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Research Article

Evaluations of Complete Bell Polynomials for Quadruple-Indexed Bernoulli Numbers in Terms of Generalized Bernoulli Polynomials

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In the paper, in light of the generating function of the complete Bell polynomials and other techniques, the author discovers several evaluations of complete Bell polynomials for quadruple-indexed Bernoulli numbers in terms of the Stirling numbers and generalized Bernoulli polynomials. These findings are related to multiple zeta functions, correct an error in previous work, and generalize some known results.

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1. Introduction

1.1. Hoffman's formula

In 1992, Hoffman [1] introduced and investigated the multiple harmonic series (now known as the multiple zeta functions)

$$\zeta^*(z_1, z_2, \dots, z_k) = \sum_{n_1 \ge n_2 \ge \cdots \ge n_k \ge 1} \frac{1}{n_1^{z_1} n_2^{z_2} \cdots n_k^{z_k}}$$

and

$$\zeta(z_1, z_2, \dots, z_k) = \sum_{n_1 > n_2 > \cdots > n_k \ge 1} \frac{1}{n_1^{z_1} n_2^{z_2} \cdots n_k^{z_k}}$$

for $k \in \mathbb{N} = \{1, 2, \dots\}$ and $\Re(z_k) > 1$. This is a natural generalization of the classical Riemann zeta function

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$$\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}, \Re(z)>1.$$

For more information about $\zeta(z)$ and its recent properties, please refer to [2] and the review article [3]. The multiple zeta function $\zeta(z_1,z_2,\dots,z_k)$ plays an important role in quantum physics and in knot theory, please refer to the monographs [4][5].

In order to derive

$$\zeta(\underbrace{2,2,\dots,2}_{k})=\frac{\pi^{2k}}{(2k+1)!}, k\in\mathrm{N}$$

in [1], Hoffman established in [1] the following elegant formula.

Theorem 1.1 ([1]). *For* $k\in\mathrm{N}$*,*

$$\sum_{\substack{\sum_{i=1}^k i\ell_i=k\\ \ell_1,\ell_2,\dots,\ell_k\in\mathrm{N}_0}}\prod_{i=1}^{k}\frac{1}{\ell_i!}\left[\frac{B_{2i}}{(2i)(2i)!}\right]^{\ell_i}=\frac{1}{2^{2k}(2k+1)!},$$

where B_{2i} *denotes the Bernoulli numbers generated* [2] *by*

$$\frac{z}{e^z-1}=\sum_{i=0}^{\infty}B_i\frac{z^i}{i!}=1-\frac{1}{2}z+\sum_{k=1}^{\infty}B_{2i}\frac{z^{2i}}{(2i)!}, |z|<2\pi.$$

1.2. Genčev's extensions of Hoffman's formula

In 2024, three decades later, among other things, Genčev [6] extended the formula (1.2) in Theorem 1.1 via specific zeta-like series in terms of the Bernoulli numbers B_{2k}, the Euler numbers E_{2k}, and the Catalan numbers C_k as follows.

Theorem 1.2 ([6]). *For* $k\in\mathrm{N}_0=\{0,1,2,\dots\}$*, we have*

$$\sum_{\substack{\sum_{j=1}^k j\ell_j=k\\ \ell_1,\ell_2,\dots,\ell_k\in\mathrm{N}_0}}\frac{1}{\prod_{j=1}^{k}\ell_j!}\prod_{j=1}^{k}\left[\frac{\epsilon B_{2j}}{(2j)(2j)!}\right]^{\ell_j}=\begin{cases}\frac{1}{2k+1}\frac{1}{(4k)!!}, & \epsilon=1;\\ \frac{2-2^{2k}}{(4k)!!}B_{2k}, & \epsilon=-1.\end{cases}$$

Theorem 1.3 ([6]). *For* $k\in\mathrm{N}_0$*, we have*

$$\sum_{\substack{\sum_{j=1}^k j\ell_j=k\\ \ell_1,\ell_2,\dots,\ell_k\in\mathrm{N}_0}}\frac{1}{\prod_{j=1}^{k}\ell_j!}\prod_{j=1}^{k}\left[\left(4^j-1\right)\frac{\epsilon B_{2j}}{(2j)(2j)!}\right]^{\ell_j}=\begin{cases}\frac{1}{(4k)!!}, & \epsilon=1;\\ \frac{1}{(4k)!!}E_{2k}, & \epsilon=-1,\end{cases}$$

where E_{2k} denotes the Euler numbers generated [7] *by*

$$\frac{2}{e^z+e^{-z}}=\sum_{k=0}^{\infty}E_k\frac{z^k}{k!}=\sum_{k=0}^{\infty}E_{2k}\frac{z^{2k}}{(2k)!},\ |z|<\frac{\pi}{2}.$$

Theorem 1.4 ([6]). *For $k\in\mathbb{N}$, we have*

$$\sum_{\substack{\sum_{j=1}^{k}j\ell_j=k\\ \ell_1,\ell_2,\dots,\ell_k\in\mathbb{N}_0}}\frac{1}{\prod_{j=1}^{k}\ell_j!}\prod_{j=1}^{k}\left[\frac{\epsilon}{2j}\binom{2j}{j}\right]^{\ell_j}=\begin{cases}C_k, & \epsilon=1;\\ -C_{k-1}, & \epsilon=-1,\end{cases}$$

where $C_k=\frac{1}{k+1}\binom{2k}{k}$ for $k\in\mathbb{N}_0$ denotes the Catalan numbers generated [8][9][10] *by*

$$\frac{2}{1+\sqrt{1-4x}}=\sum_{k=0}^{\infty}C_kx^k=1+x+2x^2+5x^3+\cdots,\ |x|<\frac{1}{4}.$$

1.3. He–Qi's reformulations of Genčev's formulas

In [11], the Bell polynomials of the second kind, also known as the partial Bell polynomials, are defined by

$$\frac{B_{k,j}(a_1,a_2,\dots,a_{k-j+1})}{k!}=\sum_{\substack{\sum_{i=1}^{k-j+1}i\ell_i=k\\ \sum_{i=1}^{k-j+1}\ell_i=j\\ \ell_1,\ell_2,\dots,\ell_{k-j+1}\in\mathbb{N}_0}}\prod_{i=1}^{k-j+1}\left[\frac{1}{\ell_i!}\left(\frac{a_i}{i!}\right)^{\ell_i}\right]$$

for $j,k\in\mathbb{N}_0$ satisfying $k\ge j$, with the special cases $B_{0,0}(a_1)=1$ and

$$B_{k,0}(a_1,a_2,\dots,a_{k+1})=0,k\in\mathbb{N}.$$

In [11] and [12], the complete Bell polynomials, denoted by $B_k(a_1,a_2,\dots,a_k)$, are defined by $B_0=1$ and

$$B_k(a_1,a_2,\dots,a_k)=\sum_{j=1}^{k}B_{k,j}(a_1,a_2,\dots,a_{k-j+1}),k\in\mathbb{N}.$$

For more information on the partial and complete Bell polynomials, please refer to [11]. Directly from the relation (1.8), it follows that

$$\frac{B_k(a_1,a_2,\dots,a_k)}{k!}=\sum_{\substack{\sum_{i=1}^{k}i\ell_i=k\\ \ell_1,\ell_2,\dots,\ell_k\in\mathbb{N}_0}}\prod_{i=1}^{k}\left[\frac{1}{\ell_i!}\left(\frac{a_i}{i!}\right)^{\ell_i}\right],k\in\mathbb{N}.$$

The complete Bell polynomials $B_k(a_1,a_2,\dots,a_k)$ have an exponential generating function

$$\exp\left(\sum_{k=1}^{\infty}a_k\frac{z^k}{k!}\right)=\sum_{k=0}^{\infty}B_k(a_1,a_2,\dots,a_k)\frac{z^k}{k!},$$

which can be found in [11] and [12].

In 2024, observing and comparing the expressions on the left-hand sides of the identities (1.2) through (1.6) with the expression on the right-hand side of (1.9), He and Qi [13] reformulated and alternatively proved the formulas (1.4), (1.5), and (1.6) in Theorems 1.2, 1.3, and 1.4 in terms of the complete Bell polynomials B_k as follows.

Theorem 1.5 ([13]). *For* $k \in \mathbb{N}_0$ *and* $\epsilon = \pm 1$, *we have*

$$B_k\left(\frac{\epsilon}{2}\frac{B_2}{2!}, \frac{\epsilon}{2}\frac{B_4}{4!}, \dots, \frac{\epsilon}{2}\frac{(k-1)!B_{2k}}{(2k)!}\right) = \begin{cases} \frac{1}{2k+1}\frac{k!}{(4k)!!}, & \epsilon = 1; \\ \frac{k!\left(2-2^{2k}\right)}{(4k)!!}B_{2k}, & \epsilon = -1. \end{cases}$$

Theorem 1.6 ([13]). *For* $k \in \mathbb{N}_0$ *and* $\epsilon = \pm 1$, *we have*

$$B_k\left(\frac{3\epsilon}{2}\frac{B_2}{2!}, \frac{15\epsilon}{2}\frac{B_4}{4!}, \dots, \frac{\left(2^{2k}-1\right)\epsilon}{2}\frac{(k-1)!B_{2k}}{(2k)!}\right) = \begin{cases} \frac{k!}{(4k)!!}, & \epsilon = 1; \\ \frac{k!}{(4k)!!}E_{2k}, & \epsilon = -1. \end{cases}$$

Theorem 1.7 ([13]). *For* $k \in \mathbb{N}$ *and* $\epsilon = \pm 1$, *we have*

$$B_k\left(\epsilon, 3\epsilon, 20\epsilon, \dots, \frac{\epsilon(k-1)!}{2}\binom{2k}{k}\right) = \begin{cases} k!C_k, & \epsilon = 1; \\ -k!C_{k-1}, & \epsilon = -1. \end{cases}$$

1.4. He–Qi's generalizations of Genčev's formulas

In light of the generating function (1.10) and other intricate techniques, He and Qi [13] not only alternatively proved Theorems 1.5 through 1.7, but also generalized Theorems 1.5 through 1.7 from the specific cases $\epsilon = \pm 1$ to the general case $\epsilon \in \mathbb{R}$. We now recite these generalizations as follows.

Theorem 1.8 ([13]). *For* $k \in \mathbb{N}$ *and* $\epsilon \in \mathbb{R}$, *we have*

$$B_k\left(\epsilon\frac{B_2}{2!}, \epsilon\frac{B_4}{4!}, \dots, \epsilon\frac{(k-1)!B_{2k}}{(2k)!}\right) = \frac{k!}{(2k)!}\sum_{\ell=1}^{2k}\frac{(-2\epsilon)_\ell}{\ell!}\sum_{j=1}^{\ell}(-1)^j\binom{\ell}{j}\frac{T(2k+j,j)}{\binom{2k+j}{j}},$$

where the rising factorial $(z)_\ell$ *for* $z \in \mathbb{C}$ *is defined* [7] *by*

$$(z)_\ell = \prod_{\ell=0}^{\ell-1}(z+\ell) = \begin{cases} z(z+1)\cdots(z+\ell-1), & \ell \ge 1 \\ 1, & \ell = 0 \end{cases}$$

and the central factorial numbers of the second kind $T(p,q)$ *for* $p, q \in \mathrm{N}_0$ *can be computed* [14] *by*

$$T(p,q) = \frac{1}{q!}\sum_{k=0}^{q}(-1)^k\binom{q}{k}\left(\frac{q}{2}-k\right)^p, p, q \in \mathrm{N}_0$$

with $T(q,q)=1$ *for* $q \in \mathrm{N}_0$ *and* $T(p,0)=0$ *for* $p \in \mathrm{N}$.

Theorem 1.9 ([13]). *For* $k \in \mathrm{N}_0$ *and* $\epsilon \in \mathrm{R}$, *we have*

$$\begin{aligned} &B_k\left(3\epsilon\frac{B_2}{2!}, 15\epsilon\frac{B_4}{4!}, \dots, \left(2^{2k}-1\right)(k-1)!\epsilon\frac{B_{2k}}{(2k)!}\right) \\ &= \frac{k!}{2^{2k}(2k)!}\sum_{\ell=0}^{2k}\frac{(-2\epsilon)_\ell}{\ell!}\sum_{m=0}^{\ell}\frac{(-1)^m}{2^m}\binom{\ell}{m}\sum_{q=0}^{m}\binom{m}{q}(2q-m)^{2k}, \end{aligned}$$

where 0^0 *is understood as* 1.

Theorem 1.10 ([13]). *For* $k \in \mathrm{N}_0$ *and* $\epsilon \in \mathrm{R}$, *we have*

$$B_k\left(2\epsilon, 6\epsilon, 40\epsilon, \dots, \epsilon(k-1)!\binom{2k}{k}\right) = \sum_{\ell=0}^{k}(2\epsilon)_{k-\ell}\binom{k+\ell-1}{2\ell}2^\ell(2\ell-1)!!.$$

1.5. Xu's concise forms for He–Qi's formulas

In early 2026, using the concept of the complete Bell polynomials $B_k(a_1, a_2, \dots, a_k)$ and the equation (1.10) again, Xu [15] presented concise forms for the formulas (1.11), (1.12), and (1.13) in Theorems 1.8, 1.9, and 1.10. We recite his results as follows.

Theorem 1.11 ([15]). *For* $k \in \mathrm{N}_0$ *and* $\epsilon \in \mathrm{R}$, *we have*

$$B_k\left(\epsilon\frac{B_2}{2!}, \epsilon\frac{B_4}{4!}, \dots, \epsilon\frac{(k-1)!B_{2k}}{(2k)!}\right) = \frac{k!}{(2k)!}B_{2k}^{(-2\epsilon)}(-\epsilon),$$

where the generalized Bernoulli polynomials $B_{2k}^{(\sigma)}$ *for* $\sigma \in \mathrm{C}$ *are generated* [2] *by*

$$\left(\frac{z}{e^z-1}\right)^\sigma e^{xz} = \sum_{k=0}^{\infty}B_k^{(\sigma)}(x)\frac{z^k}{k!}, \ |z| < 2\pi.$$

Remark 1.1. It is easy to see that the generalized Bernoulli polynomials $B_k^{(\sigma)}(x)$ generated by (1.15) satisfy

$$B_k^{(1)}(0) = B_k \quad \text{and} \quad B_k^{(1)}(x) = B_k(x),$$

where the Bernoulli numbers B_k are generated by (1.3) and $B_k(x)$ denotes the Bernoulli polynomials.

Theorem 1.12 ([15]). *For $k \in \mathrm{N}_0$ and $\epsilon \in \mathrm{R}$, we have*

$$B_k\left(3\epsilon\frac{B_2}{2!}, 15\epsilon\frac{B_4}{4!}, \dots, \left(2^{2k}-1\right)\epsilon\frac{(k-1)!B_{2k}}{(2k)!}\right) = \frac{k!}{(2k)!}E_{2k}^{(-2\epsilon)}(-\epsilon),$$

where the generalized Euler polynomials $E_{2k}^{(\sigma)}$ for $\sigma \in \mathrm{C}$ are generated [2] *by*

$$\left(\frac{2}{e^z+1}\right)^{\sigma}e^{xz} = \sum_{k=0}^{\infty}E_k^{(\sigma)}(x)\frac{z^k}{k!}, \ |z| < \pi.$$

Theorem 1.13 ([15]). *For $k \in \mathrm{N}$ and $\epsilon \in \mathrm{R}$, we have*

$$B_k\left(2\epsilon, 6\epsilon, 40\epsilon, \dots, \epsilon(k-1)!\binom{2k}{k}\right) = 2\epsilon(k-1)!\binom{2k-1+2\epsilon}{k-1},$$

where the generalized binomial coefficient $\binom{z}{k}$ for $z \in \mathrm{C}$ and $k \in \mathrm{N}_0$ is defined by

$$\binom{z}{k} = \begin{cases} \frac{(-1)^k(-z)_k}{k!}, & k \geq 0; \\ 0, & k < 0. \end{cases}$$

1.6. Alternative proofs of Xu's concise formulas

In the recent paper [16], by establishing the formulas

$$B_{2m,k}\left(0, \frac{1}{3}, 0, \frac{1}{5}, \dots, \frac{1+(-1)^{2m-k+1}}{2}\frac{1}{2m-k+2}\right)$$

$$= (-1)^k\frac{2^{2m}}{k!}\left[\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\frac{T(2m+\ell,\ell)}{\binom{2m+\ell}{\ell}}\right]$$

and

$$B_{2m+1,k}\left(0, \frac{1}{3}, 0, \frac{1}{5}, \dots, \frac{1+(-1)^{2m-k+2}}{2}\frac{1}{2m-k+3}\right) = 0$$

for $m \geq k \in \mathrm{N}_0$, where $B_{m,k}$ is defined by (1.7), verifying the identity

$$\sum_{m=0}^{j}\frac{(-1)^m}{2^m}\binom{j}{m}\sum_{q=0}^{m}\binom{m}{q}(2q-m)^k = \frac{(-1)^j}{2^j}\sum_{\ell=0}^{2j}(-1)^{\ell}\binom{2j}{\ell}(j-\ell)^k$$

for $k \in \mathbb{N}$ and $j \in \mathbb{N}_0$, and making use of the well-known Faà di Bruno formula, the author proved the formulas

$$B_{2k}^{(\epsilon)}\left(\frac{\epsilon}{2}\right) = \sum_{\ell=1}^{2k} \frac{(\epsilon)_\ell}{\ell!} \sum_{j=1}^{\ell} (-1)^j \binom{\ell}{j} \frac{T(2k+j, j)}{\binom{2k+j}{j}},$$

$$E_{2k}^{(\epsilon)}\left(\frac{\epsilon}{2}\right) = \frac{1}{4^k} \sum_{\ell=0}^{2k} \frac{(\epsilon)_\ell}{\ell!} \sum_{m=0}^{\ell} \frac{(-1)^m}{2^m} \binom{\ell}{m} \sum_{q=0}^{m} \binom{m}{q} (2q-m)^{2k},$$

and

$$\epsilon(k-1)! \binom{2k-1+\epsilon}{k-1} = \sum_{\ell=0}^{k} (\epsilon)_{k-\ell} \binom{k+\ell-1}{2\ell} 2^\ell (2\ell-1)!!$$

for $k \in \mathbb{N}$ and $\epsilon \in \mathbb{R}$, and then deduced Theorems 1.11 through 1.13.

1.7. Concise and elegant proofs of Xu's concise formulas

In [17], the author presented concise and elegant proofs of Theorems 1.5 through 1.7 and Theorems 1.11 through 1.13. In particular, the proofs of Theorems 1.11 through 1.13 are especially streamlined and elegant.

1.8. Two aims of this paper

The aims of this paper can be stated as follows.

1.8.1. First aim

Lemma 3.2 in [6] reads that the identity

$$\prod_{i=0}^{k} (x-i) = \sum_{i=0}^{k} c_{i,k} \prod_{j=0}^{i} (2x-j)$$

is true for $k \in \mathbb{N}_0$ and $x \in \mathbb{R}$, where

$$c_{i,k} = \frac{(-1)^{i+k}}{2^{2k-i+1} i!} \prod_{j=i}^{i+k-1} (2k-j), \quad 0 \le i \le k \in \mathbb{N}_0.$$

Applying (1.16), Genčev established in [6] that the formula

$$\sum_{n_1 > n_2 > \cdots > n_k \ge 1} \prod_{j=1}^{k} \frac{1}{n_j(n_j+1)} = \frac{1}{k+1} \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \binom{2k-2j}{k} \frac{\pi^{2j}}{(2j)!}$$

is true for $k \in \mathbb{N}$, where $\lfloor \lambda \rfloor$ denotes the floor function whose value is the largest integer less than or equal to $\lambda \in \mathbb{R}$.

In this paper, we will give a more concise and more meaningful form for the quantity $c_{i,k}$ in (1.17) in terms of the product of the Stirling numbers of the first and second kinds $s(n,k)$ and $S(n,k)$, which can be analytically generated by [(see [18])]

$$\left[\frac{\ln(1+x)}{x}\right]^n = \sum_{k=0}^{\infty} \frac{s(k+n,n)}{\binom{k+n}{n}} \frac{x^k}{k!}, \quad |x| < 1$$

and [(see [11])]

$$\left(\frac{e^x-1}{x}\right)^n = \sum_{k=0}^{\infty} \frac{S(k+n,n)}{\binom{k+n}{n}} \frac{x^k}{k!}, \quad |x| < \infty$$

for $n \in \mathbb{N}_0$, respectively.

Making use of the new form for the quantity $c_{i,k}$ in (1.17), we will rewrite the formula (1.18) and give a new formula for $\sum_{n_1>n_2>\cdots>n_k\ge 1}\prod_{j=1}^{k}\frac{1}{n_j(n_j+1)}$.

1.8.2. Second aim

In [6], Genčev pointed out in a concluding remark that Hoffman's formula (1.2) in Theorem 1.1 can be generalized in light of the known formulas for the multiple zeta functions

$$\zeta(\underbrace{2n,\dots,2n}_{k}) = (-1)^{k(n+1)}(2\pi)^{2nk} \sum_{\substack{\sum_{i=1}^{k} i\ell_i = k \\ \ell_1,\ell_2,\dots,\ell_k \in \mathbb{N}_0}} \prod_{i=1}^{k} \frac{1}{\ell_i!}\left[\frac{1}{2i}\frac{B_{2in}}{(2in)!}\right]^{\ell_i}$$

and

$$\zeta^*(\underbrace{2n,\dots,2n}_{k}) = (-1)^{nk}(2\pi)^{2nk} \sum_{\substack{\sum_{i=1}^{k} i\ell_i = k \\ \ell_1,\ell_2,\dots,\ell_k \in \mathbb{N}_0}} \prod_{i=1}^{k} \frac{(-1)^{\ell_i}}{\ell_i!}\left[\frac{1}{2i}\frac{B_{2in}}{(2in)!}\right]^{\ell_i}$$

for $n, k \in \mathbb{N}$; see [19]. For more closed-form evaluations of these functions, please refer to the papers [19] [20]. Concretely, Genčev claimed that

$$\sum_{\substack{\sum_{i=1}^{k} i\ell_i = k \\ \ell_1,\ell_2,\dots,\ell_k \in \mathbb{N}_0}} \prod_{i=1}^{k} \frac{1}{\ell_i!}\left[\frac{B_{4i}}{(2i)(4i)!}\right]^{\ell_i} = (-1)^k \frac{2^{2k+1}}{(4k+2)!}, \quad k \in \mathbb{N}.$$

The formula (1.23) is of a slightly different structure from the forms in Theorem 1.1 and Theorem 1.2 for $\epsilon = 1$.

As done in the paper [13], applying the definition in (1.9) for the complete Bell polynomials $B_k(a_1, a_2, \ldots, a_k)$, we can reformulate the formulas (1.21), (1.22), and (1.23) as

$$\zeta(\underbrace{2n, \ldots, 2n}_{k}) = \frac{(-1)^{k(n+1)}(2\pi)^{2nk}}{k!} B_k\left(\frac{0!}{2}\frac{B_{2n}}{(2n)!}, \frac{1!}{2}\frac{B_{4n}}{(4n)!}, \ldots, \frac{(k-1)!}{2}\frac{B_{2kn}}{(2kn)!}\right),$$

$$\zeta^*(\underbrace{2n, \ldots, 2n}_{k}) = \frac{(-1)^{nk}(2\pi)^{2nk}}{k!} B_k\left(-\frac{0!}{2}\frac{B_{2n}}{(2n)!}, -\frac{1!}{2}\frac{B_{4n}}{(4n)!}, \ldots, -\frac{(k-1)!}{2}\frac{B_{2kn}}{(2kn)!}\right),$$

and

$$B_k\left(\frac{0!}{2}\frac{B_4}{4!}, \frac{1!}{2}\frac{B_8}{8!}, \ldots, \frac{(k-1)!}{2}\frac{B_{4k}}{(4k)!}\right) = (-1)^k \frac{2^{2k+1}k!}{(4k+2)!}$$

for $k, n \in \mathbb{N}$.

From (1.8), it follows that

$$B_1(a_1) = B_{1,1}(a_1) = a_1.$$

Therefore, the formula (1.26) for $k = 1$ becomes

$$B_1\left(\frac{0!}{2}\frac{B_4}{4!}\right) = -\frac{2^3}{6!} \quad \Longleftrightarrow \quad \frac{0!}{2}\frac{B_4}{4!} = -\frac{2^3}{6!} \quad \Longleftrightarrow \quad -\frac{1}{1440} = -\frac{1}{90}.$$

This contradiction shows that the formulas (1.23) and (1.26) are incorrect.

The second aim of this paper is to correct the formulas (1.23) and (1.26) and generalize them to more general cases.

2. Achievement of the first aim

We now start out to achieve the first aim of this paper.

Theorem 2.1. *For* $0 \le i \le k \in \mathbb{N}_0$, *the sequence* $c_{i,k}$ *in (1.17) can be alternatively expressed as*

$$c_{i,k} = \sum_{j=i}^{k} \frac{s(k+1, j+1)S(j+1, i+1)}{2^{j+1}},$$

where $s(k,i)$ *and* $S(k,i)$ *for* $0\le i\le k\in\mathbb{N}_0$ *denote the Stirling numbers of the first and second kinds generated by (1.19) and (1.20).*

Proof. It is known that the Stirling numbers of the first kind $s(k,i)$ for $k\ge i\in\mathbb{N}_0$ can also be generalized by

$$\langle x\rangle_k=\sum_{i=0}^{k}s(k,i)x^i,\ k\in\mathbb{N}_0,$$

where the falling factorial $\langle z\rangle_n$ for $z\in\mathbb{C}$ and $n\in\mathbb{N}_0$ is defined [11] by

$$\langle z\rangle_n=\prod_{k=0}^{n-1}(z-k)=\begin{cases}z(z-1)\cdots(z-n+1), & n\in\mathbb{N};\\ 1, & n=0.\end{cases}$$

For details on (2.2), please refer to [2], [7], and [11]. Accordingly, due to $s(i,0)=0$ for $i\in\mathbb{N}$, we obtain

$$\langle x\rangle_{k+1}=\prod_{i=0}^{k}(x-i)=\sum_{i=0}^{k+1}s(k+1,i)x^i=\sum_{i=0}^{k}s(k+1,i+1)x^{i+1},\ k\in\mathbb{N}_0,$$
$$\langle 2x\rangle_{i+1}=\prod_{j=0}^{i}(2x-j)=\sum_{j=0}^{i+1}s(i+1,j)(2x)^j=\sum_{j=0}^{i}s(i+1,j+1)(2x)^{j+1},\ i\in\mathbb{N}_0,$$

and, by interchanging the order of summations,

$$\begin{aligned}\sum_{i=0}^{k}c_{i,k}\prod_{j=0}^{i}(2x-j)&=\sum_{i=0}^{k}c_{i,k}\sum_{j=0}^{i}s(i+1,j+1)(2x)^{j+1}\\&=\sum_{j=0}^{k}\sum_{i=j}^{k}c_{i,k}s(i+1,j+1)(2x)^{j+1}\\&=\sum_{i=0}^{k}\left[\sum_{\ell=i}^{k}c_{\ell,k}s(\ell+1,i+1)\right](2x)^{i+1}.\end{aligned}$$

Consequently, equating the coefficients of x^i in the equation

$$\sum_{i=0}^{k}s(k+1,i+1)x^{i+1}=\sum_{i=0}^{k}\left[\sum_{\ell=i}^{k}c_{\ell,k}s(\ell+1,i+1)\right](2x)^{i+1}$$

results in

$$\sum_{\ell=i}^{k}c_{\ell,k}s(\ell+1,i+1)=\frac{s(k+1,i+1)}{2^{i+1}},\ 0\le i\le k\in\mathbb{N}_0.$$

These equations can be written as a matrix equation

$$\begin{pmatrix} s(1,1) & s(2,1) & s(3,1) & \cdots & s(k+1,1) \\ 0 & s(2,2) & s(3,2) & \cdots & s(k+1,2) \\ 0 & 0 & s(3,3) & \cdots & s(k+1,3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s(k+1,k+1) \end{pmatrix} \begin{pmatrix} c_{0,k} \\ c_{1,k} \\ c_{2,k} \\ \vdots \\ c_{k,k} \end{pmatrix} = \begin{pmatrix} \frac{s(k+1,1)}{2^1} \\ \frac{s(k+1,2)}{2^2} \\ \frac{s(k+1,3)}{2^3} \\ \vdots \\ \frac{s(k+1,k+1)}{2^{k+1}} \end{pmatrix}.$$

Therefore, since $s(k,k) = 1$ for $k \in \mathbb{N}_0$, we arrive at

$$\begin{pmatrix} c_{0,k} \\ c_{1,k} \\ c_{2,k} \\ \vdots \\ c_{k,k} \end{pmatrix} = \begin{pmatrix} s(1,1) & s(2,1) & s(3,1) & \cdots & s(k+1,1) \\ 0 & s(2,2) & s(3,2) & \cdots & s(k+1,2) \\ 0 & 0 & s(3,3) & \cdots & s(k+1,3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s(k+1,k+1) \end{pmatrix}^{-1} \begin{pmatrix} \frac{s(k+1,1)}{2^1} \\ \frac{s(k+1,2)}{2^2} \\ \frac{s(k+1,3)}{2^3} \\ \vdots \\ \frac{s(k+1,k+1)}{2^{k+1}} \end{pmatrix}.$$

From the relation

$$\sum_{j=0}^{n} s(n,j)S(j,k) = \binom{0}{n-k} \quad \text{or} \quad \sum_{j=0}^{n} S(n,j)s(j,k) = \binom{0}{n-k}$$

for $n \geq k \in \mathbb{N}_0$, see [21] and [22], we deduce

$$\begin{pmatrix} s(1,1) & s(2,1) & s(3,1) & \cdots & s(k+1,1) \\ 0 & s(2,2) & s(3,2) & \cdots & s(k+1,2) \\ 0 & 0 & s(3,3) & \cdots & s(k+1,3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s(k+1,k+1) \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} S(1,1) & S(2,1) & S(3,1) & \cdots & S(k+1,1) \\ 0 & S(2,2) & S(3,2) & \cdots & S(k+1,2) \\ 0 & 0 & S(3,3) & \cdots & S(k+1,3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & S(k+1,k+1) \end{pmatrix}.$$

Consequently, we obtain

$$\begin{pmatrix} c_{0,k} \\ c_{1,k} \\ c_{2,k} \\ \vdots \\ c_{k,k} \end{pmatrix} = \begin{pmatrix} S(1,1) & S(2,1) & S(3,1) & \cdots & S(k+1,1) \\ 0 & S(2,2) & S(3,2) & \cdots & S(k+1,2) \\ 0 & 0 & S(3,3) & \cdots & S(k+1,3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & S(k+1,k+1) \end{pmatrix} \begin{pmatrix} \frac{s(k+1,1)}{2^1} \\ \frac{s(k+1,2)}{2^2} \\ \frac{s(k+1,3)}{2^3} \\ \vdots \\ \frac{s(k+1,k+1)}{2^{k+1}} \end{pmatrix}.$$

This means that

$$c_{i,k} = \sum_{j=i}^{k} S(j+1,i+1)\frac{s(k+1,j+1)}{2^{j+1}} = \sum_{j=i}^{k} \frac{s(k+1,j+1)S(j+1,i+1)}{2^{j+1}}$$

for $0 \le i \le k$. The proof of Theorem 2.1 is complete. □

Making use of the new form (2.1), we now start off to rewrite the formula (1.18) and give a new formula for the quantity $\sum_{n_1>n_2>\cdots>n_k\ge 1}\prod_{j=1}^{k}\frac{1}{n_j(n_j+1)}$.

Theorem 2.2. *For* $n \in \mathrm{N}$, *we have*

$$\sum_{n_1>n_2>\cdots>n_k\ge 1}\prod_{j=1}^{k}\frac{1}{n_j(n_j+1)}$$
$$= (-1)^k\frac{2^{2k}}{(k+1)!}\sum_{i=0}^{\lfloor k/2\rfloor}(-1)^i\left(\frac{\pi}{2}\right)^{2i}\sum_{j=2i}^{k}\frac{s(k+1,j+1)S(j+1,2i+1)}{2^j}.$$

Proof. Let

$$\varphi_k(a_n) = \sum_{n_1>n_2>\cdots>n_k\ge 1}\frac{1}{a_{n_1}a_{n_2}\cdots a_{n_k}}, k \in \mathrm{N}.$$

In [23], it was proved that

$$\prod_{n=1}^{\infty}\left(1-\frac{x}{a_n}\right) = 1 + \sum_{k=1}^{\infty}(-x)^k\varphi_k(a_n).$$

For $a_n = n(n+1)$, it is not difficult to see that

$$\begin{aligned}\prod_{n=1}^{\infty}\left[1-\frac{x}{n(n+1)}\right]&=\prod_{n=1}^{\infty}\frac{\left(n+\frac{1}{2}-\sqrt{x+\frac{1}{4}}\right)\left(n+\frac{1}{2}+\sqrt{x+\frac{1}{4}}\right)}{n(n+1)}\\&=\lim_{m\to\infty}\prod_{n=1}^{m}\frac{\left(n+\frac{1}{2}-\sqrt{x+\frac{1}{4}}\right)\left(n+\frac{1}{2}+\sqrt{x+\frac{1}{4}}\right)}{n(n+1)}\\&=\lim_{m\to\infty}\frac{\dfrac{\Gamma\left(m+\frac{3}{2}-\sqrt{x+\frac{1}{4}}\right)}{\Gamma\left(\frac{3}{2}-\sqrt{x+\frac{1}{4}}\right)}\dfrac{\Gamma\left(m+\frac{3}{2}+\sqrt{x+\frac{1}{4}}\right)}{\Gamma\left(\frac{3}{2}+\sqrt{x+\frac{1}{4}}\right)}}{\Gamma(m+1)\Gamma(m+2)}\\&=\frac{1}{\Gamma\left(\frac{3}{2}-\sqrt{x+\frac{1}{4}}\right)\Gamma\left(\frac{3}{2}+\sqrt{x+\frac{1}{4}}\right)}\\&\quad\times\lim_{m\to\infty}\frac{\Gamma\left(m+\frac{3}{2}-\sqrt{x+\frac{1}{4}}\right)\Gamma\left(m+\frac{3}{2}+\sqrt{x+\frac{1}{4}}\right)}{\Gamma(m+1)\Gamma(m+2)}\end{aligned}$$

for $|x|<\frac{1}{4}$.

In [7], we find the asymptotic relation

$$z^{b-a}\frac{\Gamma(z+a)}{\Gamma(z+b)}\sim 1+\frac{(a-b)(a+b-1)}{2z}+\frac{1}{12}\binom{a-b}{2}\frac{3(a+b-1)^2-a+b-1}{z^2}+\cdots$$

as $z\to\infty$ along any curve joint $z=0$ and $z=\infty$, providing $z\neq -a,-a-1,-a-2,\ldots$ and $z\neq -b,-b-1,-b-2,\ldots$. This implies that

$$\begin{aligned}&\lim_{m\to\infty}\frac{\Gamma\left(m+\frac{3}{2}-\sqrt{x+\frac{1}{4}}\right)\Gamma\left(m+\frac{3}{2}+\sqrt{x+\frac{1}{4}}\right)}{\Gamma(m+1)\Gamma(m+2)}\\&=\lim_{m\to\infty}\left[\frac{\Gamma\left(m+\frac{3}{2}-\sqrt{x+\frac{1}{4}}\right)}{\Gamma(m+1)}m^{-\frac{1}{2}+\sqrt{x+\frac{1}{4}}}\right]\\&\quad\times\lim_{m\to\infty}\left[\frac{\Gamma\left(m+\frac{3}{2}+\sqrt{x+\frac{1}{4}}\right)}{\Gamma(m+2)}m^{\frac{1}{2}-\sqrt{x+\frac{1}{4}}}\right]\\&=1.\end{aligned}$$

Therefore, we arrive at

$$\begin{aligned}
\prod_{n=1}^{\infty}\left[1-\frac{x}{n(n+1)}\right] &= \frac{1}{\Gamma\left(\frac{3}{2}-\sqrt{x+\frac{1}{4}}\right)\Gamma\left(\frac{3}{2}+\sqrt{x+\frac{1}{4}}\right)} \\
&= -\frac{1}{x\pi}\sin\left(\frac{\pi}{2}+\frac{\pi}{2}\sqrt{1+4x}\right) \\
&= -\frac{1}{x\pi}\cos\frac{\pi\sqrt{1+4x}}{2} \\
&= \frac{1}{x\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(2n)!}\left(\frac{\pi}{2}\right)^{2n}(1+4x)^n \\
&= \frac{1}{x\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(2n)!}\left(\frac{\pi}{2}\right)^{2n}\sum_{k=0}^{n}\binom{n}{k}(4x)^k \\
&= \frac{1}{x\pi}\sum_{k=0}^{\infty}(4x)^k\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(2n)!}\left(\frac{\pi}{2}\right)^{2n}\binom{n}{k} \\
&= \frac{1}{x\pi}\sum_{k=1}^{\infty}\frac{(4x)^k}{k!}\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(2n)!}\left(\frac{\pi}{2}\right)^{2n}\prod_{i=0}^{k-1}(n-i),
\end{aligned}$$

where we used in the second equality the recursive formula $\Gamma(z+1)=z\Gamma(z)$ and the reflection formula

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(z\pi)}, z\notin Z$$

in [2] and considered in the last equality the facts that an empty product is understood to be 1 and that

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\pi}{2}\right)^{2n}=\cos\frac{\pi}{2}=0.$$

Further applying the identity (1.16) yields

$$\begin{aligned}
\prod_{n=1}^{\infty}\left[1-\frac{x}{n(n+1)}\right] &= \frac{1}{x\pi}\sum_{k=1}^{\infty}\frac{(4x)^k}{k!}\sum_{i=0}^{k-1}c_{i,k-1}\left[\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(2n)!}\left(\frac{\pi}{2}\right)^{2n}\prod_{j=0}^{i}(2n-j)\right] \\
&\triangleq \frac{1}{x\pi}\sum_{k=1}^{\infty}\frac{(4x)^k}{k!}\sum_{i=0}^{k-1}c_{i,k-1}R_i,
\end{aligned}$$

where

$$\begin{aligned}R_i &= \sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(2n)!}\left(\frac{\pi}{2}\right)^{2n}\prod_{j=0}^{i}(2n-j)\\ &= \sum_{n=\lceil (i+1)/2\rceil}^{\infty}\frac{(-1)^{n-1}}{(2n-i-1)!}\left(\frac{\pi}{2}\right)^{2n}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^{n+\lceil (i+1)/2\rceil-1}}{[2(n+\lceil (i+1)/2\rceil)-i-1]!}\left(\frac{\pi}{2}\right)^{2(n+\lceil (i+1)/2\rceil)}\\ &= (-1)^{\lceil (i+1)/2\rceil-1}\left(\frac{\pi}{2}\right)^{2\lceil (i+1)/2\rceil-\delta_i}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(2n+\delta_i)!}\left(\frac{\pi}{2}\right)^{2n+\delta_i},\end{aligned}$$

the notation $\lceil x\rceil$ stands for the ceiling function which gives the smallest integer not less than x, and

$$\delta_i = 2\left\lceil\frac{i+1}{2}\right\rceil - i - 1 = \frac{1+(-1)^i}{2},\ i\in\mathbb{N}_0.$$

Accordingly, we have

$$\begin{aligned}R_i &= (-1)^{\lceil (i+1)/2\rceil-1}\left(\frac{\pi}{2}\right)^{i+1}\begin{cases}\sin\frac{\pi}{2}, & i \text{ is even}\\ \cos\frac{\pi}{2}, & i \text{ is odd}\end{cases}\\ &= (-1)^{\lceil (i+1)/2\rceil-1}\left(\frac{\pi}{2}\right)^{i+1}\delta_i.\end{aligned}$$

As a result, in view of (2.1), it follows that

$$\begin{aligned}\prod_{n=1}^{\infty}\left[1-\frac{x}{n(n+1)}\right] &= \frac{1}{x\pi}\sum_{k=1}^{\infty}\frac{(4x)^k}{k!}\sum_{i=0}^{k-1}c_{i,k-1}(-1)^{\lceil (i+1)/2\rceil-1}\left(\frac{\pi}{2}\right)^{i+1}\delta_i\\ &= \frac{1}{x\pi}\sum_{k=1}^{\infty}\frac{(4x)^k}{k!}\sum_{i=0}^{\lfloor (k-1)/2\rfloor}c_{2i,k-1}(-1)^i\left(\frac{\pi}{2}\right)^{2i+1}\\ &= \frac{1}{x\pi}\sum_{k=1}^{\infty}\frac{(4x)^k}{k!}\sum_{i=0}^{\lfloor (k-1)/2\rfloor}(-1)^i\left(\frac{\pi}{2}\right)^{2i+1}\sum_{j=2i}^{k-1}\frac{s(k,j+1)S(j+1,2i+1)}{2^{j+1}}\\ &= \sum_{k=0}^{\infty}\left[\sum_{i=0}^{\lfloor k/2\rfloor}(-1)^i\left(\frac{\pi}{2}\right)^{2i}\sum_{j=2i}^{k}\frac{s(k+1,j+1)S(j+1,2i+1)}{2^j}\right]\frac{2^{2k}x^k}{(k+1)!}.\end{aligned}$$

Comparing this result with (2.5) for $a_n = n(n+1)$ leads to

$$\varphi_k(n(n+1)) = (-1)^k\frac{2^{2k}}{(k+1)!}\sum_{i=0}^{\lfloor k/2\rfloor}(-1)^i\left(\frac{\pi}{2}\right)^{2i}\sum_{j=2i}^{k}\frac{s(k+1,j+1)S(j+1,2i+1)}{2^j}$$

for $k\in\mathbb{N}$. By the definition (2.4), it follows that

$$\begin{aligned}\varphi_k(n(n+1)) &= \sum_{n_1>n_2>\cdots>n_k\ge 1}\frac{1}{n_1(n_1+1)n_2(n_2+1)\cdots n_k(n_k+1)}\\ &= \sum_{n_1>n_2>\cdots>n_k\ge 1}\prod_{j=1}^{k}\frac{1}{n_j(n_j+1)},\ k\in\mathrm{N}.\end{aligned}$$

Consequently, the equality (2.3) is thus proved. The proof of Theorem 2.2 is thus complete. $\square$

By establishing a series expansion of the function $\cos\left(a\sqrt{1+bx}\right)$ for $a,b\in\mathrm{R}$, we derive a new formula for the quantity $\sum_{n_1>n_2>\cdots>n_k\ge 1}\prod_{j=1}^{k}\frac{1}{n_j(n_j+1)}$.

Theorem 2.3. *For* $k\in\mathrm{N}$, *we have*

$$\begin{aligned}&\sum_{n_1>n_2>\cdots>n_k\ge 1}\prod_{j=1}^{k}\frac{1}{n_j(n_j+1)}\\ &=\frac{1}{\pi}\frac{2^{k+1}}{(k+1)!}\sum_{j=1}^{k+1}(-1)^j\left(\frac{\pi}{2}\right)^j[2(k-j)+1]!!\binom{2k-j+1}{2(k-j+1)}\cos\frac{(j+1)\pi}{2}.\end{aligned}$$

Proof. The famous Faà di Bruno formula reads that

$$[f(h(z))]^{(k)}=\sum_{j=0}^{k}f^{(j)}(h(z))B_{k,j}\left(h'(z),h''(z),\dots,h^{(k-j+1)}(z)\right)$$

for $k\in\mathrm{N}_0$, where f is a k-time differentiable function and h is a $(k+1)$-time differentiable function; see [24] or [11]. Accordingly, we obtain

$$\begin{aligned}\left[\cos\left(a\sqrt{1+bx}\right)\right]^{(n)} &= \sum_{k=0}^{n}(\cos u)^{(k)}B_{n,k}\Bigg(ab\left\langle\frac{1}{2}\right\rangle_1(1+bx)^{1/2-1},\\ &\quad ab^2\left\langle\frac{1}{2}\right\rangle_2(1+bx)^{1/2-2},\dots,ab^{n-k+1}\left\langle\frac{1}{2}\right\rangle_{n-k+1}(1+bx)^{1/2-(k-i+1)}\Bigg)\\ &\to\sum_{k=0}^{n}\cos\left(a+\frac{k\pi}{2}\right)B_{n,k}\left(ab\left\langle\frac{1}{2}\right\rangle_1,ab^2\left\langle\frac{1}{2}\right\rangle_2,\dots,ab^{n-k+1}\left\langle\frac{1}{2}\right\rangle_{n-k+1}\right),x\to 0\\ &=\sum_{k=0}^{n}\cos\left(a+\frac{k\pi}{2}\right)a^kb^nB_{n,k}\left(\left\langle\frac{1}{2}\right\rangle_1,\left\langle\frac{1}{2}\right\rangle_2,\dots,\left\langle\frac{1}{2}\right\rangle_{n-k+1}\right)\\ &=(-1)^n\left(\frac{b}{2}\right)^n\sum_{k=0}^{n}(-1)^k\cos\left(a+\frac{k\pi}{2}\right)a^k[2(n-k)-1]!!\binom{2n-k-1}{2(n-k)}\end{aligned}$$

for $n\in\mathrm{N}_0$, where $u=u(x)=a\sqrt{1+bx}\to a$ as $x\to 0$ and we used the identity

$$B_{n,k}\left(\alpha\beta x_1,\alpha\beta^2z_2,\dots,\alpha\beta^{n-k+1}z_{n-k+1}\right)=\alpha^k\beta^nB_{n,k}(z_1,z_2,\dots,z_{n-k+1})$$

for $n\ge k\ge 0$ and $\alpha,\beta\in\mathrm{C}$, in [11] and [12], and the formula

$$B_{n,k}\left(\left\langle\frac{1}{2}\right\rangle_1,\left\langle\frac{1}{2}\right\rangle_2,\dots,\left\langle\frac{1}{2}\right\rangle_{n-k+1}\right)=(-1)^{n+k}[2(n-k)-1]!!\left(\frac{1}{2}\right)^n\binom{2n-k-1}{2(n-k)}$$

in [25] and [26]. As a result, we acquire a series expansion

$$\cos\left(a\sqrt{1+bx}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left[\sum_{k=0}^{n}(-1)^k a^k[2(n-k)-1]!!\binom{2n-k-1}{2(n-k)}\cos\left(a+\frac{k\pi}{2}\right)\right]\left(\frac{bx}{2}\right)^n$$

for $a,b\in\mathbb{R}$. Taking $a=\frac{\pi}{2}$ and $b=4$ in (2.10) leads to

$$\cos\frac{\pi\sqrt{1+4x}}{2}$$

$$=\sum_{n=1}^{\infty}(-1)^n\left[\sum_{k=1}^{n}(-1)^k\left(\frac{\pi}{2}\right)^k[2(n-k)-1]!!\binom{2n-k-1}{2(n-k)}\cos\frac{(k+1)\pi}{2}\right]\frac{(2x)^n}{n!}.$$

Substituting this series expansion into (2.6) yields

$$\prod_{n=1}^{\infty}\left[1-\frac{x}{n(n+1)}\right]=1+\frac{1}{\pi}\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n+1}(-1)^k\cos\frac{(k+1)\pi}{2}\left(\frac{\pi}{2}\right)^k\right.$$
$$\left.\times\ [2(n-k)+1]!!\binom{2n-k+1}{2(n-k+1)}\right]\frac{2^{n+1}(-x)^n}{(n+1)!}.$$

Comparing this with (2.5) for $a_n=n(n+1)$, we arrive at

$$\varphi_k(n(n+1))=\frac{1}{\pi}\frac{2^{k+1}}{(k+1)!}\sum_{j=1}^{k+1}(-1)^j\cos\frac{(j+1)\pi}{2}\left(\frac{\pi}{2}\right)^j[2(k-j)+1]!!\binom{2k-j+1}{2(k-j+1)}.$$

Combining this with (2.7) results in the formula (2.8). The proof of Theorem 2.3 is complete. □

3. Achievement of the second aim

In order to correct the formulas (1.23) and (1.26), we first present a lemma.

Lemma 3.1. *For* $n\in\mathbb{N}_0$, *we have*

$$\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!(2n-2k+1)!}=\frac{2^{n+1}}{(2n+2)!}\sin\frac{(n+1)\pi}{2}.$$

Proof. Consider the series expansions

$$\sin x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}\quad\text{and}\quad\sinh x=\sum_{m=0}^{\infty}\frac{x^{2m+1}}{(2m+1)!}.$$

Multiplying the two series gives

$$\sin x \sinh x = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!(2n-2k+1)!}\right]x^{2n+2}.$$

On the other hand, utilizing the exponential forms

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i} \quad \text{and} \quad \sinh x = \frac{e^x-e^{-x}}{2},$$

where $i=\sqrt{-1}$ is the imaginary unit, we deduce

$$\begin{aligned}\sin x \sinh x &= \frac{1}{4i}\left[e^{(1+i)x}-e^{(-1+i)x}-e^{(1-i)x}+e^{(-1-i)x}\right]\\ &= \frac{1}{4i}\sum_{m=0}^{\infty}\left[(1+i)^m-(-1+i)^m-(1-i)^m+(-1-i)^m\right]\frac{x^m}{m!}.\end{aligned}$$

Since $\sin x \sinh x$ is an even function, only even powers $m=2n+2$ for $n\in\mathbb{N}_0$ survive. The coefficient of x^{2n+2} in $\sin x \sinh x$ is

$$\begin{aligned}&\frac{1}{4i}\frac{(1+i)^{2n+2}-(-1+i)^{2n+2}-(1-i)^{2n+2}+(-1-i)^{2n+2}}{(2n+2)!}\\ &= \frac{1}{2i}\frac{(1+i)^{2n+2}-(1-i)^{2n+2}}{(2n+2)!}\\ &= \frac{1}{2i}\frac{\left(\sqrt{2}e^{i\pi/4}\right)^{2n+2}-\left(\sqrt{2}e^{-i\pi/4}\right)^{2n+2}}{(2n+2)!}\\ &= \frac{2^{n+1}}{2i}\frac{e^{i\pi(n+1)/2}-e^{-i\pi(n+1)/2}}{(2n+2)!}\\ &= \frac{2^{n+1}}{(2n+2)!}\sin\frac{(n+1)\pi}{2},\end{aligned}$$

where we used $1\pm i=\sqrt{2}e^{\pm i\pi/4}$. Consequently, we conclude

$$\sin x \sinh x = \sum_{n=0}^{\infty}\frac{2^{n+1}}{(2n+2)!}\sin\frac{(n+1)\pi}{2}x^{2n+2}.$$

Combining (3.2) and (3.3) leads to the formula (3.1). The proof of Lemma 3.1 is thus complete. □

We now in a position to give and prove a corrected version of the formulas (1.23) and (1.26).

Theorem 3.1. *For* $k\in\mathbb{N}$, *we have*

$$B_k\left(\frac{0!}{2}\frac{B_4}{4!},\frac{1!}{2}\frac{B_8}{8!},\dots,\frac{(k-1)!}{2}\frac{B_{4k}}{(4k)!}\right)=(-1)^k\frac{k!}{2^{2k-1}(4k+2)!}$$

and

$$\zeta(\underbrace{4,\ldots,4}_{k})=\frac{2^{2k+1}\pi^{4k}}{(4k+2)!}.$$

Proof. Making use of the generating function (1.10) yields

$$\exp\left(\frac{1}{2}\sum_{k=1}^{\infty}\frac{B_{4k}}{(4k)!}\frac{z^k}{k}\right)=\sum_{k=0}^{\infty}B_k\left(\frac{0!}{2}\frac{B_4}{4!},\frac{1!}{2}\frac{B_8}{8!},\ldots,\frac{(k-1)!}{2}\frac{B_{4k}}{(4k)!}\right)\frac{z^k}{k!}.$$

Therefore, it suffices to show that

$$\exp\left(\frac{1}{2}\sum_{k=1}^{\infty}\frac{B_{4k}}{(4k)!}\frac{z^k}{k}\right)=\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k-1}(4k+2)!}z^k.$$

From the series expansion

$$\ln\frac{\sin x}{x}=-\sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k}\frac{(2x)^{2k}}{(2k)!},\ |x|<\pi,$$

see [13] and [27], and the relation

$$\sinh x=\frac{\sin(ix)}{i}$$

in [27], it follows that

$$\ln\frac{\sinh x}{x}=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{|B_{2k}|}{2k}\frac{(2x)^{2k}}{(2k)!},\ |x|<\pi.$$

Accordingly, we obtain

$$\begin{aligned}\ln\left(\frac{\sin x}{x}\frac{\sinh x}{x}\right)&=\ln\frac{\sin x}{x}+\ln\frac{\sinh x}{x}\\&=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{|B_{2k}|}{2k}\frac{(2x)^{2k}}{(2k)!}-\sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k}\frac{(2x)^{2k}}{(2k)!}\\&=-\sum_{k=1}^{\infty}[(-1)^k+1]\frac{|B_{2k}|}{2k}\frac{(2x)^{2k}}{(2k)!}\\&=\sum_{k=1}^{\infty}\frac{B_{4k}}{(4k)!}\frac{(2x)^{4k}}{2k},\ |x|<\pi.\end{aligned}$$

Replacing x by $\frac{\sqrt[4]{z}}{2}$ gives

$$\ln\left(\frac{\sin\frac{\sqrt[4]{z}}{2}}{\frac{\sqrt[4]{z}}{2}}\frac{\sinh\frac{\sqrt[4]{z}}{2}}{\frac{\sqrt[4]{z}}{2}}\right)=\frac{1}{2}\sum_{k=1}^{\infty}\frac{B_{4k}}{(4k)!}\frac{z^k}{k}.$$

Equivalently,

$$\exp\left(\frac{1}{2}\sum_{k=1}^{\infty}\frac{B_{4k}}{(4k)!}\frac{z^k}{k}\right)=\frac{\sin\frac{\sqrt[4]{z}}{2}}{\frac{\sqrt[4]{z}}{2}}\frac{\sinh\frac{\sqrt[4]{z}}{2}}{\frac{\sqrt[4]{z}}{2}}.$$

On the other hand, since

$$\frac{\sin\frac{\sqrt[4]{z}}{2}}{\frac{\sqrt[4]{z}}{2}}=\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{\sqrt[4]{z}}{2}\right)^{2n}}{(2n+1)!}=\sum_{n=0}^{\infty}(-1)^n\frac{\left(\sqrt{z}\right)^n}{2^{2n}(2n+1)!}$$

and

$$\frac{\sinh\frac{\sqrt[4]{z}}{2}}{\frac{\sqrt[4]{z}}{2}}=\sum_{n=0}^{\infty}\frac{\left(\frac{\sqrt[4]{z}}{2}\right)^{2n}}{(2n+1)!}=\sum_{n=0}^{\infty}\frac{\left(\sqrt{z}\right)^n}{2^{2n}(2n+1)!},$$

straightforward computation results in

$$\begin{aligned}\frac{\sin\frac{\sqrt[4]{z}}{2}}{\frac{\sqrt[4]{z}}{2}}\frac{\sinh\frac{\sqrt[4]{z}}{2}}{\frac{\sqrt[4]{z}}{2}}&=\left[\sum_{n=0}^{\infty}(-1)^n\frac{\left(\sqrt{z}\right)^n}{2^{2n}(2n+1)!}\right]\left[\sum_{n=0}^{\infty}\frac{\left(\sqrt{z}\right)^n}{2^{2n}(2n+1)!}\right]\\&=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{(-1)^k}{2^{2k}(2k+1)!}\frac{1}{2^{2(n-k)}(2n-2k+1)!}\right]\left(\sqrt{z}\right)^n\\&=\sum_{n=0}^{\infty}\frac{1}{2^{2n}}\left[\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)!(2n-2k+1)!}\right]\left(\sqrt{z}\right)^n.\end{aligned}$$

Further employing the formula (3.1) in Lemma 3.1 reveals

$$\frac{\sin\frac{\sqrt[4]{z}}{2}}{\frac{\sqrt[4]{z}}{2}}\,\frac{\sinh\frac{\sqrt[4]{z}}{2}}{\frac{\sqrt[4]{z}}{2}} = \sum_{n=0}^{\infty}\frac{1}{2^{4n}}\left[\sum_{k=0}^{2n}\frac{(-1)^k}{(2k+1)!(4n-2k+1)!}\right]\left(\sqrt{z}\right)^{2n}$$

$$= \sum_{n=0}^{\infty}\frac{1}{2^{4n}}(-1)^n\frac{2^{2n+1}}{(4n+2)!}z^n$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{2n-1}(4n+2)!}z^n.$$

Combining this with (3.8) leads to (3.6). Consequently, the formula (3.4) follows.

Substituting the formula (3.4) into (1.24) and simplifying lead to (3.5). The proof of Theorem 3.1 is complete. □

In order to generalize Theorem 3.1, we need the following lemma.

Lemma 3.2. *For $k \in \mathrm{N}_0$ and $\epsilon \in \mathrm{C}$, we have*

$$\sum_{j=0}^{2k+1}(-1)^j\frac{B_{2j}^{(2\epsilon)}(\epsilon)}{(2j)!}\,\frac{B_{4k-2j+2}^{(2\epsilon)}(\epsilon)}{(4k-2j+2)!} = 0.$$

Proof. From the generating function (1.15), it follows that

$$\left(\frac{z}{e^z-1}\right)^{2\epsilon}e^{\epsilon z} = \sum_{k=0}^{\infty}B_k^{(2\epsilon)}(\epsilon)\frac{z^k}{k!},\ |z| < 2\pi.$$

Let

$$F(z) = \left(\frac{z}{e^z-1}\right)^{2\epsilon}e^{\epsilon z}.$$

It is easy to verify that the generating function $F(z)$ is even, that is, $F(-z) = F(z)$. Thus, the equation (3.10) becomes

$$F(z) = \left(\frac{z}{e^z-1}\right)^{2\epsilon}e^{\epsilon z} = \sum_{k=0}^{\infty}B_{2k}^{(2\epsilon)}(\epsilon)\frac{z^{2k}}{(2k)!}$$

and this implies that

$$B_{2k+1}^{(2\epsilon)}(\epsilon) = 0, k \in \mathrm{N}_0,$$

which was concluded in [17].

Straightforward computation gives

$$\begin{aligned} F(zi)F(z) &= \left[\sum_{k=0}^{\infty}(-1)^k B_{2k}^{(2\epsilon)}(\epsilon)\frac{z^{2k}}{(2k)!}\right]\left[\sum_{k=0}^{\infty}B_{2k}^{(2\epsilon)}(\epsilon)\frac{z^{2k}}{(2k)!}\right] \\ &= \sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}(-1)^j\frac{B_{2j}^{(2\epsilon)}(\epsilon)}{(2j)!}\frac{B_{2k-2j}^{(2\epsilon)}(\epsilon)}{(2k-2j)!}\right]z^{2k} \end{aligned}$$

and

$$\begin{aligned} F(z)F(zi) &= \left[\sum_{k=0}^{\infty}B_{2k}^{(2\epsilon)}(\epsilon)\frac{z^{2k}}{(2k)!}\right]\left[\sum_{k=0}^{\infty}(-1)^k B_{2k}^{(2\epsilon)}(\epsilon)\frac{z^{2k}}{(2k)!}\right] \\ &= \sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}\frac{B_{2j}^{(2\epsilon)}(\epsilon)}{(2j)!}(-1)^{k-j}\frac{B_{2k-2j}^{(2\epsilon)}(\epsilon)}{(2k-2j)!}\right]z^{2k} \\ &= \sum_{k=0}^{\infty}\left[(-1)^k\sum_{j=0}^{k}(-1)^j\frac{B_{2j}^{(2\epsilon)}(\epsilon)}{(2j)!}\frac{B_{2k-2j}^{(2\epsilon)}(\epsilon)}{(2k-2j)!}\right]z^{2k}. \end{aligned}$$

Since $F(zi)F(z) = F(z)F(zi)$, equating the coefficient of z^{2k}, we derive

$$\sum_{j=0}^{k}(-1)^j\frac{B_{2j}^{(2\epsilon)}(\epsilon)}{(2j)!}\frac{B_{2k-2j}^{(2\epsilon)}(\epsilon)}{(2k-2j)!} = (-1)^k\sum_{j=0}^{k}(-1)^j\frac{B_{2j}^{(2\epsilon)}(\epsilon)}{(2j)!}\frac{B_{2k-2j}^{(2\epsilon)}(\epsilon)}{(2k-2j)!}, \quad k \in \mathbb{N}_0.$$

Replacing k by $2k+1$ yields

$$\sum_{j=0}^{2k+1}(-1)^j\frac{B_{2j}^{(2\epsilon)}(\epsilon)}{(2j)!}\frac{B_{4k-2j+2}^{(2\epsilon)}(\epsilon)}{(4k-2j+2)!} = -\sum_{j=0}^{2k+1}(-1)^j\frac{B_{2j}^{(2\epsilon)}(\epsilon)}{(2j)!}\frac{B_{4k-2j+2}^{(2\epsilon)}(\epsilon)}{(4k-2j+2)!}, \quad k \in \mathbb{N}_0.$$

The identity (3.9) is thus proved. The proof of Lemma 3.2 is complete. □

We now in a position to generalize Theorem 3.1 by the ideas, methods, and techniques in the papers[13][15][16][17].

Theorem 3.2. *For* $k \in \mathbb{N}$ *and* $\epsilon \in \mathbb{R}$, *we have*

$$B_k\left(\epsilon 0!\frac{B_4}{4!}, \epsilon 1!\frac{B_8}{8!}, \dots, \epsilon(k-1)!\frac{B_{4k}}{(4k)!}\right) = k!\sum_{j=0}^{2k}(-1)^j\frac{B_{2j}^{(-2\epsilon)}(-\epsilon)}{(2j)!}\frac{B_{2(2k-j)}^{(-2\epsilon)}(-\epsilon)}{[2(2k-j)]!}.$$

For $k \in \mathbb{N}$, *the multiple zeta functions satisfy*

$$\zeta(\underbrace{4,\dots,4}_{k}) = (-1)^{3k}(2\pi)^{4k}\sum_{j=0}^{2k}(-1)^j\frac{B_{2j}^{(-1)}\left(-\frac{1}{2}\right)}{(2j)!}\frac{B_{2(2k-j)}^{(-1)}\left(-\frac{1}{2}\right)}{[2(2k-j)]!}$$

and

$$\zeta^*(\underbrace{4,\dots,4}_{k}) = (2\pi)^{4k}\sum_{j=0}^{2k}(-1)^j\frac{B_{2j}\left(\frac{1}{2}\right)}{(2j)!}\frac{B_{2(2k-j)}\left(\frac{1}{2}\right)}{[2(2k-j)]!},$$

where the generalized Bernoulli polynomials $B_{2k}^{(\sigma)}(x)$ *for* $\sigma \in \mathrm{C}$ *and* $k \in \mathrm{N}_0$ *are generated by (1.15).*

Proof. Making use of the generating function (1.10) yields

$$\exp\left(\epsilon\sum_{k=1}^{\infty}\frac{B_{4k}}{(4k)!}\frac{z^k}{k}\right) = \sum_{k=0}^{\infty}B_k\left(\epsilon 0!\frac{B_4}{4!}, \epsilon 1!\frac{B_8}{8!}, \dots, \epsilon(k-1)!\frac{B_{4k}}{(4k)!}\right)\frac{z^k}{k!}.$$

From (3.8), it follows that

$$\exp\left(\epsilon\sum_{k=1}^{\infty}\frac{B_{4k}}{(4k)!}\frac{z^k}{k}\right) = \left[\exp\left(\frac{1}{2}\sum_{k=1}^{\infty}\frac{B_{4k}}{(4k)!}\frac{z^k}{k}\right)\right]^{2\epsilon} = \left(\frac{\sin\frac{\sqrt[4]{z}}{2}}{\frac{\sqrt[4]{z}}{2}}\frac{\sinh\frac{\sqrt[4]{z}}{2}}{\frac{\sqrt[4]{z}}{2}}\right)^{2\epsilon}.$$

In [17], we established the series expansion

$$\left(\frac{\sin x}{x}\right)^r = \sum_{k=0}^{\infty}(-1)^k B_{2k}^{(-r)}\left(-\frac{r}{2}\right)\frac{(2x)^{2k}}{(2k)!}, r \in \mathrm{R}.$$

Utilizing the relation (3.7) and replacing x by ix in (3.16) results in

$$\left(\frac{\sinh x}{x}\right)^r = \sum_{k=0}^{\infty}B_{2k}^{(-r)}\left(-\frac{r}{2}\right)\frac{(2x)^{2k}}{(2k)!}, r \in \mathrm{R}.$$

Multiplying (3.16) and (3.17) reveals

$$\begin{aligned}\left(\frac{\sin x}{x}\frac{\sinh x}{x}\right)^r &= \left[\sum_{k=0}^{\infty}(-1)^k B_{2k}^{(-r)}\left(-\frac{r}{2}\right)\frac{(2x)^{2k}}{(2k)!}\right]\left[\sum_{k=0}^{\infty}B_{2k}^{(-r)}\left(-\frac{r}{2}\right)\frac{(2x)^{2k}}{(2k)!}\right]\\ &= \sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}(-1)^j\frac{B_{2j}^{(-r)}\left(-\frac{r}{2}\right)}{(2j)!}\frac{B_{2(k-j)}^{(-r)}\left(-\frac{r}{2}\right)}{[2(k-j)]!}\right](2x)^{2k}.\end{aligned}$$

Further replacing x by $\frac{\sqrt[4]{z}}{2}$ and r by 2ϵ leads to

$$\left(\frac{\sin\frac{\sqrt[4]{z}}{2}}{\frac{\sqrt[4]{z}}{2}}\,\frac{\sinh\frac{\sqrt[4]{z}}{2}}{\frac{\sqrt[4]{z}}{2}}\right)^{2\epsilon} = \sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}(-1)^j\frac{B_{2j}^{(-2\epsilon)}(-\epsilon)}{(2j)!}\,\frac{B_{2(k-j)}^{(-2\epsilon)}(-\epsilon)}{[2(k-j)]!}\right]z^{k/2}$$

$$= \sum_{k=0}^{\infty}\left[\sum_{j=0}^{2k}(-1)^j\frac{B_{2j}^{(-2\epsilon)}(-\epsilon)}{(2j)!}\,\frac{B_{2(2k-j)}^{(-2\epsilon)}(-\epsilon)}{[2(2k-j)]!}\right]z^{k},$$

where we used the identity (3.9) in Lemma 3.2. Combining (3.18) with (3.15) yields

$$\exp\left(\epsilon\sum_{k=1}^{\infty}\frac{B_{4k}}{(4k)!}\,\frac{z^k}{k}\right) = \sum_{k=0}^{\infty}\left[\sum_{j=0}^{2k}(-1)^j\frac{B_{2j}^{(-2\epsilon)}(-\epsilon)}{(2j)!}\,\frac{B_{2(2k-j)}^{(-2\epsilon)}(-\epsilon)}{[2(2k-j)]!}\right]z^{k}.$$

Comparing this with (3.14) produces the formula (3.11).

Substituting (3.11) for $\epsilon = \pm\frac{1}{2}$ into (1.24) and (1.25) respectively and simplifying yield (3.12) and (3.13). The proof of Theorem 3.2 is complete. □

4. Remarks

In this section, from our main results and their proofs, we deduce several values for generalized Bernoulli polynomials as by-products.

Remark 4.1. Comparing (3.12) with (3.5) results in

$$\sum_{j=0}^{2k}(-1)^j\frac{B_{2j}^{(-1)}\left(-\frac{1}{2}\right)}{(2j)!}\,\frac{B_{2(2k-j)}^{(-1)}\left(-\frac{1}{2}\right)}{[2(2k-j)]!} = \frac{(-1)^{3k}}{2^{2k-1}(4k+2)!},\ k \in \mathbb{N}_0.$$

Remark 4.2. Combining the formulas (1.1) and (1.14) for $\epsilon = \frac{1}{2}$ with (1.24) for $n = 1$ gives

$$B_{2k}^{(-1)}\left(-\frac{1}{2}\right) = \frac{1}{2^{2k}(2k+1)},\ k \in \mathbb{N}_0.$$

Remark 4.3. In [6], we find the formula

$$\zeta^{*}(\underbrace{2,\dots,2}_{k}) = (-1)^{k-1}\left(2^{2k}-2\right)\frac{B_{2k}}{(2k)!}\pi^{2k},\ k \in \mathbb{N}.$$

Combining this with (1.14) for $\epsilon = -\frac{1}{2}$ and (1.25) for $n = 1$ produces

$$B_{2k}\left(\frac{1}{2}\right) = \frac{2-2^{2k}}{2^{2k}}B_{2k},\ k \in \mathbb{N}_0,$$

which recovers [7].

5. Conclusions

In this paper, we presented the following main results.

1. In the first section, we reviewed and surveyed the history, especially those results obtained in the papers[13][15][16][17].

2. We reformulated the multiple zeta values

$$\zeta(\underbrace{2n, \dots, 2n}_{k}) \quad \text{and} \quad \zeta^{*}(\underbrace{2n, \dots, 2n}_{k})$$

in (1.21) and (1.22) as (1.24) and (1.25) in terms of the complete Bell polynomials

$$B_k\left(\pm\frac{0!}{2}\frac{B_{2n}}{(2n)!}, \pm\frac{1!}{2}\frac{B_{4n}}{(4n)!}, \dots, \pm\frac{(k-1)!}{2}\frac{B_{2kn}}{(2kn)!}\right), k, n \in \mathrm{N}.$$

3. We derived the meaningful expression (2.1) in terms of the Stirling numbers of the first kind and second kinds $s(n, k)$ and $S(n, k)$, and rewrote the formula (1.18) as (2.3) in Theorem 2.2.

4. Using the Faà di Bruno formula (2.9) and two properties of the partial Bell polynomials $B_{n,k}$, we established a series expansion (2.10). By this nice series expansion, we derived in Theorem 2.3 a new formula for the quantity

$$\sum_{n_1 > n_2 > \cdots > n_k \geq 1} \prod_{j=1}^{k} \frac{1}{n_j(n_j+1)}.$$

5. By establishing the equality (3.1) in Lemma 3.1, we corrected in Theorem 3.1 the formulas (1.23) and (1.26). Most importantly, by establishing the identity (3.9) in Lemma 3.2, we further nicely generalized the evaluation (3.4) as (3.11) in Theorem 3.2.

6. Similar to the deduction of the series expansion (2.10), we can also obtain the series expansions

$$\sin\left(a\sqrt{1+bx}\right)$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left[\sum_{k=0}^{n} (-1)^k a^k [2(n-k)-1]!!\binom{2n-k-1}{2(n-k)}\sin\left(a+\frac{k\pi}{2}\right)\right]\left(\frac{bx}{2}\right)^n$$

and

$$e^{a\sqrt{1+bx}} = e^{a}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!}\left[\sum_{k=0}^{n}(-1)^{k}a^{k}[2(n-k)-1]!!\binom{2n-k-1}{2(n-k)}\right]\left(\frac{bx}{2}\right)^{n}.$$

The ideas, methods, and techniques of this paper can be applied to discover more evaluations of the complete Bell polynomials of the form in (5.1) and others.

Notes

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References

1. [a, b, c, d] *Hoffman ME (1992). "Multiple Harmonic Series." Pac J Math. **152**(2):275–290. http://projecteuclid.org/euclid.pjm/1102636166.*
2. [a, b, c, d, e, f] *Temme NM (1996). Special Functions: An Introduction to Classical Functions of Mathematical Physics. New York: John Wiley & Sons, Inc. doi:10.1002/9781118032572.*
3. ^ *Bao LY, He CY, Qi F (2026). "Monotonic Sequences and Inequalities Involving the Ratio Between Two Adjacent Nonzero Bernoulli Numbers." Math Inequal Appl. **29**(1):1–14. doi:10.7153/mia-2026-29-01.*
4. ^ *Gil JIB, Fresán J (2020). Multiple Zeta Values: From Numbers to Motives. Clay Mathematics Proceedings. https://javier.fresan.perso.math.cnrs.fr/mzv.pdf.*
5. ^ *Zhao J (2016). Multiple Zeta Functions, Multiple Polylogarithms and Their Special Values. Hackensack, NJ: World Scientific Publishing Co. Pte. Ltd. doi:10.1142/9634.*
6. [a, b, c, d, e, f, g, h] *Genčev M (2024). "Extension of Hoffman's Combinatorial Identity Via Specific Zeta-Like Series." Results Math. **79**(1):Paper No. 2, 25 pages. doi:10.1007/s00025-023-02035-w.*
7. [a, b, c, d, e] *Abramowitz M, Stegun IA (Eds) (1992). Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables. New York: Dover Publications, Inc.*
8. ^ *Koshy T (2009). Catalan Numbers With Applications. Oxford: Oxford University Press.*
9. ^ *Qi F (2025). "Some Properties of the Catalan Numbers." Appl Anal Discrete Math. **19**(1):176–184. doi:10.2298/AADM240130002Q.*
10. ^ *Stanley RP (2015). Catalan Numbers. New York: Cambridge University Press. doi:10.1017/CBO9781139871495.*

11. a, b, c, d, e, f, g, h, i *Comtet L (1974). Advanced Combinatorics: The Art of Finite and Infinite Expansions. Dordrecht and Boston: D. Reidel Publishing Co. doi:10.1007/978-94-010-2196-8.*
12. a, b, c *Charalambides CA (2002). Enumerative Combinatorics. Boca Raton, FL: Chapman & Hall/CRC.*
13. a, b, c, d, e, f, g, h, i, j, k, l *He CY, Qi F (2024). "Reformulations and Generalizations of Hoffman's and Genčev's Combinatorial Identities." Results Math.* ***79****(4):Paper No. 131, 17 pages. doi:10.1007/s00025-024-02160-0.*
14. ^ *Riordan J (1979). Combinatorial Identities. Huntington, N.Y.: Robert E. Krieger Publishing Co.*
15. a, b, c, d, e, f *Xu A (2026). "Concise Forms of He–Qi Combinatorial Identities in Terms of Generalized Bernoulli Polynomials, Euler Polynomials, and Catalan Numbers." ResearchGate Preprint. doi:10.13140/RG.2.2.14137.63849.*
16. a, b, c *Qi F (2026). "Alternative Proofs of Xu's Forms for He–Qi's Combinatorial Identities Reformulating and Generalizing Hoffman's and Genčev's Combinatorial Identities for Complete Bell Polynomials." Filomat.* ***40****: accepted on 10 February 2026. doi:10.13140/RG.2.2.14137.63849.*
17. a, b, c, d, e *Qi F (submitted). "Concise and Elegant Proofs of Three Formulas for Complete Bell Polynomials."*
18. ^ *Mansour T, Schork M (2016). Commutation Relations, Normal Ordering, and Stirling Numbers. Boca Raton, FL: CRC Press.*
19. a, b *Chen KW, Chung CL, Eie M (2017). "Sum Formulas of Multiple Zeta Values With Arguments Multiples of a Common Positive Integer." J Number Theory.* ***177****:479–496. doi:10.1016/j.jnt.2017.01.029.*
20. ^ *Chen KW, Chung CL (2017). "Sum Relations of Multiple Zeta Star Values With Even Arguments." Mediterr J Math.* ***14****(3):Paper No. 110, 13 pp. doi:10.1007/s00009-017-0911-0.*
21. ^ *Jin S, Guo BN, Qi F (2022). "Partial Bell Polynomials, Falling and Rising Factorials, Stirling Numbers, and Combinatorial Identities." CMES Comput Model Eng Sci.* ***132****(3):781–799. doi:10.32604/cmes.2022.019941.*
22. ^ *Quaintance J, Gould HW (2016). Combinatorial Identities for Stirling Numbers. Singapore: World Scientific Publishing Co. Pte. Ltd.*
23. ^ *Genčev M (2016). "On Restricted Sum Formulas for Multiple Zeta Values With Even Arguments." Arch Math (Basel).* ***107****(1):9–22. doi:10.1007/s00013-016-0912-4.*
24. ^ *Chiţescu I (2017). Around the Formula of Faà Di Bruno. Mauritius: Éditions universitaires européennes.*
25. ^ *Qi F, Niu DW, Lim D, Guo BN (2020). "Closed Formulas and Identities for the Bell Polynomials and Falling Factorials." Contrib Discrete Math.* ***15****(1):163–174. https://doi.org/10.11575/cdm.v15i1.68111.*
26. ^ *Qi F, Niu DW, Lim D, Yao YH (2020). "Special Values of the Bell Polynomials of the Second Kind for Some Sequences and Functions." J Math Anal Appl.* ***491****(2):Art. 124382, 31 pages. https://doi.org/10.1016/j.jmaa.2020.1*

24382.

27. [a], [b]*Gradshteyn IS, Ryzhik IM (2015). Table of Integrals, Series, and Products. Amsterdam: Elsevier/Academic Press. doi:10.1016/B978-0-12-384933-5.00013-8.*

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