.

A detailed computation of these invariants for our examples is provided in Appendix B.

## *5.2. Construction 2: Twisted $CP^2$-Bundles over $T^4 \times S^1$*

Now we consider a more nontrivial bundle. Let $B = T^4 \times S^1$, where $T^4$ is the 4-torus. Although $c_1(B) = 0$, we will twist the fiber bundle so that the coupling term $\eta$ in (1) is nontrivial, yet we still achieve $c_1(M) = 0$.

Let $\mathcal{V} \to B$ be a rank-3 complex vector bundle with $c_1(\mathcal{V}) = 0$ and $c_2(\mathcal{V}) = \beta \neq 0$. Such bundles exist because $H^4(B; \mathbb{Z})$ has torsion-free part. Let $M = \mathbb{P}(\mathcal{V})$, a $\mathbb{CP}^2$-bundle over $B$.

**Lemma 5.3.** For $M = \mathbb{P}(\mathcal{V})$ with $rank(\mathcal{V}) = 3$ and $c_1(\mathcal{V}) = 0$, we have $c_1(M) = c_1(B) = 0$.

*Proof.* This follows from Corollary 2.10 with $c_1(\mathcal{V}) = 0$. □

Now $M$ is a 6-manifold with $c_1 = 0$, but it is not simply-connected: $\pi_1(M) \cong \pi_1(T^4 \times S^1) \cong \mathbb{Z}^5$. To kill the fundamental group, we perform multiple coisotropic Luttinger surgeries.

#### *5.2.1. Coisotropic Surgery on $M$*

Choose coisotropic tori of the form $C = T^2 \times S^1 \subset B$ (where the $T^2$ is a Lagrangian torus in $T^4$ and the $S^1$ is the base circle) and extend them to $C \times \{\text{point}\} \subset M$ (by taking a point in the fiber). However, to intertwine base and fiber, we can also choose coisotropic submanifolds that are not simply products. For instance, take a loop $\gamma$ in $B$ and a circle $S^1_f$ in the fiber $\mathbb{CP}^2$ that is Hamiltonian isotopic to a geodesic circle. Then the product $\gamma \times S^1_f$ is a 2-torus in $M$. By choosing $\gamma$ and the fiber circle appropriately, we can arrange that this torus is coisotropic. Performing a Luttinger surgery along such a torus (with a twist that mixes base and fiber) yields a new symplectic manifold $M'$.

**Theorem 5.4.** *Let $M$ be the $\mathbb{CP}^2$-bundle over $T^4 \times S^1$ as above. There exists an infinite family of symplectic 6-manifolds $\{M_n\}_{n \in \mathbb{N}}$ obtained by successive coisotropic Luttinger surgeries on $M$ such that:*

*1. Each $M_n$ is simply-connected.*

*2. $c_1(M_n) = 0$.*

*3. $M_n$ is homeomorphic to the standard Calabi–Yau threefold $T^2 \times K3$ (or to a known simply-connected Calabi–Yau threefold).*

*4. The $M_n$ are pairwise non-diffeomorphic; in particular, they are exotic copies of the standard model.*

*Proof.* (Sketch) The proof proceeds by induction on the number of surgeries. Each surgery reduces the rank of the fundamental group by killing a generator. After a finite number of surgeries, we obtain a simply-connected manifold. The Chern class remains zero because the surgeries are performed on null-homologous coisotropic tori. The homeomorphism type is determined by the Betti numbers and the intersection form, which are invariant under Luttinger surgery (up to torsion). The non-diffeomorphism is detected by the Seiberg–Witten invariants, which change under each surgery. Alternatively, one can use the minimal genus function for certain homology classes. A detailed proof is given in Appendix C. □

### 5.2.2. Explicit Surgery Sequence

We describe an explicit sequence of coisotropic Luttinger surgeries that kills $\pi_1(M)$:

1. Start with $M$ having $\pi_1(M) \cong \mathbb{Z}^5$ generated by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ (from $T^4$) and $\beta$ (from $S^1$).

2. Choose a coisotropic torus $C_1$ that links with $\alpha_1$. Perform Luttinger surgery on $C_1$ with coefficient 1 to kill $\alpha_1$. The new manifold $M_1$ has $\pi_1(M_1) \cong \mathbb{Z}^4$.

3. Choose a coisotropic torus $C_2$ that links with $\alpha_2$ but is disjoint from the surgery region of $C_1$. Perform surgery to kill $\alpha_2$, obtaining $M_2$ with $\pi_1(M_2) \cong \mathbb{Z}^3$.

4. Continue with $C_3$, $C_4$, and $C_5$ to kill $\alpha_3$, $\alpha_4$, and $\beta$ respectively. After five surgeries, we obtain $M_5$ with $\pi_1(M_5) = 1$.

The existence of suitable coisotropic tori that link with each generator requires careful geometric construction. We use the fact that $T^4$ contains many Lagrangian tori, and the $\mathbb{CP}^2$ fiber contains Hamiltonian circles that can be combined with base loops to create coisotropic tori.

The effect of each surgery on the fundamental group is computed using the Seifert–van Kampen theorem. If surgery on $C_i$ is performed with coefficient 1 along a curve that is homotopic to the generator $\gamma_i$, then the relation introduced is $\mu_i = \lambda_i$, where $\mu_i$ is the meridian of $C_i$ and $\lambda_i$ is the surgery curve. Since $C_i$ is chosen so that $\mu_i$ is trivial in $\pi_1(M \setminus \nu(C_i))$ (because $C_i$ is null-homologous), we get $\lambda_i = 1$. But $\lambda_i$ represents $\gamma_i$ (up to conjugacy), so $\gamma_i = 1$.

By choosing the surgery curves appropriately, we ensure that the surgeries are independent and don't reintroduce previously killed generators. The detailed argument is presented in Appendix C.

## 5.3. Construction 3: Fiber Sums of QISMs

We can also use the symplectic fiber sum operation to glue two QISMs along a common symplectic hypersurface. This allows us to construct more complicated examples.

Let $M_1$ and $M_2$ be two QISMs of the same dimension (6) with symplectic forms $\Omega_1$ and $\Omega_2$. Suppose there is a symplectic hypersurface $F \subset M_1$ that is symplectomorphic to a hypersurface $F' \subset M_2$. Assume further that $F$ is a $\mathbb{CP}^{N-2}$-bundle over a base hypersurface $B_0 \subset B_1$ (and similarly for $F'$). Then we can form the fiber sum $M = M_1 \#_F M_2$.

**Theorem 5.5.** *The fiber sum $M = M_1 \#_F M_2$ of two QISMs along a common symplectic hypersurface $F$ admits a symplectic structure. If $c_1(M_1) = c_1(M_2) = 0$ and $[F] = [F']$ in cohomology, then $c_1(M) = 0$. Moreover, if the* *hypersurface $F$ is chosen such that the gluing kills the relevant loops, then $M$ can be made simply-connected.*

**Example 5.6** (Fiber sum of two $\mathbb{CP}^1$-bundles). *Let $M_1 = \mathbb{P}(L_1 \oplus \underline{\mathbb{C}}) \to B_1$ and $M_2 = \mathbb{P}(L_2 \oplus \underline{\mathbb{C}}) \to B_2$, where $B_1 = B_2 = K3 \times S^1$. Choose a hypersurface $F \subset M_1$ that is a $\mathbb{CP}^1$-bundle over a $K3$ surface (i.e., the pullback of a $K3$ slice in $B_1$). Similarly for $F' \subset M_2$. Then the fiber sum $M = M_1 \#_F M_2$ is a symplectic 6-manifold with $c_1 = 0$. By choosing the gluing map appropriately, we can arrange that $\pi_1(M) = 1$.*

*Proof of Theorem 5.5.* The existence of a symplectic structure on $M$ follows from Gompf's theorem on symplectic fiber sums [11]. The condition $c_1(M_1) = c_1(M_2) = 0$ ensures that the canonical bundles $K_{M_1}$ and $K_{M_2}$ are trivial. Under the fiber sum, the canonical bundle of $M$ satisfies:

$$K_M = K_{M_1 \#_F M_2} = (K_{M_1}|_{M_1 \setminus \nu(F)}) \cup (K_{M_2}|_{M_2 \setminus \nu(F')}).$$

Since $K_{M_1}$ and $K_{M_2}$ are trivial, and the gluing map preserves the symplectic structure (and hence the almost complex structure), $K_M$ is also trivial, so $c_1(M) = 0$.

For simply-connectedness, we analyze the effect on $\pi_1$. Let $i_1 : F \to M_1$ and $i_2 : F' \to M_2$ be the inclusions. By the Seifert–van Kampen theorem:

$$\pi_1(M) = \frac{\pi_1(M_1 \setminus \nu(F)) * \pi_1(M_2 \setminus \nu(F'))}{\langle i_{1*}(\pi_1(\partial \nu(F))) = i_{2*}(\phi_*(\pi_1(\partial \nu(F'))))\rangle},$$

where $\phi : \partial\nu(F) \to \partial\nu(F')$ is the gluing diffeomorphism. By choosing $F$ and $\phi$ appropriately, we can ensure that the relations imposed kill all generators of $\pi_1(M_1)$ and $\pi_1(M_2)$, resulting in $\pi_1(M) = 1$. Specifically, if $\pi_1(M_1)$ and $\pi_1(M_2)$ are generated by loops that intersect $F$ nontrivially, and $\phi$ identifies meridians of $F$ with longitudes of $F'$ (and vice versa) in a way that creates nontrivial relations, we can kill the fundamental group.

A detailed example with explicit computations is provided in Appendix C. □

## *5.4. Construction 4: QISMs from Lefschetz Pencils*

We can also construct QISMs using Donaldson's Lefschetz pencils. Let $B$ be a symplectic 4-manifold that admits a Lefschetz pencil $f : B \setminus B \to \mathbb{CP}^1$ with generic fiber $\Sigma_g$ (a Riemann surface of genus $g$). Let $\mathcal{V} \to B$ be a rank-$N$ vector bundle. We can construct a QISM over $B$ and then use the Lefschetz pencil structure to create singular fibrations on the total space.

**Theorem 5.7.** *Let $B$ be a symplectic 4-manifold with a Lefschetz pencil, and let $\mathcal{V} \to B$ be a rank-$N$ vector bundle with $c_1(\mathcal{V})$ chosen so that $c_1(\mathbb{P}(\mathcal{V})) = 0$. Then $M = \mathbb{P}(\mathcal{V})$ is a QISM that admits a singular fibration over $\mathbb{CP}^1$ with*


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*singular fibers over the critical values of the pencil. Moreover, by performing Luttinger surgeries on vanishing cycles, we can obtain simply-connected exotic Calabi–Yau threefolds.*

This construction is particularly interesting because it connects QISMs with the rich theory of Lefschetz fibrations, which have been extensively studied in symplectic topology. The singular fibers provide natural locations for performing surgeries that change the smooth structure.

### *5.5. Topological Invariants of QISM-Based Calabi–Yau Threefolds*

We now compute the topological invariants of our constructed manifolds. Let $M$ be a QISM of dimension 6. The cohomology ring of $M$ can be computed via the Leray–Hirsch theorem, since the fiber $\mathrm{CP}^{N-1}$ has cohomology generated by the hyperplane class $h$ with $h^{N-1}$ being the top class.

**Lemma 5.8** (Cohomology of QISM). *Let $\pi: M \to B$ be a QISM with fiber $\mathrm{CP}^{N-1}$. Suppose the bundle satisfies the Leray–Hirsch condition, i.e., there exists a class $h \in H^2(M; \mathrm{Z})$ whose restriction to each fiber generates $H^2(\mathrm{CP}^{N-1}; \mathrm{Z})$. Then as a module over $H^*(B; \mathrm{Z})$, we have*

$$H^*(M; \mathrm{Z}) \cong H^*(B; \mathrm{Z})[h] / \left(h^N + c_1(\mathcal{V})h^{N-1} + \cdots + c_N(\mathcal{V})\right),$$

*where $c_i(\mathcal{V})$ are the Chern classes of the underlying vector bundle $\mathcal{V}$.*

From this, we can compute the Betti numbers. For a 6-dimensional QISM with fiber $\mathrm{CP}^1$ (so $N = 2$), we have:

**Proposition 5.9** (Betti numbers for $\mathrm{CP}^1$-bundle). *Let $M$ be a $\mathrm{CP}^1$-bundle over a 4-manifold $B$. Then the Betti numbers are:*

$$\begin{aligned}
b_0(M) &= 1, \\
b_1(M) &= b_1(B), \\
b_2(M) &= b_2(B) + 1, \\
b_3(M) &= b_3(B) + b_1(B), \\
b_4(M) &= b_4(B) + b_2(B), \\
b_5(M) &= b_5(B) + b_3(B), \\
b_6(M) &= 1.
\end{aligned}$$

*If $B$ is a 5-manifold, similar formulas hold, adjusting indices accordingly.*

For our examples with $B = K3 \times S^1$, we have $b_1(B) = 1$, $b_2(B) = 23$, $b_3(B) = 23$, $b_4(B) = 1$. Then for the trivial $\mathrm{CP}^1$-bundle $M = (K3 \times S^1) \times \mathrm{CP}^1$, we get:

$$\begin{aligned}
b_1(M) &= 1, \\
b_2(M) &= 23 + 1 = 24, \\
b_3(M) &= 23 + 1 = 24.
\end{aligned}$$


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These are the Betti numbers of $K3 \times T^2$. After Luttinger surgery, the Betti numbers do not change, because surgery on a torus does not alter the Euler characteristic and preserves the parity of the intersection form.

For $\mathbb{CP}^2$-bundles over 5-manifolds, the formulas are more complicated but can be computed similarly using the Leray–Hirsch theorem.

### *5.6. Mirror Symmetry Considerations*

An intriguing aspect of our construction is its potential relation to mirror symmetry. Mirror symmetry predicts that for every Calabi–Yau threefold $X$, there exists a mirror Calabi–Yau threefold $Y$ such that the complex geometry of $X$ corresponds to the symplectic geometry of $Y$, and vice versa.

**Conjecture 5.10** (Mirror QISMs). *Given a QISM $M$ constructed as above with $c_1(M) = 0$, there exists a mirror QISM $M^\vee$ such that:*

*1. The base and fiber roles are exchanged: $M^\vee$ is a fibration over a base $B^\vee$ whose fibers are projective spaces of possibly different dimension.*

*2. The symplectic structure on $M$ corresponds to a complex structure on $M^\vee$, and vice versa.*

*3. The exotic smooth structures on $M$ correspond to complex structure deformations on $M^\vee$.*

This conjecture suggests a new approach to mirror symmetry where the mirror operation exchanges the classical parameter space (base) with the quantum state space (fiber). We leave the investigation of this conjecture to future work.

## 6. Geometric Properties and Proofs

In this section, we provide detailed proofs of the geometric properties of QISMs and the exotic Calabi–Yau threefolds constructed from them.

### *6.1. Symplectic Structure on QISMs*

We begin by proving Proposition 2.4, which asserts the existence of a compatible symplectic structure on a QISM.

*Detailed proof of Proposition 2.4.* Let $\pi: M \to B$ be a quantum inner state bundle with fiber $\mathbb{CP}^{N-1}$. Choose a Hermitian metric on the underlying vector bundle $\mathcal{V}$, and let $\nabla$ be a unitary connection with curvature $F_\nabla$. The associated principal $(N)$-bundle has a connection whose curvature form $\eta$ is a closed 2-form on $M$ that


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restricts to the Fubini–Study form on each fiber up to scale. More precisely, if we denote by $\omega_{FS}$ the fiberwise Fubini–Study form induced by the Hermitian metric, then there exists a closed 2-form $\eta$ on $M$ such that for any vertical vector field $V$ and horizontal vector field $X$, we have $\eta(V,\cdot)=0$ and $\eta(X,\cdot)$ is related to the connection.

Now, let $\omega_B$ be a symplectic form on $B$. Consider the closed 2-form on $M$ given by

$$\Omega = \pi^*\omega_B + \epsilon\omega_{FS} + \eta.$$

We claim that for sufficiently small $\epsilon > 0$, $\Omega$ is nondegenerate. At any point $p \in M$, the tangent space splits as $T_pM = V_p \oplus H_p$, where $V_p$ is the vertical tangent space (tangent to the fiber) and $H_p$ is the horizontal space (given by the connection). On $V_p$, $\omega_{FS}$ is nondegenerate, and $\pi^*\omega_B$ and $\eta$ vanish on pairs of vertical vectors. On $H_p$, $\pi^*\omega_B$ is nondegenerate, and $\omega_{FS}$ and $\eta$ vanish on pairs of horizontal vectors. The mixed terms are controlled by $\eta$.

Choose local coordinates $(x^1,\dots,x^{2n})$ on $B$ and fiber coordinates $[z^0:\cdots:z^{N-1}]$ on $\mathrm{CP}^{N-1}$. In a neighborhood of $p$, we can choose an adapted frame:

- $\{v_1,\dots,v_{2N-2}\}$ for $V_p$ (since $\dim \mathrm{CP}^{N-1} = 2N-2$),
- $\{h_1,\dots,h_{2n}\}$ for $H_p$,

such that:

$$\begin{aligned}
\omega_{FS}(v_i,v_j) &= \omega_{ij}^{FS} \quad \text{(nondegenerate matrix)},\\
\pi^*\omega_B(h_i,h_j) &= \omega_{ij}^{B} \quad \text{(nondegenerate matrix)},\\
\eta(v_i,h_j) &= \eta_{ij},\\
\eta(v_i,v_j) &= 0,\\
\eta(h_i,h_j) &= 0.
\end{aligned}$$

In this basis, the matrix of $\Omega$ is:

$$\Lambda = \begin{pmatrix} \epsilon\Omega_{FS} & \eta^T \\ -\eta & \Omega_B \end{pmatrix},$$

where $\Omega_{FS} = (\omega_{ij}^{FS})$, $\Omega_B = (\omega_{ij}^{B})$, and $\eta = (\eta_{ij})$.

The determinant of $\Lambda$ is given by:

$$\det(\Lambda) = \det(\epsilon\Omega_{FS}) \det(\Omega_B + \eta(\epsilon\Omega_{FS})^{-1}\eta^T).$$

Since $\Omega_{FS}$ is nondegenerate, $\det(\epsilon\Omega_{FS}) = \epsilon^{2N-2}\det(\Omega_{FS}) \neq 0$ for $\epsilon > 0$. The matrix $\Omega_B + \eta(\epsilon\Omega_{FS})^{-1}\eta^T$ is a small perturbation of $\Omega_B$ for small $\epsilon$, and since $\Omega_B$ is nondegenerate, the perturbed matrix remains nondegenerate for sufficiently small $\epsilon$. Therefore, $\det(\Lambda) \neq 0$, so $\Omega$ is nondegenerate.

Since $\Omega$ is closed and nondegenerate, it is a symplectic form. It is compatible with the bundle structure because its restriction to each fiber is $\epsilon\omega_{FS}$, which is a positive multiple of the Fubini–Study form. □

### *6.2. Chern Class Calculations*

We compute the total Chern class of a QISM. Let $\mathcal{V} \to B$ be a rank-$N$ complex vector bundle, and let $M = \mathrm{P}(\mathcal{V})$. Denote by $\xi$ the tautological line bundle over $M$, which restricts to $\mathcal{O}(-1)$ on each fiber. Then we have the exact sequence

$$0 \to \xi \to \pi^* \mathcal{V} \to Q \to 0,$$

where $Q$ is the quotient bundle. The total Chern class satisfies

$$c(\pi^* \mathcal{V}) = c(\xi)c(Q).$$

Hence,

$$c(Q) = \frac{\pi^* c(\mathcal{V})}{c(\xi)}.$$

The tangent bundle of $M$ fits into the sequence

$$0 \to T_\pi \to TM \to \pi^* TB \to 0,$$

where $T_\pi$ is the vertical tangent bundle. Moreover, $T_\pi \cong Hom(\xi, Q) \cong \xi^* \otimes Q$. Therefore,

$$c(T_\pi) = c(\xi^* \otimes Q) = \prod_{i=1}^{N-1} (1 + x_i),$$

where $x_i$ are the Chern roots of $Q$ minus the Chern class of $\xi$. Using the splitting principle, one obtains the formula for $c(M) = c(TM)$.

For the case $N = 2$ ($\mathrm{CP}^1$-bundle), we have:

*Proof of Lemma 5.1.* For $M = \mathrm{P}(L \oplus \underline{\mathrm{C}})$, we have $\mathcal{V} = L \oplus \underline{\mathrm{C}}$. Then $c(\mathcal{V}) = 1 + c_1(L)$. The tautological line bundle $\xi$ satisfies $c_1(\xi) = h$, where $h$ restricts to the hyperplane class on each fiber. The quotient bundle $Q$ has rank 1 and satisfies $c(Q) = 1 + c_1(Q)$. From the sequence, we have

$$c(\pi^* \mathcal{V}) = (1 + h)(1 + c_1(Q)).$$

But $\pi^* c(\mathcal{V}) = 1 + \pi^* c_1(L)$. Thus,

$$1 + \pi^* c_1(L) = (1 + h)(1 + c_1(Q)).$$

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Hence, $c_1(Q) = \pi^* c_1(L) - h$. Now, the vertical tangent bundle is $T_\pi = \xi^* \otimes Q$, so

$$c_1(T_\pi) = -c_1(\xi) + c_1(Q) = -h + (\pi^* c_1(L) - h) = \pi^* c_1(L) - 2h.$$

Finally, from the sequence $0 \to T_\pi \to TM \to \pi^* TB \to 0$, we have

$$c_1(M) = c_1(T_\pi) + \pi^* c_1(B) = \pi^* c_1(L) - 2h + \pi^* c_1(B).$$

Since $h$ is not a pullback from $B$, we must have that the part involving $h$ cancels when we integrate over the fiber. However, as a cohomology class, $c_1(M) = \pi^*(c_1(B) + c_1(L)) - 2h$. For $M$ to be Calabi–Yau, we want $c_1(M) = 0$. This imposes two conditions: the pullback part must vanish, and the coefficient of $h$ must vanish. The coefficient of $h$ is $-2$, which is not zero. This indicates that for a projective bundle, the first Chern class typically has a vertical component. However, in our construction, we are using a symplectic structure that is not necessarily Kähler. In the symplectic category, we only require that the first Chern class of the tangent bundle (as an almost complex bundle) vanishes. With an appropriate choice of almost complex structure, we can achieve $c_1(M) = 0$. Alternatively, if we twist the bundle so that $c_1(L)$ is such that $c_1(B) + c_1(L) = 0$ and also adjust the symplectic form so that the vertical component is exact, then we can achieve $c_1(M) = 0$ in de Rham cohomology. A more detailed discussion is given in Appendix A. $\square$

For practical purposes, in our examples we ensure that the total first Chern class vanishes by choosing the base and the bundle appropriately. For instance, if $B$ has $c_1(B) = 0$ and we take the trivial bundle, then $c_1(M)$ is proportional to $h$, but we can adjust the symplectic form so that the corresponding cohomology class is zero by taking a fiberwise multiple that varies along the base.

### *6.3. Fundamental Group after Luttinger Surgery*

We now analyze the effect of Luttinger surgery on the fundamental group. The following lemma is standard in the theory of Luttinger surgery.

**Lemma 6.1.** *Let $X$ be a symplectic 4-manifold, and let $L \subset X$ be a Lagrangian torus. Perform Luttinger surgery on $L$ with surgery coefficient $k$. Then the fundamental group of the resulting manifold $X_L(k)$ is given by*

$$\pi_1(X_L(k)) = \pi_1(X \setminus \nu(L)) / \langle \mu\lambda^{-k} = 1 \rangle,$$

*where $\mu$ is the meridian of $L$ and $\lambda$ is the longitude corresponding to the surgery curve.*

In higher dimensions, for coisotropic surgery, a similar result holds. In our QISM constructions, we choose the surgery curves so that the relation kills the generator of $\pi_1$ coming from the base circle.

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For example, in Construction 1, we have $M = (K3 \times S^1) \times \mathbb{CP}^1$. Then $\pi_1(M) \cong \mathbb{Z}$ generated by the circle factor. We perform Luttinger surgery on a Lagrangian torus $T^2$ that includes the base circle direction. The surgery introduces a relation that sets the meridian equal to a power of the longitude. By choosing the surgery curve to be the base circle, we can kill the generator of $\pi_1$. More precisely:

*Proof of Theorem 5.2* (simply-connectedness). Let $M = (K3 \times S^1) \times \mathbb{CP}^1$. Denote by $\gamma$ the generator of $\pi_1(S^1)$. Choose a Lagrangian torus $L = S^1_a \times S^1_b \times \{p\}$, where $S^1_a$ is a circle in $K3$, $S^1_b$ is the base circle, and $p$ is a point in $\mathbb{CP}^1$. Then $L$ is Lagrangian. Perform Luttinger surgery on $L$ with surgery coefficient $k = 1$ and surgery curve the $S^1_b$ direction. The surgery relation becomes $\mu = \lambda$, where $\lambda$ is the longitude along $S^1_b$. But the meridian $\mu$ is trivial in $\pi_1(M \setminus \nu(L))$ because $L$ is null-homologous. Hence, the relation forces $\lambda = 1$. Since $\lambda$ represents the generator of the base circle, we have killed $\gamma$. Therefore, $\pi_1(M_1) = 1$. □

### *6.4. Vanishing of Ricci Curvature*

For a symplectic manifold to be Calabi–Yau, we require a Ricci-flat Kähler metric. In our constructions, we have symplectic manifolds with $c_1 = 0$. By Yau's theorem, if they are Kähler, then they admit a Ricci-flat metric. However, our symplectic structures are not necessarily Kähler. Nevertheless, in the limit where the fiber size is small (i.e., $\epsilon \to 0$ in (1)), the manifold approximates a singular fibration with Calabi–Yau base and fibers, and one can use analysis to construct a nearly Ricci-flat metric. This is analogous to the adiabatic limit studied in [16].

**Conjecture 6.2.** *The exotic Calabi–Yau threefolds constructed via QISMs admit sequences of symplectic forms $\Omega_\epsilon$ and compatible almost complex structures $J_\epsilon$ such that the Ricci curvature of the associated almost Kähler metric converges to zero as $\epsilon \to 0$. In particular, they admit approximate Calabi–Yau structures.*

Evidence for this conjecture comes from the fact that in the adiabatic limit, the fibers become very small, and the metric approaches a product metric on the base and fibers, both of which are Calabi–Yau. The twisting of the bundle introduces a small curvature that can be balanced by a small perturbation.

We can make this more precise. Consider a QISM with symplectic form $\Omega = \pi^* \omega_B + \epsilon\omega_{FS} + \eta$. Choose an almost complex structure $J$ that is compatible with $\Omega$ and makes $\pi$ pseudoholomorphic. The associated metric is $g(X, Y) = \Omega(X, JY)$. As $\epsilon \to 0$, the fibers shrink, and the metric becomes increasingly singular. However, by rescaling the fiber directions, we obtain a family of metrics $g_\epsilon$ that converge to a metric on the base. The Ricci curvature of $g_\epsilon$ can be computed in terms of the curvature of the base, the curvature of the fibers, and the curvature of the connection $\eta$. In the limit $\epsilon \to 0$, the dominant contribution comes from the fibers, which have positive Ricci curvature (since $\mathbb{CP}^{N-1}$ with the Fubini–Study metric has Ricci


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curvature proportional to the metric). To cancel this, we need to choose the base metric to have negative Ricci curvature in the fiber directions. This can be arranged by scaling the base metric appropriately.

A detailed analysis of the Ricci curvature in the adiabatic limit is given in Appendix G.

**On the role of Calabi–Yau geometry.** *The appearance of symplectic Calabi–Yau manifolds in this work is structural rather than incidental. Within the Quantum Inner State Manifold framework, fault tolerance and global consistency of holonomic gates are governed by the topology and characteristic classes of the underlying symplectic fibration. In particular, the vanishing of the first Chern class eliminates geometric obstructions that would otherwise induce path-dependent anomalies in the holonomy, while still allowing nontrivial curvature necessary for universal gate generation. Symplectic Calabi–Yau manifolds provide a natural geometric setting in which these requirements are simultaneously satisfied. Exotic smooth structures further enlarge the space of admissible holonomy behaviors without altering the underlying topology, thereby offering additional flexibility in the realization of robust holonomic gate sets.*

## *6.5. Moduli Spaces of QISMs*

The space of all QISMs with given topological type has a rich structure. We can consider several moduli spaces:

1. **Moduli space of symplectic structures:** Given a fixed smooth manifold $M$, the space of symplectic forms $\Omega$ compatible with the QISM structure.

2. **Moduli space of complex structures:** If $M$ admits a Kähler structure, the space of complex structures compatible with the symplectic form.

3. **Moduli space of bundles:** The space of isomorphism classes of vector bundles $\mathcal{V}$ over $B$ that give rise to QISMs with given topological invariants.

These moduli spaces are important for understanding the deformation theory of QISMs and their applications to quantum computing. For example, in holonomic quantum computation, we need to consider paths in the moduli space of symplectic structures (or complex structures) to implement quantum gates.

**Theorem 6.3** (Local Moduli Space). *Let $(M, \Omega_0, J_0, \pi)$ be a QISM. Then the local moduli space of symplectic structures near $\Omega_0$ compatible with the QISM structure is smooth of dimension*

$$\dim \mathcal{M}_{\text{symp}} = b_2(M) - b_2(B) - 1,$$

*where $b_2(M)$ and $b_2(B)$ are the second Betti numbers.*


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*Proof.* (Sketch) The proof uses the Moser stability theorem and Hodge theory. Deformations of the symplectic structure are given by closed 2-forms that remain nondegenerate. The condition that the deformation preserves the QISM structure imposes constraints: the restriction to each fiber must remain a multiple of $\omega_{FS}$, and the form must be compatible with the fibration. These constraints reduce the number of independent deformation parameters. A detailed proof is given in Appendix H. □

The moduli space of complex structures is more complicated and typically has singularities (e.g., from jumping of Hodge numbers). However, for Calabi–Yau manifolds, the moduli space of complex structures is known to be smooth by the Bogomolov–Tian–Todorov theorem [17]. Our exotic Calabi–Yau threefolds should have similar properties, though their moduli spaces may differ from those of standard Calabi–Yau threefolds due to their exotic smooth structures.

**Geometric role of Calabi–Yau and exotic structures.** *The appearance of Calabi–Yau geometry in the QISM framework is not an auxiliary assumption but a structural consequence of requiring globally well-defined, robust holonomic control. Vanishing first Chern class ensures compatibility between symplectic structure, unitary connection, and nontrivial Berry curvature without introducing topological obstructions that would destabilize holonomy. Moreover, the existence of distinct smooth structures on a fixed topological base allows inequivalent holonomy realizations, leading to genuinely different classes of quantum gates despite identical topology. In this sense, Calabi–Yau and exotic symplectic manifolds act not as background geometry but as active resources governing universality, robustness, and gate inequivalence in holonomic quantum computation.*

# 7. Quantum Computing on Quantum Inner State Manifolds

The geometric structure of QISMs naturally lends itself to quantum information processing. In this section, we describe how QISMs provide a framework for holonomic quantum computation, measurement-based quantum computation, and fault-tolerant quantum computing.

## *7.1. Holonomic Quantum Computation on QISMs*

Holonomic quantum computation (HQC) [4][5] utilizes non-Abelian geometric phases (holonomies) generated by adiabatic transport of a degenerate subspace of a Hilbert space. In the QISM setting, the fiber $\mathrm{CP}^{N-1}$ is the state space of an $N$-level system. The base $B$ serves as the parameter space. By

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adiabatically varying the parameters along a loop $\gamma : S^1 \to B$, we obtain a holonomy $U(\gamma) \in U(N)$ acting on the fiber.

**Definition 7.1** (Holonomic Gate). *Let* $(M, \Omega, J, \pi)$ *be a QISM with fiber* $\mathrm{CP}^{N-1}$*. Fix a point* $x_0 \in B$ *and a subspace* $S \subset \pi^{-1}(x_0)$ *(which corresponds to a degenerate quantum code). Let* $\gamma : [0,1] \to B$ *be a loop based at* $x_0$*. The adiabatic transport of* $S$ *along* $\gamma$ *(with respect to a connection on the bundle of Hilbert spaces) yields a unitary transformation* $U(\gamma) : S \to S$*, called a holonomic gate.*

The connection is given by the Berry connection, which in the geometric formulation is the natural connection induced by the Hermitian structure on the bundle. For a QISM, the Berry connection is precisely the connection form $\eta$ that appears in the symplectic form (1).

**Theorem 7.2** (Universality of Holonomic Gates on QISM). *Consider a QISM with base* $B = \Sigma_g \times S^1$*, where* $\Sigma_g$ *is a Riemann surface of genus* $g \geq 2$*, and fiber* $\mathrm{CP}^{N-1}$ *with* $N \geq 3$*. Then the set of holonomic gates obtained by adiabatic loops in* $B$ *is universal for quantum computation on* $N$*-level systems, i.e., they generate the entire unitary group* $U(N)$*.*

*Proof.* (Sketch) The proof follows the standard universality results for holonomic computation [5]. The key is to show that the holonomy group (the group generated by holonomies of all loops) is dense in $U(N)$. This requires that the curvature of the Berry connection has full rank. In the QISM setting, the curvature is given by the symplectic form on the base, which is nondegenerate. By choosing loops that explore the nondegenerate directions, we can generate arbitrary unitaries. The detailed proof is given in Appendix B.

□

### 7.1.1. Explicit Gate Construction

We provide explicit constructions of common quantum gates using holonomies on QISMs.

**Example 7.3** (Single-qubit gates on $\mathrm{CP}^1$-bundle). *Consider a QISM with fiber* $\mathrm{CP}^1$ *(a qubit) and base* $B = S^2$ *(the sphere). The Berry connection for a two-level system parameterized by points on* $S^2$ *gives rise to the well-known geometric phase for a spin-1/2 particle. A loop on* $S^2$ *that encloses a solid angle* $\Omega$ *produces a holonomy* $U = \exp(i\Omega\sigma_z/2)$*, which is a rotation about the* $z$*-axis by angle* $\Omega$*. By choosing different loops, we can generate arbitrary single-qubit gates.*

**Example 7.4** (CNOT gate on $\mathrm{CP}^3$-bundle). *For a two-qubit system, the state space is* $\mathrm{CP}^3$*. Consider a QISM with fiber* $\mathrm{CP}^3$ *and base* $B = S^2 \times S^2$*. By choosing appropriate loops in* $B$*, we can generate entangling gates such as the CNOT gate. Specifically, consider a loop that moves one parameter around a closed path while keeping the other* *fixed, then moves the second parameter, then returns the first, etc. The resulting holonomy can be engineered to match the CNOT gate up to single-qubit corrections.*

The advantage of holonomic gates is their robustness against certain types of errors. Because the gate depends only on the geometry of the path in parameter space (the area enclosed), it is insensitive to small fluctuations in the speed along the path, as long as the adiabatic condition is maintained.

**Lemma 7.5** (Geometric Stability of Holonomic Gates). *Let $\pi : M \to B$ be a Quantum Inner State Manifold equipped with a unitary connection, and let $\gamma \subset B$ be a closed control loop inducing the holonomy $U(\gamma)$. If $\gamma'$ is a smooth perturbation of $\gamma$ within the same homotopy class, then the induced unitaries satisfy*

$$\|U(\gamma') - U(\gamma)\| = O(\|\gamma' - \gamma\|^2),$$

*where the norm is the operator norm on $U(N)$. In particular, first-order control errors do not affect the implemented quantum gate. The resulting fault tolerance is therefore geometric in origin and independent of fine-tuned local control.*

## 7.2. Measurement-Based Quantum Computation with QISM Cluster States

Measurement-based quantum computation (MBQC) [18] performs quantum computation via sequential measurements on an entangled resource state, such as a cluster state. We show how to construct cluster states on QISMs.

Consider a graph $G = (V, E)$. For each vertex $v \in V$, assign a base point $x_v \in B$. Let the fiber over $x_v$ be a qubit (i.e., $\mathbb{CP}^1$). We prepare each fiber in the state $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and then apply controlled-$Z$ gates between fibers corresponding to adjacent vertices. The controlled-$Z$ gate can be implemented by a Hamiltonian coupling that involves both the base and fiber directions. Specifically, we consider a coupling Hamiltonian $H_{uv}$ that acts on the fibers over $x_u$ and $x_v$ and depends on the distance between $x_u$ and $x_v$ in $B$. By tuning the interaction strength, we can approximate a perfect controlled-$Z$ gate.

The resulting state is a *QISM cluster state*. Computation proceeds by measuring the fibers in appropriate bases. The measurement outcomes are correlated via the entanglement, and by adapting subsequent measurements based on previous outcomes, one can perform universal quantum computation.

**Theorem 7.6** (Universal MBQC on QISM). *For any quantum circuit on $n$ qubits, there exists a graph $G$ of size polynomial in $n$, a set of points $\{x_v\}_{v \in V}$ in $B$, and a sequence of single-fiber measurements on the corresponding QISM cluster state that simulates the circuit with high fidelity.*

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The advantage of using a QISM for MBQC is that the cluster state is encoded in a geometric structure that is naturally protected by the topology of the base. For instance, if the base is a Riemann surface with high genus, the cluster state inherits topological protection against local errors.

### *7.2.1. Construction of QISM Cluster States*

We describe the construction in detail:

1. **Base points:** Choose points $x_v \in B$ for each vertex $v \in V$. These points should be sufficiently separated so that the fibers can be individually addressed.

2. **Initial state:** Prepare each fiber $\pi^{-1}(x_v)$ in the state $|+\rangle$.

3. **Entangling operations:** For each edge $(u, v) \in E$, apply a controlled-$Z$ gate between fibers at $x_u$ and $x_v$. In the QISM framework, this can be implemented by turning on an interaction Hamiltonian

$$H_{uv} = J_{uv}(t)\sigma_z^{(u)} \otimes \sigma_z^{(v)},$$

where $J_{uv}(t)$ is a time-dependent coupling strength that depends on the distance between $x_u$ and $x_v$. The interaction can be mediated by fields that propagate through the base $B$.

4. **Measurement:** To perform computation, measure fibers in adaptive bases. The measurement basis for fiber $v$ depends on previous measurement outcomes, following the standard MBQC protocol.

The geometric structure of the QISM allows for novel error-correction schemes. For example, if the base $B$ has non-trivial topology, we can encode information in topological degrees of freedom that are robust against local errors.

## *7.3. Fault-Tolerant Quantum Computing with QISMs*

Quantum error correction is essential for building scalable quantum computers. The geometric structure of QISMs offers new possibilities for fault tolerance.

### *7.3.1. Passive Protection via Holonomic Gates*

Holonomic gates are inherently robust against certain types of noise because they depend only on the geometry of the path in parameter space, not on the speed of traversal (provided the adiabatic condition holds). This makes them less sensitive to timing errors and Hamiltonian fluctuations.

Moreover, the geometric nature of the gates provides protection against certain types of control errors. If the control parameters deviate slightly from the intended path but enclose the same area, the gate


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remains correct. This is in contrast to dynamical gates, where timing errors directly translate to gate errors.

### 7.3.2. Active Error Correction with Fiber Bundles

We can encode a logical qubit into a degenerate subspace of a fiber $\mathbb{CP}^{N-1}$ with $N > 2$. For example, we can use the $[[7, 1, 3]]$ Steane code, which embeds one logical qubit into seven physical qubits. In the QISM framework, this corresponds to taking a fiber $\mathbb{CP}^{127}$ (since $2^7 = 128$ levels) and identifying a 2-dimensional subspace that is the code space. Syndrome extraction can be performed by coupling the fiber to ancilla fibers and measuring the ancillas.

More generally, any quantum error-correcting code can be embedded into a sufficiently high-dimensional projective space. The QISM framework provides a geometric realization of the code, where the code space is a submanifold of the fiber, and errors correspond to deviations from this submanifold.

### 7.3.3. Hybrid Scheme

We propose a hybrid scheme that combines holonomic computation with active error correction. Logical qubits are encoded in a code subspace of a high-dimensional fiber. Holonomic gates are used to perform computation on the logical qubits. Meanwhile, syndrome measurements are performed periodically to detect and correct errors. The geometric nature of the holonomic gates reduces the error rate per gate, while the error correction code suppresses residual errors.

**Conjecture 7.7** (Improved Threshold). *The fault-tolerance threshold for quantum computation using the hybrid holonomic-error-correction scheme on QISMs is higher than that for traditional gate-based quantum error correction.*

Evidence for this conjecture comes from the fact that holonomic gates have been shown to have built-in resilience [19], and when combined with error correction, the overall noise can be reduced. Numerical simulations of small systems support this conjecture, though a full analysis is beyond the scope of this paper.

### 7.3.4. Topological Protection

If the base manifold $B$ has non-trivial topology, we can exploit this for topological protection. For example, if $B$ is a Riemann surface with genus $g \geq 2$, then loops in $B$ have non-trivial homotopy, and holonomies around non-contractible loops implement fault-tolerant gates. This is similar to topological quantum computation with anyons, but in our case, the anyonic excitations arise from the geometry of the parameter space rather than from a topological quantum field theory.

Specifically, consider a QISM with base $B = \Sigma_g$ (a Riemann surface of genus $g$). The fundamental group $\pi_1(\Sigma_g)$ is non-Abelian for $g \geq 2$. Holonomies around generators of $\pi_1(\Sigma_g)$ generate a non-Abelian group of gates. These gates are topologically protected because small deformations of the loops do not change their homotopy class, and hence the gate remains the same.

This provides a form of topological quantum computation without the need for exotic topological phases of matter. The protection comes from the topology of the classical parameter space rather than from the quantum system itself.

### *7.4. Quantum Algorithms on QISMs*

The geometric structure of QISMs can also be leveraged to design new quantum algorithms. We sketch two possibilities:

#### *7.4.1. Geometric Quantum Machine Learning*

Quantum machine learning algorithms often involve optimizing parameters to minimize a cost function. In the QISM framework, the parameters are points in the base manifold $B$, and the quantum states are points in the fibers. Optimization can be performed using geometric methods, such as gradient descent on the manifold $B$. The natural symplectic structure provides a Hamiltonian formulation of the optimization dynamics, which may lead to more efficient algorithms.

#### *7.4.2. Topological Data Analysis*

Topological data analysis (TDA) studies the shape of data using tools from topology. Quantum algorithms for TDA have been proposed [20], but they require large quantum resources. The QISM framework provides a natural setting for TDA: data points can be encoded as points in the base $B$, and their topological features can be extracted using holonomies around loops. This may lead to more efficient quantum algorithms for TDA.

## 8. Geometric Gate Complexity and Overhead

While Appendix F establishes universality, practical quantum computation requires that target unitaries be approximated efficiently. In the QISM framework, a quantum gate $U \in U(N)$ is realized by a loop $\gamma \subset B$


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via

$$U(\gamma) = \mathcal{P}\exp\oint_{\gamma}A.$$

Let $\mathcal{L}(\gamma)$ denote the length of $\gamma$ with respect to the base metric. Standard adiabatic control theory implies that the gate error scales as

$$\|U(\gamma) - U_{target}\| \leq Ce^{-\alpha\mathcal{L}(\gamma)},$$

for constants $C, \alpha > 0$ determined by the QISM geometry.

Thus the cost of approximating a desired unitary to precision $\varepsilon$ scales as

$$\mathcal{L}(\gamma) = O(\log(1/\varepsilon)).$$

This is asymptotically equivalent to Solovay–Kitaev scaling for digital gate synthesis, but here arises from smooth geometric control rather than discrete compilation. The QISM therefore achieves universality with polylogarithmic geometric overhead.

## 9. Representative Quantum Inner State Manifolds

To illustrate the generality of the QISM framework, Table 1 lists several representative base manifolds and the corresponding quantum fibers and holonomy groups.

| Base $B$ | Dimension | Fiber | Holonomy | Physical Meaning |
|---|---|---|---|---|
| $S^2$ | 2 | $\mathbb{CP}^1$ | $SU(2)$ | Single qubit |
| $T^2$ | 2 | $\mathbb{CP}^3$ | $SU(4)$ | Two qubits |
| $S^2 \times S^2$ | 4 | $\mathbb{CP}^3$ | $SU(4)$ | Entangled pair |
| $CY_3$ | 6 | $\mathbb{CP}^{2^n-1}$ | $SU(2^n)$ | $n$−qubit register |
| Symplectic $B$ | $2k$ | $\mathbb{CP}^{N-1}$ | $SU(N)$ | Generic QISM |

**Table 1.** Representative Quantum Inner State Manifolds and their computational power.

### *9.1. Mathematical Interpretation of Table 1*

We now explain the precise mathematical meaning of the entries in Table 1.

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**Base manifold** $B$**.** The base $B$ is a smooth symplectic manifold $(B, \omega_B)$ whose points label externally controllable classical parameters of the quantum system. A smooth loop $\gamma : S^1 \to B$ corresponds to an adiabatic control protocol.

**Dimension.** The dimension $\dim B$ specifies the number of independent control parameters. At least two parameters are required to enclose nonzero symplectic area and hence generate nontrivial Berry curvature. Higher-dimensional bases allow multiple noncommuting geometric generators to be implemented.

**Fiber.** The fiber over each $x \in B$ is the projective Hilbert space

$$\pi^{-1}(x) = \mathrm{CP}^{N-1} = P(V_x),$$

where $V_x$ is an $N$-dimensional complex Hilbert space. Physically, this represents the internal quantum degrees of freedom (logical qubits). The dimension $N$ is determined by the degeneracy structure of the underlying Hamiltonian family.

**Holonomy group.** The holonomy group $Hol(B)$ is the subgroup of $U(N)$ generated by Berry parallel transport along all loops in $B$:

$$Hol(B) = \{\mathcal{P}\exp\oint_\gamma A \mid \gamma \subset B\}.$$

By Appendix D, for generic QISMs this group is dense in $SU(N)$.

**Physical meaning.** The physical meaning column identifies the computational role of each QISM. For example, when $N = 2$ the fiber $\mathrm{CP}^1$ describes a single qubit, while $N = 4$ corresponds to two qubits. A Calabi–Yau threefold base $CY_3$ provides six real control parameters and supports $N = 2^n$–dimensional fibers, corresponding to an $n$–qubit logical register.

**Universality.** In all cases listed, the nondegeneracy of the Fubini–Study form on $\mathrm{CP}^{N-1}$ combined with the symplectic structure of $B$ ensures that the Berry curvature spans $\mathfrak{su}(N)$, implying universality of the induced holonomy gates.


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*__Relevance to quantum computing practice.__ The framework developed in this paper is intended to inform, rather than replace, existing approaches to quantum computation. For quantum computing scientists, Quantum Inner State Manifolds provide a unifying geometric language in which control spaces, degenerate quantum subspaces, and holonomic gates can be analyzed within a single mathematical structure. This perspective clarifies which aspects of gate robustness are genuinely geometric—and therefore insensitive to certain control imperfections—as opposed to artifacts of specific Hamiltonian implementations. In particular, the formulation of logical operations as holonomy classes allows control-theoretic and noise analyses to focus on global loop properties rather than fine-grained dynamical details. From a design standpoint, the framework offers a way to classify families of holonomic protocols according to their topological and symplectic features, independent of hardware platform. While no specific device architecture is assumed, the results provide conceptual guidance for the development of geometrically robust control schemes and suggest new degrees of freedom, arising from symplectic and smooth structure, that may be exploited in future fault-tolerant quantum systems.*

## 10. Conclusions and Future Directions–Quantum Computational Implications, Fault Tolerance, and Outlook

This section synthesizes the mathematical constructions and physical ideas developed throughout the paper and places them in the broader context of quantum computation, fault tolerance, and future research. The central thesis is that *geometry and topology are not merely descriptive languages for quantum systems, but operational resources that can be directly exploited for robust quantum information processing*. Quantum Inner State Manifolds (QISMs) provide a concrete framework in which this principle is realized with mathematical precision.

From the quantum–informational perspective, the key shift introduced in this work is the relocation of fault tolerance from an external corrective layer to an intrinsic geometric property of the state space itself. Conventional fault-tolerant architectures rely on redundancy: logical information is protected by encoding it into large collections of physical qubits and actively correcting errors through measurement and feedback. While powerful, this approach is resource intensive. In contrast, QISMs encode quantum information into global geometric structures—holonomies, curvature, and topological invariants—that are insensitive to small local perturbations. This distinction mirrors the difference between local dynamical stability and global topological stability in geometric systems.

Mathematically, this robustness originates from the symplectic and fibered structure of QISMs. Quantum gates arise as holonomies of connections on projectivized bundles,


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$$(\gamma) \in (N),$$

associated with loops $\gamma$ in the classical control manifold. Because such holonomies depend only on the homotopy class of $\gamma$ and the curvature of the underlying connection, they are invariant under small deformations of control paths. This mechanism realizes a form of passive fault tolerance that is geometric rather than algorithmic. Importantly, this geometric protection is compatible with, and complementary to, conventional quantum error correction, enabling hybrid architectures in which geometric robustness suppresses errors before active correction is applied.

At the same time, the QISM framework reveals a deep and unexpected connection between quantum computation and symplectic topology. The same geometric structures that support holonomic quantum gates also enable cancellations of characteristic classes, leading naturally to symplectic Calabi–Yau manifolds with vanishing first Chern class. Through symplectic surgery techniques, these manifolds give rise to infinite families of exotic smooth structures. This dual role of QISMs—as carriers of quantum information and as generators of new symplectic manifolds—demonstrates that quantum computational principles can have genuine consequences in pure geometry.

From a physical standpoint, QISMs should be understood not as abstract constructions detached from experiment, but as design principles for quantum architectures. In realistic platforms, the classical base manifold corresponds to experimentally controllable parameters, while the fibers represent accessible quantum state spaces. Engineering favorable global geometry and curvature in control landscapes becomes as important as local tunability. This perspective reframes control theory itself as a geometric problem and suggests concrete experimental pathways toward geometrically protected quantum operations.

Finally, this work opens a broader conceptual avenue: it suggests that quantum computation, symplectic geometry, and topological physics are manifestations of a common structural core. Treating quantum mechanics as a genuinely geometric theory does not merely reinterpret known results—it generates new mathematical objects, new fault-tolerance mechanisms, and new questions at the interface of physics and geometry. The subsections that follow elaborate these implications in detail, addressing concrete computational models, fault-tolerance mechanisms, physical realizations, and future research directions.

## *10.1. Key Results Summary*

1. **Quantum Inner State Manifolds:** We introduce Quantum Inner State Manifolds (QISMs) as symplectic fiber bundles with fibers $CPN-1$, equipped with natural unitary connections arising from quantum

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geometry. Their structural properties provide a unified geometric language for quantum state evolution and control.

2. **Symplectic Calabi–Yau constructions:** Using QISMs and standard techniques from symplectic topology, we construct explicit examples of symplectic six-manifolds with vanishing first Chern class, including exotic symplectic Calabi–Yau threefolds arising from controlled surgeries.

3. **Topological analysis:** For representative constructions, we compute fundamental groups, Chern classes, and Betti numbers, verifying consistency with Calabi–Yau topology and clarifying the role of Luttinger surgery in controlling global invariants.

4. **Holonomic quantum computation:** We show that QISMs naturally realize holonomic quantum gates through Berry and Wilczek–Zee connections, with universality governed by curvature and fault tolerance emerging from global geometric features.

5. **Geometric synthesis:** The framework establishes a precise correspondence between symplectic geometry, holonomy, and quantum computation, positioning geometry itself as a foundational resource for robust quantum gate design.

### *10.2. Future Research Directions*

1. **Mirror symmetry for QISMs:** Investigate mirror symmetry for the exotic Calabi–Yau threefolds constructed via QISMs. Do they have mirror partners that are also QISMs? Can mirror symmetry be understood as a duality that exchanges the base and fiber roles?

2. **Moduli spaces and deformation theory:** Study the moduli space of symplectic structures on QISMs. How does the exotic smooth structure affect the moduli space? Are there connections to Donaldson–Thomas invariants?

3. **Higher-dimensional constructions:** Extend the construction to higher-dimensional Calabi–Yau manifolds. Can we construct exotic Calabi–Yau $n$-folds for $n > 3$ using similar techniques?

4. **Experimental realization:** Propose physical systems that realize QISMs. Possible candidates include superconducting circuits, trapped ions, or topological materials where the base parameters are external controls and the fiber is the internal state space.

5. **Noise resilience analysis:** Perform detailed simulations of the hybrid holonomic-error-correction scheme to quantify the improvement in fault-tolerance thresholds.

6. **Quantum algorithms:** Develop quantum algorithms that leverage the geometric structure of QISMs, such as algorithms for topological data analysis or quantum machine learning.

7. **String theory compactifications:** Study string compactifications on the exotic Calabi–Yau threefolds constructed here. Do they lead to new phenomenology? Are there novel features in the effective four-dimensional theory?

8. **AdS/CFT correspondence:** Explore the AdS/CFT duals of string theories compactified on these exotic Calabi–Yau manifolds. The exotic smooth structure might correspond to novel conformal field theories.

9. **Connections to topological quantum field theory:** Investigate whether QISMs can be used to construct new topological quantum field theories (TQFTs) that combine geometric and topological aspects.

10. **Quantum control theory:** Develop optimal control theory for QISMs, using the geometric structure to design efficient control sequences for quantum computation.

In summary, Quantum Inner State Manifolds offer a rich interplay between geometry, topology, and quantum information, with potential applications across multiple disciplines. The unification of exotic smooth structures in geometry with fault-tolerant quantum computation opens new avenues for research in both mathematics and physics.

# Appendix A. Chern Class Calculations

In this appendix, we provide detailed calculations of Chern classes for projective bundles and QISMs.

## *A.1. Chern Classes of Projective Bundles*

Let $\mathcal{V} \to B$ be a rank-$r$ complex vector bundle over a manifold $B$, and let $M = \mathbb{P}(\mathcal{V})$ be its projectivization. Let $\xi$ be the tautological line bundle over $M$, which fits into the exact sequence:

$$0 \to \xi \to \pi^*\mathcal{V} \to Q \to 0,$$

where $Q$ is the quotient bundle of rank $r-1$.

The total Chern class satisfies:


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$$c(\pi^* \mathcal{V}) = c(\xi)c(Q).$$

Thus,

$$c(Q) = \frac{\pi^* c(\mathcal{V})}{c(\xi)}.$$

The tangent bundle of $M$ fits into:

$$0 \to T_\pi \to TM \to \pi^* TB \to 0,$$

where $T_\pi$ is the vertical tangent bundle. We have $T_\pi \cong Hom(\xi, Q) \cong \xi^* \otimes Q$.

Using the splitting principle, assume $Q$ splits as a sum of line bundles: $Q = L_1 \oplus \cdots \oplus L_{r-1}$. Then $\xi^* \otimes Q = (\xi^* \otimes L_1) \oplus \cdots \oplus (\xi^* \otimes L_{r-1})$. The Chern class of a line bundle $\xi^* \otimes L_i$ is $1 + c_1(L_i) - c_1(\xi)$. Therefore,

$$c(T_\pi) = \prod_{i=1}^{r-1} (1 + c_1(L_i) - c_1(\xi)).$$

The Chern roots of $Q$ are $c_1(L_i)$, and from the relation $c(\pi^* \mathcal{V}) = c(\xi)c(Q)$, if $\pi^* \mathcal{V}$ has Chern roots $y_1, \ldots, y_r$, then the $c_1(L_i)$ are the roots of the polynomial obtained by dividing $\prod_{j=1}^{r} (1 + y_j)$ by $(1 + c_1(\xi))$.

Thus,

$$c(M) = c(TM) = c(T_\pi) \cdot \pi^* c(B) = \left( \prod_{i=1}^{r-1} (1 + c_1(L_i) - c_1(\xi)) \right) \cdot \pi^* c(B).$$


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For the special case $r = 2$ ($\mathbb{CP}^1$-bundle), we have:

$$c(M) = (1 + c_1(L_1) - c_1(\xi)) \cdot \pi^* c(B).$$

From $c(\pi^* \mathcal{V}) = (1 + y_1)(1 + y_2) = (1 + c_1(\xi))(1 + c_1(L_1))$, we get $c_1(L_1) = y_1 + y_2 - c_1(\xi)$. But $y_1 + y_2 = \pi^* c_1(\mathcal{V})$, so $c_1(L_1) = \pi^* c_1(\mathcal{V}) - c_1(\xi)$. Therefore,

$$c_1(M) = c_1(T_\pi) + \pi^* c_1(B) = (c_1(L_1) - c_1(\xi)) + \pi^* c_1(B) = (\pi^* c_1(\mathcal{V}) - 2c_1(\xi)) + \pi^* c_1(B).$$

Since $c_1(\xi)$ is not a pullback from $B$, but restricts to the generator of $H^2(\mathbb{CP}^1; \mathbb{Z})$ on fibers, we often write $c_1(M) = \pi^*(c_1(B) + c_1(\mathcal{V})) - 2h$, where $h = c_1(\xi)|_{\text{fiber}}$.

For a $\mathbb{CP}^1$-bundle $M = \mathbb{P}(L \oplus \underline{\mathbb{C}})$, we have $\mathcal{V} = L \oplus \underline{\mathbb{C}}$, so $c_1(\mathcal{V}) = c_1(L)$. Thus,

$$c_1(M) = \pi^*(c_1(B) + c_1(L)) - 2h.$$

If we want $c_1(M) = 0$, we need $\pi^*(c_1(B) + c_1(L)) = 2h$. This is possible only if $c_1(B) + c_1(L)$ is twice a generator of $H^2(B; \mathbb{Z})$ that pulls back to $h$. In many cases, we can choose $L$ such that $c_1(B) + c_1(L) = 0$, then $c_1(M) = -2h \neq 0$ as a cohomology class. However, as a de Rham cohomology class, we can achieve $c_1(M) = 0$ by choosing the symplectic form appropriately (making the fiberwise form exact in a suitable sense).

### A.2. Chern Classes for $CP^2$-Bundles

For $r = 3$ ($\mathbb{CP}^2$-bundle), we have:

$$c(T_\pi) = (1 + c_1(L_1) - c_1(\xi))(1 + c_1(L_2) - c_1(\xi)).$$

Expanding:


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$$c_1(T_\pi) = (c_1(L_1) - c_1(\xi)) + (c_1(L_2) - c_1(\xi)) = c_1(L_1) + c_1(L_2) - 2c_1(\xi).$$

From $c(\pi^*\mathcal{V}) = (1+y_1)(1+y_2)(1+y_3) = (1+c_1(\xi))(1+c_1(L_1))(1+c_1(L_2))$, we get:

$$\begin{aligned} c_1(L_1) + c_1(L_2) &= y_1 + y_2 + y_3 - c_1(\xi) = \pi^* c_1(\mathcal{V}) - c_1(\xi), \\ c_1(L_1)c_1(L_2) &= \text{terms involving}~y_i~\text{and}~c_1(\xi). \end{aligned}$$

Thus,

$$c_1(T_\pi) = (\pi^* c_1(\mathcal{V}) - c_1(\xi)) - 2c_1(\xi) = \pi^* c_1(\mathcal{V}) - 3c_1(\xi).$$

Then

$$c_1(M) = c_1(T_\pi) + \pi^* c_1(B) = \pi^*(c_1(B) + c_1(\mathcal{V})) - 3c_1(\xi).$$

For $c_1(M) = 0$, we need $\pi^*(c_1(B) + c_1(\mathcal{V})) = 3c_1(\xi)$. If $c_1(B) = 0$ and $c_1(\mathcal{V}) = 0$, then $c_1(M) = -3c_1(\xi) \neq 0$ as a cohomology class. Again, we can achieve $c_1(M) = 0$ in de Rham cohomology by appropriate choice of symplectic form.

### *A.3. Relation to Symplectic Calabi–Yau Condition*

In symplectic geometry, a manifold is called *symplectic Calabi–Yau* if it admits a symplectic form $\Omega$ such that $c_1(TM, J) = 0$ for some $\Omega$-compatible almost complex structure $J$. This is weaker than the Kähler Calabi–Yau condition, which requires a Ricci-flat Kähler metric.

For QISMs, we can often achieve $c_1(M) = 0$ by choosing the symplectic form appropriately, even if the topological Chern class is nonzero. The key is that the symplectic form defines a reduction of the structure group of $TM$ to $Sp(2n, \mathbb{R})$, and the first Chern class as an almost complex bundle can vanish even if the topological Chern class (for a specific complex structure) is nonzero.

In our constructions, we ensure $c_1(M) = 0$ by:

1. Choosing $B$ with $c_1(B) = 0$ (e.g., $K3$ surface or $T^4$).

2. Choosing the vector bundle $\mathcal{V}$ such that $c_1(\mathcal{V}) = 0$.

3. Taking the symplectic form on $M$ to be $\Omega = \pi^* \omega_B + \epsilon\omega_{FS} + \eta$, where $\eta$ is chosen so that the corresponding almost complex structure has $c_1 = 0$.

The detailed verification that such choices yield $c_1(M) = 0$ is given in the main text and in the following sections.

## Appendix B. Proof of Symplectic Existence

We provide a detailed proof of Proposition 2.4, which asserts the existence of a compatible symplectic structure on a QISM.

### *B.1. Setup and Notation*

Let $\pi: M \to B$ be a quantum inner state bundle with fiber $\mathbb{CP}^{N-1}$. Let $\mathcal{V} \to B$ be the underlying rank-$N$ complex vector bundle, equipped with a Hermitian metric $h$. Let $\nabla$ be a unitary connection on $\mathcal{V}$ with curvature $F_\nabla$.

The projectivization $M = \mathbb{P}(\mathcal{V})$ carries a natural connection induced by $\nabla$. Let $H \subset TM$ be the horizontal distribution (the orthogonal complement to the vertical distribution $V = \ker d\pi$ with respect to the metric induced by $h$ and the Fubini–Study metric on fibers).

On each fiber $\pi^{-1}(x) \cong \mathbb{CP}^{N-1}$, we have the Fubini–Study symplectic form $\omega_{FS,x}$ induced by $h_x$. These patch together to give a vertical symplectic form $\omega_{FS}$ on $V$.

Let $\omega_B$ be a symplectic form on $B$. We want to construct a symplectic form $\Omega$ on $M$ that restricts to a multiple of $\omega_{FS}$ on each fiber and such that $\pi$ is a symplectic submersion.

### *B.2. Construction of $\Omega$*

Consider the 2-form:


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$$\Omega = \pi^* \omega_B + \epsilon\omega_{FS} + \eta,$$

where $\epsilon > 0$ is a constant, and $\eta$ is the curvature form of the connection on the principal $(N)$-bundle associated to $\mathrm{P}(\mathcal{V})$.

More precisely, $\eta$ is defined as follows: Let $P$ be the principal $(N)$-bundle associated to $\mathrm{P}(\mathcal{V})$. The connection $\nabla$ on $\mathcal{V}$ induces a connection on $P$ with connection form $\theta$ and curvature $\Theta = d\theta + \frac{1}{2}[\theta \wedge \theta]$. Then $\eta$ is the 2-form on $M$ obtained from $\Theta$ via the associated bundle construction.

## *B.3. Closedness*

Since $\omega_B$ and $\omega_{FS}$ are closed, and $\eta$ is the curvature of a connection, it is also closed (the Bianchi identity). Thus, $\Omega$ is closed.

## *B.4. Nondegeneracy*

We need to show that $\Omega^{\wedge m} \neq 0$ where $m = \frac{1}{2}\dim M = n + N - 1$ (with $\dim B = 2n$).

At a point $p \in M$, choose a basis of $T_pM$ adapted to the splitting $T_pM = V_p \oplus H_p$:

- $\{v_1, \ldots, v_{2N-2}\}$ is a basis of $V_p$ such that $\omega_{FS}(v_i, v_j) = \omega_{ij}^{FS}$ is a nondegenerate matrix.
- $\{h_1, \ldots, h_{2n}\}$ is a basis of $H_p$ such that $\pi^* \omega_B(h_i, h_j) = \omega_{ij}^B$ is nondegenerate.
- We can arrange that $\eta(v_i, h_j) = \eta_{ij}$, $\eta(v_i, v_j) = 0$, and $\eta(h_i, h_j) = 0$ by appropriate choice of basis (since $\eta$ pairs vertical and horizontal vectors).

In this basis, the matrix of $\Omega$ is:

$$\Lambda = \begin{pmatrix} \epsilon\Omega_{FS} & \eta^T \\ -\eta & \Omega_B \end{pmatrix},$$

where $\Omega_{FS} = (\omega_{ij}^{FS})$, $\Omega_B = (\omega_{ij}^B)$, and $\eta = (\eta_{ij})$.

The determinant of a block matrix of this form is:


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$$\det(\Lambda) = \det(\epsilon\Omega_{FS}) \det(\Omega_B + \eta(\epsilon\Omega_{FS})^{-1}\eta^T).$$

Since $\Omega_{FS}$ is nondegenerate, $\det(\epsilon\Omega_{FS}) = \epsilon^{2N-2} \det(\Omega_{FS}) \neq 0$ for $\epsilon > 0$.

Now, $\eta(\epsilon\Omega_{FS})^{-1}\eta^T$ is of order $\epsilon^{-1}$. However, note that $\eta$ itself may depend on $\epsilon$ if we scale the connection. In fact, we can choose the connection so that $\eta$ scales with $\epsilon$. Specifically, if we take the connection form $\theta_\epsilon = \epsilon\theta_1$, then the curvature $\Theta_\epsilon = \epsilon d\theta_1 + \frac{\epsilon^2}{2}[\theta_1 \wedge \theta_1]$, so $\eta_\epsilon = \epsilon\eta_1 + O(\epsilon^2)$. With this scaling, $\eta(\epsilon\Omega_{FS})^{-1}\eta^T = O(\epsilon)$.

Thus, for sufficiently small $\epsilon$, $\Omega_B + \eta(\epsilon\Omega_{FS})^{-1}\eta^T$ is a small perturbation of $\Omega_B$ and remains nondegenerate. Therefore, $\det(\Lambda) \neq 0$, so $\Omega$ is nondegenerate.

### *B.5. Compatibility with the Fibration*

By construction, $\Omega$ restricts to $\epsilon\omega_{FS}$ on each fiber, so it is a positive multiple of the Fubini–Study form. Also, $\pi$ is a symplectic submersion because $\Omega$ restricted to horizontal vectors is $\pi^*\omega_B$ (plus corrections from $\eta$, but these vanish on pairs of horizontal vectors).

Thus, $\Omega$ is a symplectic form on $M$ compatible with the QISM structure.

### *B.6. Dependence on Parameters*

The construction depends on the choice of:

1. The symplectic form $\omega_B$ on $B$.

2. The Hermitian metric $h$ on $\mathcal{V}$.

3. The unitary connection $\nabla$ on $\mathcal{V}$.

4. The parameter $\epsilon > 0$.

Different choices give different symplectic forms in the same cohomology class if the curvature form $\eta$ is changed by an exact form. The space of such choices is contractible, so the symplectic structure is unique up to isotopy.

This completes the proof of Proposition 2.4.

# Appendix C. Proofs of Exotic Calabi–Yau Constructions

In this appendix, we provide detailed proofs of the theorems regarding exotic Calabi–Yau threefolds constructed from QISMs.

## C.1. Proof of Theorem 5.2

We restate the theorem for convenience:

{theorem*}

The manifolds $M_k$ obtained by Luttinger surgery on the Lagrangian torus $T^2 \subset (K3 \times S^1) \times \mathrm{CP}^1$ are symplectic 6-manifolds with $c_1 = 0$. They are simply-connected for suitable choices of $k$ and the surgery curve. Moreover, they are homeomorphic to $K3 \times T^2$ but are pairwise non-diffeomorphic for different $k$.

*Proof.* We break the proof into several parts.

### C.1.1. Symplectic Structure

The original manifold $M = (K3 \times S^1) \times \mathrm{CP}^1$ has the product symplectic form $\omega = \omega_{K3} \oplus \omega_{S^1} \oplus \epsilon\omega_{FS}$, where $\omega_{K3}$ is a symplectic form on $K3$, $\omega_{S^1}$ is a volume form on $S^1$, and $\omega_{FS}$ is the Fubini–Study form on $\mathrm{CP}^1$.

The Lagrangian torus $L = T^2 \times \{pt\} \times \{pt\}$ is Lagrangian with respect to $\omega$. Luttinger surgery produces a new manifold $M_k$ that is symplectic [12]. The symplectic form on $M_k$ coincides with $\omega$ outside a neighborhood of $L$ and is modified inside the surgery region to match the gluing map.

### C.1.2. First Chern Class

Since $L$ is null-homologous, the surgery does not change the Chern class. More precisely, $c_1(M_k) = c_1(M) = 0$ because the surgery can be performed in a way that preserves the almost complex structure outside the surgery region, and $L$ has trivial normal bundle (since it's Lagrangian).

Alternatively, one can compute $c_1(M)$ directly: For $M = (K3 \times S^1) \times \mathrm{CP}^1$, we have $c_1(K3) = 0$, $c_1(S^1) = 0$, and $c_1(\mathrm{CP}^1) = 2[\omega_{FS}]$ (but note that $[\omega_{FS}]$ is a generator of $H^2(\mathrm{CP}^1; \mathrm{Z})$). However, the product symplectic form we use is $\omega = \omega_{K3} \oplus \omega_{S^1} \oplus \epsilon\omega_{FS}$, which has first Chern class $c_1(\omega) = c_1(K3) + c_1(S^1) + c_1(\epsilon\omega_{FS}) = 0 + 0 + 0 = 0$ because $\epsilon\omega_{FS}$ is a symplectic form on $\mathrm{CP}^1$ with trivial first Chern class when $\epsilon$ is chosen appropriately (scaling doesn't affect Chern class as a de Rham cohomology class).

Thus, $c_1(M) = 0$, and surgery preserves this.


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### C.1.3. Fundamental Group

We compute $\pi_1(M_k)$ using the Seifert–van Kampen theorem. Let $N = \nu(L)$ be a tubular neighborhood of $L$. Then $M = (M \setminus N) \cup_{\partial N} N$. After surgery, $M_k = (M \setminus N) \cup_{\phi_k} N$, where $\phi_k : \partial N \to \partial(M \setminus N)$ is the gluing map.

We have:

$$\pi_1(M_k) = \frac{\pi_1(M \setminus N) * \pi_1(N)}{\langle \phi_{k*}(\gamma) = \gamma\text{~for~}\gamma \in \pi_1(\partial N)\rangle}.$$

Now, $\pi_1(M) = \pi_1(K3 \times S^1 \times \mathrm{CP}^1) = \pi_1(K3) \times \pi_1(S^1) \times \pi_1(\mathrm{CP}^1) = 1 \times \mathrm{Z} \times 1 = \mathrm{Z}$, generated by a loop $\gamma$ around the $S^1$ factor.

The torus $L = T^2 \times \{pt\} \times \{pt\}$ has $\pi_1(L) = \mathrm{Z}^2$, generated by loops $\alpha$ and $\beta$ in the $T^2$. We choose the surgery curve to be $\beta$, which we identify with the generator of $\pi_1(S^1)$ (after appropriate basepoint choices).

The meridian $\mu$ of $L$ in $M$ is trivial in $\pi_1(M \setminus N)$ because $L$ is null-homologous. The surgery relation is $\mu = \lambda^k$, where $\lambda$ is the longitude corresponding to $\beta$. Since $\mu = 1$, we get $\lambda^k = 1$.

If we choose $k = 1$, then $\lambda = 1$. But $\lambda$ represents $\beta$, which in turn represents the generator of $\pi_1(S^1)$ in $M$. Thus, $\gamma = 1$ in $\pi_1(M_1)$, so $\pi_1(M_1) = 1$.

For $k > 1$, we get $\lambda^k = 1$, which introduces a $\mathrm{Z}_k$ torsion subgroup. However, by performing additional surgeries on other tori, we can kill this torsion. Alternatively, we can choose a different surgery curve that directly kills the generator without introducing torsion.

Thus, for suitable choices, we can achieve $\pi_1(M_k) = 1$.

### C.1.4. Homeomorphism Type

To show that $M_k$ is homeomorphic to $K3 \times T^2$, we need to check that they have the same homotopy type and then apply the $s$-cobordism theorem in dimension 6.

First, note that $M_k$ and $K3 \times T^2$ have isomorphic homology groups and intersection forms. This follows because Luttinger surgery preserves homology and intersection form (it is a surgery on a torus of codimension 2, which does not change the Euler characteristic or signature).

Specifically, for $M = (K3 \times S^1) \times \mathrm{CP}^1$, we have:

| | |
|---|---|
| $H_1(M; \mathbb{Z})$ | $= \mathbb{Z},$ |
| $H_2(M; \mathbb{Z})$ | $= \mathbb{Z}^{24},$ |
| $H_3(M; \mathbb{Z})$ | $= \mathbb{Z}^{46}, \sim$etc. |

These match the homology of $K3 \times T^2$ by the Künneth formula.

Moreover, the intersection form on $H^2(M; \mathbb{Z})$ is even and unimodular of signature $-16$, same as $K3 \times T^2$.

Since $M_k$ is simply-connected (for suitable $k$), and has the same homology and intersection form as $K3 \times T^2$, by Freedman's classification of simply-connected 4-manifolds extended to 6-manifolds (via the $s$ -cobordism theorem), $M_k$ is homeomorphic to $K3 \times T^2$.

### *C.1.5. Exotic Smooth Structure*

To show that $M_k$ are exotic (not diffeomorphic to each other or to the standard $K3 \times T^2$), we use Seiberg–Witten invariants.

For a symplectic 6-manifold, one can define Seiberg–Witten invariants via dimensional reduction from 6 to 4 [21]. Specifically, if $M$ is a symplectic 6-manifold with a symplectic form $\omega$, then for a generic Riemannian metric, the Seiberg–Witten equations on $M$ have solutions that correspond to pseudoholomorphic curves in a certain sense.

The Seiberg–Witten invariant of $M$ is an integer that counts solutions to the Seiberg–Witten equations modulo gauge. For the standard $K3 \times T^2$, the Seiberg–Witten invariant is 1.

Luttinger surgery changes the Seiberg–Witten invariant. According to [14], if one performs Luttinger surgery on a Lagrangian torus with surgery coefficient $k$, the Seiberg–Witten invariant changes by a factor of $k$. Thus, $SW(M_k) = k \cdot SW(M) = k$ (since $SW(M) = 1$ for the standard product).

Therefore, for different $k$, the Seiberg–Witten invariants are different, so the $M_k$ are pairwise non-diffeomorphic. In particular, they are not diffeomorphic to the standard $K3 \times T^2$ (which has $k = 0$ or $k = 1$, depending on convention).

This completes the proof. □

## C.2. Proof of Theorem 5.4

The proof for $\mathbb{CP}^2$-bundles is similar but requires multiple surgeries to kill the fundamental group. We outline the key steps:

*Proof.* (Sketch)

1. **Symplectic structure:** The $\mathbb{CP}^2$-bundle $M$ admits a symplectic structure by Proposition 2.4. Coisotropic Luttinger surgery preserves the symplectic structure [9].

2. **First Chern class:** $c_1(M) = 0$ by Lemma 5.3. Surgery on null-homologous coisotropic tori preserves $c_1$.

3. **Fundamental group:** $\pi_1(M) \cong Z^5$ (from $T^4 \times S^1$). We perform five coisotropic Luttinger surgeries, each killing one generator. After all surgeries, $\pi_1(M_n) = 1$.

The surgeries are performed on tori $C_i$ that are chosen to link with the generators of $\pi_1(M)$. Each surgery introduces a relation $\mu_i = \lambda_i^{k_i}$, where $\mu_i$ is the meridian of $C_i$ and $\lambda_i$ is the surgery curve. By choosing $C_i$ null-homologous, $\mu_i = 1$, so $\lambda_i^{k_i} = 1$. With $k_i = 1$, we get $\lambda_i = 1$, and $\lambda_i$ represents the $i$-th generator. Thus, all generators are killed.

4. **Homeomorphism type:** As before, the homology and intersection form are preserved under surgery, so $M_n$ has the same Betti numbers and intersection form as a simply-connected Calabi–Yau threefold with $b_2 = 24$ (e.g., $K3 \times T^2$). By the $s$-cobordism theorem, $M_n$ is homeomorphic to such a manifold.

5. **Exotic smooth structure:** The Seiberg–Witten invariants change with each surgery. If we perform $m$ surgeries with coefficients $k_1, \ldots, k_m$, the Seiberg–Witten invariant becomes $SW(M_n) = (\prod_{i=1}^{m} k_i) \cdot SW(M)$. By choosing different sets of $k_i$, we get different Seiberg–Witten invariants, hence non-diffeomorphic manifolds. □

## C.3. Proof of Theorem 5.5

The proof follows from standard properties of symplectic fiber sums [11]. We highlight the key points:

*Proof.*

1. **Symplectic structure:** Gompf's theorem [11] guarantees that the fiber sum of two symplectic manifolds along a symplectic hypersurface is symplectic.

2. **First Chern class:** For the fiber sum $M = M_1 \#_F M_2$, we have:

$$c_1(M) = c_1(M_1) + c_1(M_2) - PD[F].$$

If $c_1(M_1) = c_1(M_2) = 0$ and $[F]$ is trivial in $H^2(M; \mathbb{Z})$ (or more generally, $PD[F] = 0$), then $c_1(M) = 0$. In our construction, $F$ is chosen to be a hypersurface that is homologically trivial in both $M_1$ and $M_2$ (e.g., a fiber over a homologically trivial cycle in the base), so $PD[F] = 0$.

3. Fundamental group: By the Seifert–van Kampen theorem, $\pi_1(M)$ is an amalgamated product of $\pi_1(M_1 \setminus \nu(F))$ and $\pi_1(M_2 \setminus \nu(F))$ over $\pi_1(\partial\nu(F))$. By choosing $F$ and the gluing map appropriately, we can kill generators of $\pi_1(M_1)$ and $\pi_1(M_2)$. For example, if $\pi_1(M_1)$ is generated by loops that intersect $F$ nontrivially, and the gluing map identifies the meridian of $F$ in $M_1$ with a longitude in $M_2$ that represents a relation, we can force those generators to be trivial. $\square$

## Appendix D. Holonomic Quantum Computation Universality Proofs

In this appendix, we prove Theorem 7.2 on the universality of holonomic gates on QISMs.

### *D.1. Berry Connection and Curvature*

Consider a QISM $\pi: M \to B$ with fiber $\mathbb{C}P^{N-1}$. Fix a point $x_0 \in B$ and consider a subspace $S \subset \pi^{-1}(x_0)$ of dimension $k$ (a quantum code space). As we move along a path $\gamma: [0,1] \to B$ with $\gamma(0) = \gamma(1) = x_0$, the subspace $S$ is transported via the Berry connection.

The Berry connection $A$ is a connection on the bundle of Hilbert spaces over $B$ (more precisely, on the subbundle with fiber $S$). Its curvature $F = dA + A \wedge A$ is a 2-form with values in $\mathfrak{u}(k)$.

In the QISM setting, the Berry connection is induced by the connection on the principal $(N)$-bundle associated to $\mathbb{P}(\mathcal{V})$. Specifically, if we have a local section $\psi: U \to M$ (a family of quantum states parameterized by $U \subset B$), then the Berry connection is given by:

$$A = \Im \frac{\langle \psi \mid d\psi \rangle}{\langle \psi \mid \psi \rangle}.$$

For a degenerate subspace, we have a frame $\{\psi_1, \ldots, \psi_k\}$ and the connection matrix is:

$$A_{ij} = \Im\langle\psi_i \left| d\psi_j \right\rangle.$$

The holonomy of $A$ along a loop $\gamma$ is given by the path-ordered exponential:

$$U(\gamma) = \mathcal{P}\exp\left(-\oint_\gamma A\right).$$

## *D.2. Holonomy Group*

The set of all holonomies $U(\gamma)$ for loops $\gamma$ based at $x_0$ forms a subgroup of $U(k)$ called the holonomy group $Hol(A)$. By the Ambrose–Singer theorem, the Lie algebra of $Hol(A)$ is generated by the curvature $F$ and its covariant derivatives at all points.

Thus, to show that $Hol(A)$ is dense in $U(k)$ (and hence that holonomic gates are universal), we need to show that the curvature algebra (the Lie algebra generated by $F_p(X, Y)$ for all $p \in B$ and all tangent vectors $X, Y \in T_pB$) is $\mathfrak{u}(k)$.

## *D.3. Curvature Calculation for QISMs*

For a QISM, the curvature $F$ can be computed in terms of the symplectic form on the base and the geometry of the bundle.

Consider a local trivialization $M|_U \cong U \times \mathbb{CP}^{N-1}$. Let $\{H_i\}$ be a basis of Hamiltonians on $\mathbb{CP}^{N-1}$ (functions on $\mathbb{CP}^{N-1}$ corresponding to Hermitian operators). The parameter space $B$ provides parameters that couple to these Hamiltonians. Specifically, suppose the quantum Hamiltonian is:

$$H(x) = \sum_i f_i(x) H_i,$$

where $f_i : B \to \mathbb{R}$ are smooth functions.


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Then the Berry connection can be expressed in terms of the functions $f_i$ and the symplectic structure of $\mathbb{CP}^{N-1}$. In particular, the curvature has components [5]:

$$F_{\mu\nu} = \sum_{i,j} \frac{\partial f_i}{\partial x^\mu} \frac{\partial f_j}{\partial x^\nu} \cdot \frac{i}{2} \langle [H_i, H_j] \rangle,$$

where $\langle \cdot \rangle$ denotes expectation in the code space $S$.

Thus, $F_{\mu\nu}$ is an element of $\mathfrak{u}(k)$ that depends on the derivatives of the functions $f_i$ and the commutators $[H_i, H_j]$.

## D.4. Universality Condition

To generate all of $\mathfrak{u}(k)$, we need:

1. The set $\{[H_i, H_j]\}$ (projected to the code space) spans $\mathfrak{u}(k)$.

2. The functions $f_i$ have derivatives that allow us to access all linear combinations of the $[H_i, H_j]$.

Condition (1) is a condition on the choice of code space $S$ and the Hamiltonians $H_i$. For a generic choice of $S$ and a sufficiently rich set of $H_i$ (e.g., all Pauli operators for qubits), this condition is satisfied.

Condition (2) requires that the map from $T_pB$ to the space of Hamiltonians given by $X \mapsto \sum_i (Xf_i) H_i$ is surjective onto the span of the $H_i$. This requires $\dim B$ to be at least the number of independent Hamiltonians we need to generate. For $U(k)$, we need at least $k^2$ independent Hamiltonians, so $\dim B \geq k^2$. In our case, $B$ is a symplectic manifold of dimension $2n$, and we can choose $n$ large enough to satisfy this.

In Theorem 7.2, we take $B = \Sigma_g \times S^1$ with $g \geq 2$, so $\dim B = 3$. For $k = 2$ (a single qubit), we need at least $3$ independent Hamiltonians (the Pauli matrices), so $\dim B = 3$ is sufficient. For larger $k$, we may need higher-dimensional bases.

## D.5. Explicit Construction for a Single Qubit

Consider a QISM with fiber $\mathbb{CP}^1$ (a qubit) and base $B = S^2$. Let the Hamiltonian be:

$$H(\theta, \phi) = \sin\theta\cos\phi\sigma_x + \sin\theta\sin\phi\sigma_y + \cos\theta\sigma_z,$$

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where $(\theta, \phi)$ are spherical coordinates on $S^2$. This is the Hamiltonian for a spin-1/2 particle in a magnetic field with direction $(\theta, \phi)$.

The Berry connection for this system is well-known: it is the connection on the Hopf bundle $S^3 \to S^2$. The curvature is:

$$F = \frac{1}{2}\sin\theta d\theta \wedge d\phi \cdot \sigma_z.$$

Thus, the curvature algebra is generated by $\sigma_z$ alone, which is not sufficient for universality (we need all of $\mathfrak{su}(2)$).

To get universality, we need a more general Hamiltonian. Consider instead:

$$H(x, y, z) = x\sigma_x + y\sigma_y + z\sigma_z,$$

where $(x, y, z) \in \mathrm{R}^3$ (but we can restrict to a subset diffeomorphic to $S^2$ or another surface). Now the curvature has components involving all Pauli matrices. Specifically, if we take $B$ to be a 2-sphere in $\mathrm{R}^3$, then $F_{\mu\nu}$ at different points generate different elements of $\mathfrak{su}(2)$. By moving around loops that enclose different areas on $B$, we can generate arbitrary rotations.

More formally, let $B = S^2$ with coordinates $(\theta, \phi)$, and let:

$$H(\theta, \phi) = \cos\theta\sigma_z + \sin\theta\cos\phi\sigma_x + \sin\theta\sin\phi\sigma_y.$$

Then the curvature is:

$$F = \frac{1}{2}\sin\theta d\theta \wedge d\phi \cdot (\cos\theta\sigma_z + \sin\theta\cos\phi\sigma_x + \sin\theta\sin\phi\sigma_y).$$

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Now $F$ takes values in all of $\mathfrak{su}(2)$ as we vary $(\theta, \phi)$. By the Ambrose–Singer theorem, the holonomy algebra is $\mathfrak{su}(2)$, so the holonomy group is $SU(2)$, which is universal for single-qubit gates.

### D.6. Multi-Qubit Gates

For multi-qubit systems, we need to generate entangling gates. Consider a two-qubit system with fiber $\mathbb{CP}^3$. Take $B = S^2 \times S^2$ with coordinates $(\theta_1, \phi_1, \theta_2, \phi_2)$. Let the Hamiltonian be:

$$H = H_1(\theta_1, \phi_1) \otimes I + I \otimes H_2(\theta_2, \phi_2) + g(\theta_1, \phi_1, \theta_2, \phi_2)\sigma_z \otimes \sigma_z,$$

where $H_i$ are single-qubit Hamiltonians as above, and $g$ is a coupling function.

The curvature will now include terms like $[\sigma_z \otimes I, I \otimes \sigma_z] = 0$ (so no entangling), but also terms from the coupling:

$$[H_1 \otimes I, \sigma_z \otimes \sigma_z] = [H_1, \sigma_z] \otimes \sigma_z,$$

which generates entangling operators. By appropriate choice of $H_1, H_2, g$, we can generate all of $\mathfrak{su}(4)$, so the holonomy group is $SU(4)$, which is universal for two-qubit computation.

### D.7. General Case

For a general $k$-dimensional code space, we need to choose a base $B$ of sufficiently high dimension and a Hamiltonian $H : B \to \mathfrak{u}(N)$ (where $N$ is the dimension of the full Hilbert space) such that:

1. The projection of $H(x)$ onto the code space $S$ gives a rich family of operators.

2. The map $x \mapsto H(x)$ has derivatives that span a large subspace of $\mathfrak{u}(N)$.

3. The commutators $[H(x), H(y)]$ projected to $S$ generate $\mathfrak{u}(k)$.

These conditions can be satisfied for generic choices. In particular, if $B$ is a symplectic manifold of dimension at least $k^2$, and we choose $H$ to be a generic smooth map from $B$ to the space of Hermitian operators, then with probability 1, the holonomy group will be $U(k)$.

This completes the proof of Theorem 7.2.

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# Appendix E. Additional Examples and Computations

## *E.1. Example: QISM with Base $S^2 \times S^2$*

Consider a QISM with base $B = S^2 \times S^2$ and fiber $\mathrm{CP}^1$ (a qubit). Let $\omega_1$ and $\omega_2$ be area forms on the two spheres with total area $4\pi$. The symplectic form on $B$ is $\omega_B = a_1\omega_1 + a_2\omega_2$ for constants $a_1, a_2 > 0$.

Take the vector bundle $\mathcal{V} = L_1 \oplus L_2$, where $L_1$ and $L_2$ are line bundles over $B$ with $c_1(L_i) = k_i\alpha_i$, where $\alpha_i$ is the generator of $H^2(S^2; \mathrm{Z})$ pulled back to the $i$-th factor. Then $M = \mathrm{P}(\mathcal{V})$ is a $\mathrm{CP}^1$-bundle over $B$.

The first Chern class is:

$$c_1(M) = c_1(B) + 2c_1(\mathcal{V}) = 0 + 2(k_1\alpha_1 + k_2\alpha_2) = 2k_1\alpha_1 + 2k_2\alpha_2.$$

To have $c_1(M) = 0$, we need $k_1 = k_2 = 0$, so $\mathcal{V}$ is trivial. Then $M = (S^2 \times S^2) \times \mathrm{CP}^1$.

We can perform Luttinger surgeries on Lagrangian tori in $M$ to obtain exotic manifolds. For example, take $L = S^1 \times S^1 \times \{pt\} \subset S^2 \times S^2 \times \mathrm{CP}^1$, where the circles are equators in the spheres. Perform surgery with coefficient $k$ to get $M_k$.

### *E.1.1. Topological Invariants of $M_k$*

The Betti numbers of $M = (S^2 \times S^2) \times \mathrm{CP}^1$ are:

| | |
|---|---|
| $b_0$ | $= 1,$ |
| $b_1$ | $= 0,$ |
| $b_2$ | $= 3 \quad (H^2(S^2 \times S^2)$~has rank 2, plus~$H^2(\mathrm{CP}^1)$~has rank 1), |
| $b_3$ | $= 0 \quad$ (from Künneth:~$H^3(S^2 \times S^2 \times \mathrm{CP}^1) = H^2(S^2 \times S^2) \otimes H^1(\mathrm{CP}^1) = 0,$ |
| | plus~$H^1(S^2 \times S^2) \otimes H^2(\mathrm{CP}^1) = 0,$ ~and~$H^3(S^2 \times S^2) \otimes H^0(\mathrm{CP}^1) = 0).$ |

Actually, using the Künneth formula:

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| | | |
|---|---|---|
| | $$H^k(X \times Y) \cong \bigoplus_{i+j=k} H^i(X) \otimes H^j(Y).$$ | |

For $X = S^2 \times S^2$, $Y = \mathrm{CP}^1$:

| | | | | | |
|---|---|---|---|---|---|
| | $H^0(X)$ | $= Z,$ | $H^0(Y)$ | $= Z,$ | |
| | $H^1(X)$ | $= 0,$ | $H^1(Y)$ | $= 0,$ | |
| | $H^2(X)$ | $= Z^2,$ | $H^2(Y)$ | $= Z,$ | |
| | $H^3(X)$ | $= 0,$ | $H^3(Y)$ | $= 0,$ | |
| | $H^4(X)$ | $= Z.$ | | | |

Thus:

| | | | |
|---|---|---|---|
| | $H^0(M)$ | $= Z \otimes Z = Z,$ | |
| | $H^1(M)$ | $= 0 \otimes Z + Z \otimes 0 = 0,$ | |
| | $H^2(M)$ | $= (Z^2 \otimes Z) \oplus (0 \otimes 0) \oplus (Z \otimes Z) = Z^3,$ | |
| | $H^3(M)$ | $= (0 \otimes Z) \oplus (Z^2 \otimes 0) \oplus (0 \otimes Z) = 0,$ | |
| | $H^4(M)$ | $= (Z \otimes Z) \oplus (0 \otimes 0) \oplus (Z^2 \otimes Z) = Z^3,$ | |
| | $H^5(M)$ | $= 0 \otimes Z + Z \otimes 0 = 0,$ | |
| | $H^6(M)$ | $= Z \otimes Z = Z.$ | |

So $b_0 = 1, b_1 = 0, b_2 = 3, b_3 = 0, b_4 = 3, b_5 = 0, b_6 = 1$.

After Luttinger surgery, $b_i$ remain the same (since surgery on a torus of codimension 2 does not change Euler characteristic, and Poincaré duality forces the Betti numbers to be symmetric). However, the fundamental group may change. For $M$, $\pi_1(M) = 1$ (since $\pi_1(S^2) = 1$). After surgery, $\pi_1(M_k)$ may become nontrivial if we introduce relations. In fact, if we perform surgery on a null-homologous torus, the fundamental group becomes:

$$\pi_1(M_k) = \langle \mu, \lambda \mid \mu = \lambda^k \rangle,$$

where $\mu$ is the meridian and $\lambda$ is the surgery curve. Since $\mu$ is trivial in $\pi_1(M \setminus \nu(L))$ (because $L$ is null-homologous), we get $\lambda^k = 1$. If $k = 1$, then $\lambda = 1$, so $\pi_1(M_1) = 1$. If $k > 1$, then $\lambda$ is a torsion element of order $k$, so $\pi_1(M_k) = \mathbb{Z}_k$.

Thus, $M_k$ for $k > 1$ are not simply-connected. To get simply-connected manifolds, we need to kill this torsion by additional surgeries or choose $k = 1$.

## *E.2. Example: QISM with Base a Riemann Surface*

Let $B = \Sigma_g$ be a Riemann surface of genus $g$, with symplectic form $\omega_B$ the area form. Take $\mathcal{V} = L \oplus \underline{\mathbb{C}}$, where $L$ is a line bundle with $c_1(L) = d[\omega_B]$, where $[\omega_B]$ is the generator of $H^2(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}$. Then $M = \mathbb{P}(\mathcal{V})$ is a $\mathbb{CP}^1$-bundle over $\Sigma_g$.

We have:

$$c_1(M) = c_1(\Sigma_g) + 2c_1(L) = (2 - 2g)[\omega_B] + 2d[\omega_B] = (2 - 2g + 2d)[\omega_B].$$

For $c_1(M) = 0$, we need $2 - 2g + 2d = 0$, i.e., $d = g - 1$. So if we take $L$ with degree $g - 1$, then $c_1(M) = 0$.

Note that $\dim M = 2 + 2 = 4$, so this gives a symplectic 4-manifold. In fact, $M$ is a ruled surface over $\Sigma_g$. For $g = 1$ (torus), $d = 0$, so $L$ is trivial, and $M = T^2 \times \mathbb{CP}^1$, which has $c_1 = 0$ (a $K3$ surface is not of this form; $K3$ has $b_2 = 22$, while $T^2 \times \mathbb{CP}^1$ has $b_2 = 3$). For $g > 1$, $d > 0$, and $M$ is a nontrivial ruled surface.

These 4-manifolds are symplectic Calabi–Yau surfaces (complex surfaces with $c_1 = 0$ are called $K3$ surfaces or tori, but ruled surfaces have $c_1 \neq 0$ typically; wait, for $g > 1$, $2 - 2g$ is negative, so $c_1(M) = (2 - 2g + 2d)[\omega_B] = (2 - 2g + 2(g - 1))[\omega_B] = 0$, indeed). So for any $g$, if we choose $d = g - 1$, we get a symplectic 4-manifold with $c_1 = 0$. For $g = 2$, this gives a $K3$ surface (since $K3$ has $b_2 = 22$, but our manifold has $b_2 = b_2(\Sigma_2) + 1 = 2 + 1 = 3$, so it's not $K3$; actually $K3$ has Euler characteristic 24, while our manifold has $\chi = \chi(\Sigma_g) \cdot \chi(\mathbb{CP}^1) = (2 - 2g) \cdot 2 = 4 - 4g$, which for $g = 2$ gives $-4$, so not $K3$. So these are not $K3$ surfaces but other symplectic Calabi–Yau 4-manifolds.


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This illustrates that QISMs can also be used to construct symplectic Calabi–Yau manifolds in dimension 4.

# Appendix F. Explicit Examples and Physical Interpretation of Quantum Inner State Manifolds

This appendix provides explicit low-dimensional examples and physical interpretations of Quantum Inner State Manifolds (QISMs). The goal is to complement the abstract constructions developed in the main text by illustrating how QISMs arise concretely and how their geometric features may be interpreted in experimentally relevant settings.

## *F.1. Minimal Example: Two-Level Systems*

The simplest nontrivial QISM arises from a two-level quantum system. In this case, the quantum fiber is

$$\mathrm{CP}^1 \cong S^2,$$

equipped with the Fubini–Study symplectic form. Let $B$ be a smooth classical control manifold, for example a two- or three-dimensional parameter space describing externally tunable fields. A QISM in this setting is a fiber bundle

$$\pi : M \to B,$$

with fiber $\mathrm{CP}^1$ and structure group $(2) \cong SO(3)$.

Physically, points of $B$ correspond to distinct Hamiltonians of a qubit system, while points of the fiber represent pure quantum states modulo phase. Loops in $B$ induce rotations of the Bloch sphere via holonomy, realizing geometric quantum gates. Because these gates depend only on the global geometry of the loop, they are robust against small control imperfections.


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### *F.2. Higher-Dimensional Fibers and Multi-Level Systems*

For an $N$-level quantum system, the fiber becomes $\mathbb{CP}^{N-1}$. In this case, the holonomy group naturally generalizes to $(N)$, allowing for non-Abelian geometric gates. Such systems arise, for example, in:

- multi-level atoms or ions,
- superconducting circuits with higher excited states,
- photonic systems with internal mode structure.

In these settings, the QISM framework organizes the available quantum states into a coherent geometric object, with curvature encoding the structure of admissible fault-tolerant operations.

### *F.3. Physical Meaning of the Base Manifold*

The base manifold $B$ represents the space of classical control parameters. Typical coordinates on $B$ may include:

- external magnetic or electric field strengths,
- coupling constants between subsystems,
- geometric parameters of a device or lattice,
- adiabatic control knobs in experimental protocols.

From this perspective, a QISM unifies classical control theory and quantum state geometry into a single fibered structure. Smooth paths in $B$ correspond to experimentally realizable control sequences, while their associated holonomies encode the resulting quantum operations.

### *F.4. Relation to Berry Phases and Holonomic Gates*

Berry phases arise in QISMs as the Abelian limit of the general holonomy construction. When the relevant eigenspaces are one-dimensional, the holonomy reduces to a phase factor determined by the curvature of the connection. In higher-dimensional eigenspaces, the resulting holonomies are genuinely non-Abelian and implement quantum gates.

This viewpoint clarifies that holonomic quantum computation is not an isolated technique, but a natural consequence of the fiber-bundle structure underlying quantum mechanics when parameter dependence is treated geometrically.


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### *F.5. Outlook and Experimental Relevance*

Although the full realization of QISM-based architectures remains a long-term goal, partial implementations already exist in current experimental platforms. The key insight provided by this appendix is that QISMs should be viewed as *design templates*: they suggest how control spaces and quantum state spaces should be engineered so that robustness emerges from geometry rather than active correction alone.

In this sense, QISMs offer a conceptual bridge between abstract symplectic topology and concrete quantum technologies, reinforcing the central thesis of this paper.

## Appendix G. Ricci Curvature in the Adiabatic Limit

In this appendix we give a detailed derivation of the Ricci curvature of the rescaled metrics associated to a QISM fibration in the adiabatic limit. This justifies the claims made in §5.4 concerning the dominance of the fiber curvature and its cancellation by a suitable scaling of the base metric.

### *G.1. Geometric setup*

Let

$$\pi : (M^{2n}, \Omega) \to (B^{2m}, \omega_B)$$

be a QISM whose fibers are biholomorphic to

$$F = \mathrm{CP}^k, k = n - m - 1,$$

equipped with the Fubini–Study form $\omega_{FS}$. The symplectic form is written in the standard form

$$\Omega = \pi^* \omega_B + \varepsilon \omega_{FS} + \eta,$$


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where $\eta$ is the curvature 2-form of a chosen symplectic connection and $\varepsilon > 0$ is the adiabatic parameter.

We choose an almost complex structure $J$ compatible with $\Omega$ making $\pi$ pseudoholomorphic. The associated Riemannian metric is

$$g_\varepsilon(X, Y) = \Omega(X, JY).$$

The tangent bundle splits as

$$TM = \mathcal{H} \oplus \mathcal{V},$$

where $\mathcal{V} = \ker d\pi$ is the vertical bundle and $\mathcal{H}$ is the horizontal distribution defined by the symplectic connection.

## G.2. Rescaled metrics

Let $g_B$ be a fixed metric on $B$ compatible with $\omega_B$, and let $g_F$ be the Fubini–Study metric on the fibers normalized so that

$$Ric_{g_F} = (k+1)g_F.$$

The induced metric has the block form

$$g_\varepsilon = \pi^* g_B \oplus \varepsilon g_F + O(\varepsilon).$$

To study the adiabatic limit, introduce the rescaled metric

$$\tilde{g}_\varepsilon = g_B \oplus \varepsilon^{-1} g_F.$$


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If $\{e_i\}$ is an orthonormal frame for $(B, g_B)$ and $\{f_\alpha\}$ an orthonormal frame for $(F, g_F)$, then

$$E_i = e_i^H, E_\alpha = \varepsilon^{1/2} f_\alpha$$

is an orthonormal frame for $(M, g_\varepsilon)$.

### G.3. O'Neill tensors

Let $\nabla$ be the Levi–Civita connection of $g_\varepsilon$. Define the O'Neill tensors

$$A_X Y = (\nabla_{X^H} Y^H)^V, T_U V = (\nabla_{U^V} V^V)^H.$$

In our situation:

- The fibers are totally geodesic, hence

$$T \equiv 0.$$

- The tensor $A$ measures the curvature of the connection and satisfies

$$A_X Y = -\frac{1}{2}(\eta(X, Y))^\sharp.$$

Thus $A = O(1)$ as $\varepsilon \to 0$.

### G.4. Curvature decomposition

O'Neill's formulas give the Ricci curvature decomposition:

Horizontal-horizontal:


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$$Ric^M(X, Y) = Ric^B(d\pi X, d\pi Y) - 2\sum_{\alpha} \langle A_X E_\alpha, A_Y E_\alpha \rangle + O(\varepsilon).$$

Vertical-vertical:

$$Ric^M(U, V) = Ric^F(U, V) - \sum_{i} \langle A_{E_i} U, A_{E_i} V \rangle + O(\varepsilon).$$

Mixed terms:

$$Ric^M(X, U) = O(\varepsilon^{1/2}).$$

### G.5. Asymptotic scaling

Since the fiber metric is scaled by $\varepsilon$, we have

$$Ric^F_{g_\varepsilon} = \varepsilon^{-1} Ric_{g_F} = \varepsilon^{-1}(k+1) g_F.$$

Thus, in vertical directions,

$$Ric^M(U, U) = \frac{k+1}{\varepsilon} |U|^2 + O(1).$$

Hence:

> *The Ricci curvature diverges like $+\varepsilon^{-1}$ in the fiber directions.*

In horizontal directions:

$$Ric^M(X, X) = Ric^B(X, X) - \|A_X\|^2 + O(\varepsilon).$$

### *G.6. Cancellation mechanism*

Now scale the base metric:

$$g_B \mapsto \lambda g_B.$$

Then:

$$Ric_B \mapsto \lambda^{-1} Ric_B.$$

Choose:

$$\lambda = \varepsilon^{-1}.$$

Then the horizontal Ricci contributes:

$$Ric^M(X, X) \sim -C\varepsilon^{-1} |X|^2,$$

for suitable negative curvature base.

Thus:

> *The positive fiber Ricci term of order $+\varepsilon^{-1}$ can be cancelled by a negative base Ricci term of order $-\varepsilon^{-1}$.*


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### *G.7. Conclusion*

We have shown:

**Theorem G.1.** *In the adiabatic limit $\varepsilon \to 0$, the Ricci curvature of $g_\varepsilon$ satisfies:*

$$Ric^M = \frac{k+1}{\varepsilon} g_F + Ric_B^{scaled} + O(1).$$

*By scaling the base metric to have sufficiently negative Ricci curvature, the leading positive fiber contribution can be cancelled.*

This completes the justification of the argument in §5.4.

## Appendix H. Deformations of QISM Symplectic Structures

In this appendix we give a detailed proof of the deformation statement used in Section 5.5, namely that symplectic structures compatible with a fixed QISM structure form a finite-dimensional moduli space and that small deformations are controlled by cohomological data together with fiberwise constraints. The proof combines Moser's stability theorem with Hodge theory and the special structure of QISMs.

### *H.1. Setup*

Let $\pi: M^{2n} \to B^{2m}$ be a QISM with fiber $FcongCP^{N-1}$ and let

$$\Omega = \pi^* \omega_B + \varepsilon \omega_{FS} + \eta \qquad (2)$$

be a symplectic form on $M$ compatible with the QISM structure, where:

- $\omega_B$ is a symplectic form on the base $B$,
- $\omega_{FS}$ is the Fubini–Study form on the fibers,
- $\eta$ is a closed $2$-form that vanishes on purely vertical vectors and encodes the coupling to a chosen connection,
- $\varepsilon > 0$ is a constant.

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We consider smooth families of symplectic forms

$$\Omega_t = \Omega + \alpha_t,\ t \in (-\delta, \delta), \tag{3}$$

such that:

1. Each $\Omega_t$ is closed and nondegenerate,

2. Each $\Omega_t$ is compatible with the fixed QISM fibration $\pi$,

3. The restriction to each fiber satisfies

$$\Omega_t|_{F_b} = c(t)\omega_{FS} \tag{4}$$

for some positive function $c(t)$ independent of $b$.

We show that such deformations are classified, up to isotopy through QISM-preserving diffeomorphisms, by a finite-dimensional space of cohomology classes subject to explicit constraints.

### *H.2. Reduction to closed 2-forms*

Since each $\Omega_t$ is symplectic, we may write

$$\frac{d}{dt}\Omega_t = \dot{\Omega}_t = \beta_t, \tag{5}$$

where $\beta_t$ is a closed $2$-form on $M$.

Thus infinitesimal deformations are parameterized by closed $2$-forms. However, not all such deformations preserve the QISM structure.

The compatibility conditions impose:

1. **Fiberwise condition:** For each fiber $F_b$,


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$$\beta_t|_{F_b} = \lambda(t)\omega_{FS} \quad (6)$$

for some scalar $\lambda(t)$.

2. **No vertical–vertical mixing:** The mixed term $\eta_t$ must continue to vanish on purely vertical vectors. Hence $\beta_t$ must lie in the subspace

$$\Omega^2(M) = \pi^*\Omega^2(B) \oplus \Omega^{1,1}_{\text{mix}} \oplus \langle\omega_{FS}\rangle. \quad (7)$$

Thus allowed infinitesimal deformations lie in a finite-rank subbundle of $\Lambda^2 T^* M$.

## *H.3. Application of Moser's theorem*

Suppose that $[\Omega_t] = [\Omega_0]$ in $H^2(M, \mathcal{R})$ for all $t$. Then

$$\beta_t = \frac{d}{dt}\Omega_t = d\sigma_t \quad (8)$$

for some family of 1-forms $\sigma_t$.

Moser's method seeks a time-dependent vector field $X_t$ satisfying

$$\iota_{X_t}\Omega_t = -\sigma_t. \quad (9)$$

Since $\Omega_t$ is nondegenerate, this equation has a unique solution $X_t$. Let $\varphi_t$ be the flow of $X_t$. Then

$$\frac{d}{dt}(\varphi_t^*\Omega_t) = 0, \quad (10)$$


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hence

$$\varphi_t^* \Omega_t = \Omega_0. \quad (11)$$

Thus any cohomologically trivial deformation is symplectomorphic to the original one.

To preserve the fibration, we require $X_t$ to be fiber-preserving, i.e. tangent to the horizontal distribution. This is achieved by choosing $\sigma_t$ with no vertical component, which is possible precisely because $\beta_t$ satisfies the QISM compatibility constraints.

Hence:

**Theorem H.1** (QISM Moser stability). *Any smooth family $\Omega_t$ of QISM-compatible symplectic forms with fixed cohomology class is related by a fiber-preserving isotopy of $M$.*

## *H.4. Hodge-theoretic parametrization of deformations*

Fix a Riemannian metric on $M$ compatible with the QISM splitting. By Hodge theory, every closed $2$-form admits a unique decomposition

$$\beta = \beta^{\text{harm}} + d\gamma. \quad (12)$$

Modulo Moser isotopy, only the harmonic part matters. Hence the true deformation space is a subspace of $H^2(M, \mathcal{R})$.

The QISM constraints impose linear conditions:

1. The restriction of $[\beta]$ to the fiber must lie in the span of $[\omega_{FS}]$.

2. The class must lie in the image of

$$H^2(B, \mathcal{R}) \oplus \mathcal{R}[\omega_{FS}] \oplus H^1(B) \otimes H^1(F) \to H^2(M, \mathcal{R}). \quad (13)$$

Therefore the allowed deformation space is a finite-dimensional vector subspace


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$$\mathcal{D}_{\text{QISM}} \subset H^2(M, \mathcal{R}). \tag{14}$$

### H.5. Dimension count

Let $b_k(X)$ denote Betti numbers. Since $H^2(F, \mathcal{R}) = \mathcal{R}[\omega_{FS}]$, we obtain:

$$\dim \mathcal{D}_{\text{QISM}} \leq b_2(B) + b_1(B) b_1(F) + 1. \tag{15}$$

In particular, the deformation space is finite-dimensional.

### H.6. Nondegeneracy condition

Finally, nondegeneracy is an open condition. Hence any sufficiently small element of $\mathcal{D}_{\text{QISM}}$ yields a genuine QISM symplectic structure.

### H.7. Conclusion

We have shown:

**Theorem H.2.** *The moduli space of QISM-compatible symplectic structures near a fixed QISM structure is a finite-dimensional manifold locally modeled on a subspace of $H^2(M, \mathcal{R})$. Any cohomologically trivial deformation is induced by a fiber-preserving isotopy.*

This completes the detailed proof of the deformation statement used in Section 5.5.

## Appendix I. Lie–Algebraic Controllability of QISM Holonomies

We give a complementary controllability proof using geometric control theory.

Let $\{X_i\}$ be control vector fields on the base manifold $B$ generating horizontal motion. Their induced Hamiltonians on the quantum fiber are

$$H_i = F(X_i, \cdot) \in \mathfrak{u}(N).$$


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Define the dynamical Lie algebra

$$\mathfrak{g} = Lie\{H_i\}.$$

**Theorem I.1.** *If* $\mathfrak{g} = \mathfrak{u}(N)$*, then the reachable holonomy group is dense in* $U(N)$*.*

This follows from the Chow–Rashevskii theorem applied to the lifted horizontal distribution on the QISM. Since Appendix D shows that $F(X, Y)$ spans $\mathfrak{u}(N)$, it follows that $\mathfrak{g} = \mathfrak{u}(N)$ and full controllability is achieved.

## Notes

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### *Conflicts of Interest*

The author declares that there are no competing interests or conflicts of interest.


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