Research Article

# The negativity of a polynomial of degree forty-three

**Da-Wei Niu**[1], **Dongkyu Lim**[2], **Feng Qi**[3]

1. Department of Science, Henan University of Animal Husbandry and Economy (HUAHE), Zhengzhou, China; 2. Department of Mathematics Education, Andong National University, Korea, Republic of; 3. Institute of Mathematics, Henan Polytechnic University, Henan'an, China

**By virtue of two approaches, the authors prove that the function $G(t)=5t^{43}-218t^{30}+720t^{17}-455t^{13}-52$ is negative on $[0,1)$ and $G(1)=0$.**

**Corresponding authors:** Dongkyu Lim, dgrim84@gmail.com; Feng Qi, honest.john.china@gmail.com

1991 *Mathematics Subject Classification.* Primary 26C05; Secondary 26C10.

## 1. Motivations and main results

On 21 March 2023, Professor Chao-Ping Chen claimed that the function $52+455t^{13}-720t^{17}+218t^{30}-5t^{43}$ is positive on $[0,1)$.

In this paper, we prove the following proposition.

**Proposition 1.** *The function*

$$G(t)=5t^{43}-218t^{30}+720t^{17}-455t^{13}-52$$

*is negative on* $[0,1)$ *and* $G(1)=0$.

## 2. Proofs of Proposition 1

In this section, we give two proofs of Proposition 1.

*First proof.* The function $G(t)$ can be factorized as

$$G(t)=(t-1)^2H(t),$$

where

$$
\begin{aligned}
H(t) = {} & 5t^{41} + 10t^{40} + 15t^{39} + 20t^{38} + 25t^{37} + 30t^{36} + 35t^{35} + 40t^{34} \\
& + 45t^{33} + 50t^{32} + 55t^{31} + 60t^{30} + 65t^{29} - 148t^{28} - 361t^{27} \\
& - 574t^{26} - 787t^{25} - 1000t^{24} - 1213t^{23} - 1426t^{22} - 1639t^{21} \\
& - 1852t^{20} - 2065t^{19} - 2278t^{18} - 2491t^{17} - 2704t^{16} - 2197t^{15} \\
& - 1690t^{14} - 1183t^{13} - 676t^{12} - 624t^{11} - 572t^{10} - 520t^{9} - 468t^{8} \\
& - 416t^{7} - 364t^{6} - 312t^{5} - 260t^{4} - 208t^{3} - 156t^{2} - 104t - 52.
\end{aligned}
\tag{1}
$$

The derivatives of $H^{(k)}(t)$ for $1 \leq k \leq 29$ are


doi.org/10.32388/ID6RWM

$$
\begin{aligned}
H'(t) &= 205t^{40} + 400t^{39} + 585t^{38} + 760t^{37} + 925t^{36} + 1080t^{35} \\
&\quad + 1225t^{34} + 1360t^{33} + 1485t^{32} + 1600t^{31} + 1705t^{30} \\
&\quad + 1800t^{29} + 1885t^{28} - 4144t^{27} - 9747t^{26} - 14924t^{25} \\
&\quad - 19675t^{24} - 24000t^{23} - 27899t^{22} - 31372t^{21} - 34419t^{20} \\
&\quad - 37040t^{19} - 39235t^{18} - 41004t^{17} - 42347t^{16} - 43264t^{15} \\
&\quad - 32955t^{14} - 23660t^{13} - 15379t^{12} - 8112t^{11} - 6864t^{10} \\
&\quad - 5720t^{9} - 4680t^{8} - 3744t^{7} - 2912t^{6} - 2184t^{5} - 1560t^{4} \\
&\quad - 1040t^{3} - 624t^{2} - 312t - 104, \\
H''(t) &= 8200t^{39} + 15600t^{38} + 22230t^{37} + 28120t^{36} + 33300t^{35} \\
&\quad + 37800t^{34} + 41650t^{33} + 44880t^{32} + 47520t^{31} + 49600t^{30} \\
&\quad + 51150t^{29} + 52200t^{28} + 52780t^{27} - 111888t^{26} - 253422t^{25} \\
&\quad - 373100t^{24} - 472200t^{23} - 552000t^{22} - 613778t^{21} \\
&\quad - 658812t^{20} - 688380t^{19} - 703760t^{18} - 706230t^{17} \\
&\quad - 697068t^{16} - 677552t^{15} - 648960t^{14} - 461370t^{13} \\
&\quad - 307580t^{12} - 184548t^{11} - 89232t^{10} - 68640t^{9} \\
&\quad - 51480t^{8} - 37440t^{7} - 26208t^{6} - 17472t^{5} - 10920t^{4} \\
&\quad - 6240t^{3} - 3120t^{2} - 1248t - 312, \\
H'''(t) &= 319800t^{38} + 592800t^{37} + 822510t^{36} + 1012320t^{35} \\
&\quad + 1165500t^{34} + 1285200t^{33} + 1374450t^{32} + 1436160t^{31} \\
&\quad + 1473120t^{30} + 1488000t^{29} + 1483350t^{28} + 1461600t^{27} \\
&\quad + 1425060t^{26} - 2909088t^{25} - 6335550t^{24} - 8954400t^{23} \\
&\quad - 10860600t^{22} - 12144000t^{21} - 12889338t^{20} - 13176240t^{19} \\
&\quad - 13079220t^{18} - 12667680t^{17} - 12005910t^{16} - 11153088t^{15} \\
&\quad - 10163280t^{14} - 9085440t^{13} - 5997810t^{12} - 3690960t^{11} \\
&\quad - 2030028t^{10} - 892320t^{9} - 617760t^{8} - 411840t^{7} - 262080t^{6} \\
&\quad - 157248t^{5} - 87360t^{4} - 43680t^{3} - 18720t^{2} - 6240t - 1248, \\
H^{(4)}(t) &= 12152400t^{37} + 21933600t^{36} + 29610360t^{35} + 35431200t^{34} \\
&\quad + 39627000t^{33} + 42411600t^{32} + 43982400t^{31} + 44520960t^{30} \\
&\quad + 44193600t^{29} + 43152000t^{28} + 41533800t^{27} + 39463200t^{26} \\
&\quad + 37051560t^{25} - 72727200t^{24} - 152053200t^{23} - 205951200t^{22} \\
&\quad - 238933200t^{21} - 255024000t^{20} - 257786760t^{19} - 250348560t^{18} \\
&\quad - 235425960t^{17} - 215350560t^{16} - 192094560t^{15} - 167296320t^{14} \\
&\quad - 142285920t^{13} - 118110720t^{12} - 71973720t^{11} - 40600560t^{10} \\
&\quad - 20300280t^{9} - 8030880t^{8} - 4942080t^{7} - 2882880t^{6} - 1572480t^{5} \\
&\quad - 786240t^{4} - 349440t^{3} - 131040t^{2} - 37440t - 6240, \\
H^{(5)}(t) &= 449638800t^{36} + 789609600t^{35} + 1036362600t^{34} \\
&\quad + 1204660800t^{33} + 1307691000t^{32} + 1357171200t^{31} \\
&\quad + 1363454400t^{30} + 1335628800t^{29} + 1281614400t^{28} \\
&\quad + 1208256000t^{27} + 1121412600t^{26} + 1026043200t^{25} \\
&\quad + 926289000t^{24} - 1745452800t^{23} - 3497223600t^{22} \\
&\quad - 4530926400t^{21} - 5017597200t^{20} - 5100480000t^{19} \\
&\quad - 4897948440t^{18} - 4506274080t^{17} - 4002241320t^{16} \\
&\quad - 3445608960t^{15} - 2881418400t^{14} - 2342148480t^{13} \\
&\quad - 1849716960t^{12} - 1417328640t^{11} - 791710920t^{10} \\
&\quad - 406005600t^{9} - 182702520t^{8} - 64247040t^{7} \\
&\quad - 34594560t^{6} - 17297280t^{5} - 7862400t^{4} - 3144960t^{3} \\
&\quad - 1048320t^{2} - 262080t - 37440, \\
H^{(6)}(t) &= 16186996800t^{35} + 27636336000t^{34} + 35236328400t^{33} \\
&\quad + 39753806400t^{32} + 41846112000t^{31} + 42072307200t^{30}
\end{aligned}
$$

$$
\begin{aligned}
&+ 40903632000t^{29} + 38733235200t^{28} + 35885203200t^{27}\\
&+ 32622912000t^{26} + 29156727600t^{25} + 25651080000t^{24}\\
&+ 22230936000t^{23} - 40145414400t^{22} - 76938919200t^{21}\\
&- 95149454400t^{20} - 100351944000t^{19} - 96909120000t^{18}\\
&- 88163071920t^{17} - 76606659360t^{16} - 64035861120t^{15}\\
&- 51684134400t^{14} - 40339857600t^{13} - 30447930240t^{12}\\
&- 22196603520t^{11} - 15590615040t^{10} - 7917109200t^{9}\\
&- 3654050400t^{8} - 1461620160t^{7} - 449729280t^{6} - 207567360t^{5}\\
&- 86486400t^{4} - 31449600t^{3} - 9434880t^{2} - 2096640t - 262080,\\
H^{(7)}(t) &= 566544888000t^{34} + 939635424000t^{33} + 1162798837200t^{32}\\
&+ 1272121804800t^{31} + 1297229472000t^{30} + 1262169216000t^{29}\\
&+ 1186205328000t^{28} + 1084530585600t^{27} + 968900486400t^{26}\\
&+ 848195712000t^{25} + 728918190000t^{24} + 615625920000t^{23}\\
&+ 511311528000t^{22} - 883199116800t^{21} - 1615717303200t^{20}\\
&- 1902989088000t^{19} - 1906686936000t^{18} - 1744364160000t^{17}\\
&- 1498772222640t^{16} - 1225706549760t^{15} - 960537916800t^{14}\\
&- 723577881600t^{13} - 524418148800t^{12} - 365375162880t^{11}\\
&- 244162638720t^{10} - 155906150400t^{9} - 71253982800t^{8}\\
&- 29232403200t^{7} - 10231341120t^{6} - 2698375680t^{5}\\
&- 1037836800t^{4} - 345945600t^{3} - 94348800t^{2}\\
&- 18869760t - 2096640,\\
H^{(8)}(t) &= 19262526192000t^{33} + 31007968992000t^{32} + 37209562790400t^{31}\\
&+ 39435775948800t^{30} + 38916884160000t^{29} + 36602907264000t^{28}\\
&+ 33213749184000t^{27} + 29282325811200t^{26} + 25191412646400t^{25}\\
&+ 21204892800000t^{24} + 17494036560000t^{23} + 14159396160000t^{22}\\
&+ 11248853616000t^{21} - 18547181452800t^{20} - 32314346064000t^{19}\\
&- 36156792672000t^{18} - 34320364848000t^{17} - 29654190720000t^{16}\\
&- 23980355562240t^{15} - 18385598246400t^{14} - 13447530835200t^{13}\\
&- 9406512460800t^{12} - 6293017785600t^{11} - 4019126791680t^{10}\\
&- 2441626387200t^{9} - 1403155353600t^{8} - 570031862400t^{7}\\
&- 204626822400t^{6} - 61388046720t^{5} - 13491878400t^{4}\\
&- 4151347200t^{3} - 1037836800t^{2} - 188697600t - 18869760,\\
H^{(9)}(t) &= 635663364336000t^{32} + 992255007744000t^{31}\\
&+ 1153496446502400t^{30} + 1183073278464000t^{29}\\
&+ 1128589640640000t^{28} + 1024881403392000t^{27}\\
&+ 896771227968000t^{26} + 761340471091200t^{25}\\
&+ 629785316160000t^{24} + 508917427200000t^{23}\\
&+ 402362840880000t^{22} + 311506715520000t^{21}\\
&+ 236225925936000t^{20} - 370943629056000t^{19}\\
&- 613972575216000t^{18} - 650822268096000t^{17}\\
&- 583446202416000t^{16} - 474467051520000t^{15}\\
&- 359705333433600t^{14} - 257398375449600t^{13}\\
&- 174817900857600t^{12} - 112878149529600t^{11}\\
&- 69223195641600t^{10} - 40191267916800t^{9}\\
&- 21974637484800t^{8} - 11225242828800t^{7} - 3990223036800t^{6}\\
&- 1227760934400t^{5} - 306940233600t^{4} - 53967513600t^{3}\\
&- 12454041600t^{2} - 2075673600t - 188697600,\\
H^{(10)}(t) &= 20341227658752000t^{31} + 30759905240064000t^{30}\\
&+ 34604893395072000t^{29} + 34309125075456000t^{28}\\
&+ 31600509937920000t^{27} + 27671797891584000t^{26}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2331605192716800 0t^{25} + 1903351177728000 0t^{24} \\
&+ 1511484758784000 0t^{23} + 1170510082560000 0t^{22} \\
&+ 885198249936000 0t^{21} + 654164102592000 0t^{20} \\
&+ 472451851872000 0t^{19} - 704792895206400 0t^{18} \\
&- 1105150635388800 0t^{17} - 1106397855763200 0t^{16} \\
&- 933513923865600 0t^{15} - 711700577280000 0t^{14} \\
&- 503587466807040 0t^{13} - 334617888084480 0t^{12} \\
&- 209781481029120 0t^{11} - 124165964482560 0t^{10} \\
&- 692231956416000t^{9} - 361721411251200t^{8} \\
&- 175797099878400t^{7} - 78576699801600t^{6} - 23941338220800t^{5} \\
&- 6138804672000t^{4} - 1227760934400t^{3} - 161902540800t^{2} \\
&- 24908083200t - 2075673600, \\
H^{(11)}(t) = {}& 630578057421312000t^{30} + 922797157201920000t^{29} \\
&+ 1003541908457088000t^{28} + 960655502112768000t^{27} \\
&+ 853213768323840000t^{26} + 719466745181184000t^{25} \\
&+ 582901298179200000t^{24} + 456804282654720000t^{23} \\
&+ 347641494520320000t^{22} + 257512218163200000t^{21} \\
&+ 185891632486560000t^{20} + 130832820518400000t^{19} \\
&+ 89765851855680000t^{18} - 126862721137152000t^{17} \\
&- 187875608016096000t^{16} - 177023656922112000t^{15} \\
&- 140027088579840000t^{14} - 99638080819200000t^{13} \\
&- 65466370684915200t^{12} - 40154146570137600t^{11} \\
&- 23075962913203200t^{10} - 12416596448256000t^{9} \\
&- 6230087607744000t^{8} - 2893771290009600t^{7} \\
&- 1230579699148800t^{6} - 471460198809600t^{5} \\
&- 119706691104000t^{4} - 24555218688000t^{3} \\
&- 3683282803200t^{2} - 323805081600t - 24908083200, \\
H^{(12)}(t) = {}& 18917341722639360000t^{29} + 26761117558855680000t^{28} \\
&+ 28099173436798464000t^{27} + 25937698557044736000t^{26} \\
&+ 22183557976419840000t^{25} + 17986668629529600000t^{24} \\
&+ 13989631156300800000t^{23} + 10506498501058560000t^{22} \\
&+ 7648112879447040000t^{21} + 5407756581427200000t^{20} \\
&+ 3717832649731200000t^{19} + 2485823589849600000t^{18} \\
&+ 1615785333402240000t^{17} - 2156666259331584000t^{16} \\
&- 3006009728257536000t^{15} - 2655354853831680000t^{14} \\
&- 1960379240117760000t^{13} - 1295295050649600000t^{12} \\
&- 785596448218982400t^{11} - 441695612271513600t^{10} \\
&- 230759629132032000t^{9} - 111749368034304000t^{8} \\
&- 49840700861952000t^{7} - 20256399030067200t^{6} \\
&- 7383478194892800t^{5} - 2357300994048000t^{4} \\
&- 478826764416000t^{3} - 73665656064000t^{2} \\
&- 7366565606400t - 323805081600, \\
H^{(13)}(t) = {}& 548602909956541440000t^{28} + 749311291647959040000t^{27} \\
&+ 758677682793558528000t^{26} + 674380162483163136000t^{25} \\
&+ 554588949410496000000t^{24} + 431680047108710400000t^{23} \\
&+ 321761516594918400000t^{22} + 231142967023288320000t^{21} \\
&+ 160610370468387840000t^{20} + 108155131628544000000t^{19} \\
&+ 70638820344892800000t^{18} + 44744824617292800000t^{17} \\
&+ 27468350667838080000t^{16} - 34506660149305344000t^{15} \\
&- 45090145923863040000t^{14} - 37174967953643520000t^{13}
\end{aligned}
$$

$$
\begin{aligned}
& - 45090145923803040000t - 37174907953043520000t \\
& - 254849301215308800000t^{12} - 155435406077952000000t^{11} \\
& - 86415609304088064000t^{10} - 44169561227151360000t^{9} \\
& - 20768366621882880000t^{8} - 8939949442744320000t^{7} \\
& - 3488849060336640000t^{6} - 1215383941804032000t^{5} \\
& - 369173909744640000t^{4} - 94292039761920000t^{3} \\
& - 14364802932480000t^{2} - 1473313121280000t - 73665656064000,
\end{aligned}
$$

$$
\begin{aligned}
H^{(14)}(t) = {} & 15360881478783160320000t^{27} + 202314048744948940800000t^{26} \\
& + 1972561975263252172800000t^{25} + 16859504062079078400000t^{24} \\
& + 13310134785851904000000t^{23} + 9928641083500339200000t^{22} \\
& + 7078753365088204800000t^{21} + 4854002307489054720000t^{20} \\
& + 3212207409367756800000t^{19} + 2054947500942336000000t^{18} \\
& + 1271498766208070400000t^{17} + 760662018493977600000t^{16} \\
& + 439493610685409280000t^{15} - 517599902239580160000t^{14} \\
& - 631262042934082560000t^{13} - 483274583397365760000t^{12} \\
& - 305819161458370560000t^{11} - 170978946685747200000t^{10} \\
& - 86415609304088064000t^{9} - 39752605104436224000t^{8} \\
& - 16614693297506304000t^{7} - 6257964609921024000t^{6} \\
& - 2093309436201984000t^{5} - 607691970902016000t^{4} \\
& - 147669563897856000t^{3} - 28287611928576000t^{2} \\
& - 2872960586496000t - 147331312128000,
\end{aligned}
$$

$$
\begin{aligned}
H^{(15)}(t) = {} & 4147437999271453286400000t^{26} + 52601652673686724608000000t^{25} \\
& + 493140493815813043200000t^{24} + 404628097489897881600000t^{23} \\
& + 306133100074593792000000t^{22} + 218430103837007462400000t^{21} \\
& + 148653820666852300800000t^{20} + 97080046149781094400000t^{19} \\
& + 61031940777987379200000t^{18} + 36989055016962048000000t^{17} \\
& + 21615479025537196800000t^{16} + 12170592295903641600000t^{15} \\
& + 6592404160281139200000t^{14} - 7246398631354122240000t^{13} \\
& - 8206406558143073280000t^{12} - 5799295000768389120000t^{11} \\
& - 3364010776042076160000t^{10} - 1709789466857472000000t^{9} \\
& - 777740483736792576000t^{8} - 318020840835489792000t^{7} \\
& - 116302853082544128000t^{6} - 37547787659526144000t^{5} \\
& - 10466547181009920000t^{4} - 2430767883608064000t^{3} \\
& - 443008691693568000t^{2} - 56575223857152000t \\
& - 2872960586496000,
\end{aligned}
$$

$$
\begin{aligned}
H^{(16)}(t) = {} & 107833387981057785446400000t^{25} \\
& + 1315041316842168115200000000t^{24} \\
& + 11835371851579513036800000t^{23} \\
& + 9306446242267651276800000t^{22} \\
& + 6734928201641063424000000t^{21} \\
& + 4587032180577156710400000t^{20} \\
& + 2973076413337046016000000t^{19} \\
& + 1844520876845840793600000t^{18} \\
& + 1098574934003772825600000t^{17} \\
& + 628813935288354816000000t^{16} \\
& + 345847664408595148800000t^{15} + 182558884438554624000000t^{14} \\
& + 92293658243935948800000t^{13} - 94203182207603589120000t^{12} \\
& - 98476878697716879360000t^{11} - 63792245008452280320000t^{10} \\
& - 33640107760420761600000t^{9} - 15388105201717248000000t^{8} \\
& - 6221923869894340608000t^{7} - 2226145885848428544000t^{6}
\end{aligned}
$$

$$
\begin{aligned}
& - 697817118495264768000t^5 - 1877389382976307200000t^4 \\
& - 418661887240396800000t^3 - 72923036508241920000t^2 \\
& - 8860173833871360000t - 565752238571520000, \\
H^{(17)}(t) = {} & 2695834699526444636160000000t^{24} \\
& + 31560991604212034764800000000t^{23} \\
& + 272221355258632879984640000000t^{22} \\
& + 2047418173298883280896000000t^{21} \\
& + 1414334922344623319040000000t^{20} \\
& + 917406436115431342080000000t^{19} \\
& + 564884518534038743040000000t^{18} \\
& + 332013757832251342848000000t^{17} \\
& + 186757738780641380352000000t^{16} \\
& + 100610229646136770560000000t^{15} \\
& + 51877149661289272320000000t^{14} \\
& + 25558243821397647360000000t^{13} \\
& + 11998175571711673344000000t^{12} \\
& - 1130438186491243069440000t^{11} \\
& - 10832456656748856729600000t^{10} \\
& - 637922450084522803200000t^9 - 3027609698437868544000000t^8 \\
& - 1231048416137379840000000t^7 - 435534670892603842560000t^6 \\
& - 133568753150905712640000t^5 - 34890855924763238400000t^4 \\
& - 7509557531905228800000t^3 - 1255985661721190400000t^2 \\
& - 145846073016483840000t - 8860173833871360000, \\
H^{(18)}(t) = {} & 64700032788634671267840000000t^{23} \\
& + 725902806896876799590400000000t^{22} \\
& + 598869815689923359662080000000t^{21} \\
& + 42995781639276548898816000000t^{20} \\
& + 2828669844689246638080000000000t^{19} \\
& + 1743072228619319549952000000t^{18} \\
& + 1016792133361269737472000000t^{17} \\
& + 564423388314827282841600000t^{16} \\
& + 298812382049026208563200000t^{15} \\
& + 1509153444692051558400000000t^{14} \\
& + 72628009525804981248000000t^{13} \\
& + 33225716967816941568000000t^{12} \\
& + 14397810686054008012800000t^{11} \\
& - 1243482005140367376384000t^{10} \\
& - 1083245665674885672960000t^9 \\
& - 57413020507607052288000000t^8 \\
& - 24220877587502948352000000t^7 - 8617338912961658880000000t^6 \\
& - 2613208025355623055360000t^5 - 667843765754528563200000t^4 \\
& - 139563423699052953600000t^3 - 22528672595715686400000t^2 \\
& - 2511971323442380800000t - 145846073016483840000, \\
H^{(19)}(t) = {} & 1488100754138597439160320000000t^{22} \\
& + 15969861751731289590988800000000t^{21} \\
& + 125762661294883905529036800000t^{20} \\
& + 8599156327855309779763200000000t^{19} \\
& + 5374472704909568612352000000000t^{18} \\
& + 3137530011514775189913600000000t^{17} \\
& + 1728546626714158553702400000000t^{16}
\end{aligned}
$$

$$
\begin{aligned}
&+ 903077421303723652546560000 0t^{15}\\
&+ 4482185730735393128448000000t^{14}\\
&+ 2112814822568872181760000000t^{13}\\
&+ 944164123835464756224000000t^{12}\\
&+ 398708603613803298816000000t^{11}\\
&+ 158375917546594088140800000t^{10}\\
&- 124348200514036737638400000t^{9}\\
&- 97492109910739710566400000t^{8}\\
&- 45930416406085641830400000t^{7}\\
&- 16954614311252063846400000t^{6}\\
&- 5170403347776995328000000t^{5}\\
&- 1306604012677811527680000t^{4}\\
&- 267137506301811425280000t^{3} - 41869027109715886080000t^{2}\\
&- 4505734519143137280000t - 251197132344238080000,\\
H^{(20)}(t) = {}& 327382165910491436615270400000 0t^{21}\\
&+ 3353670967863570814107648000000t^{20}\\
&+ 2515253225897678110580736000000t^{19}\\
&+ 1633839702292508858155008000000t^{18}\\
&+ 967405086883722350223360000000t^{17}\\
&+ 533380101957511782285312000000t^{16}\\
&+ 276567460274265368592384000000t^{15}\\
&+ 135461613195558547881984000000t^{14}\\
&+ 62750600230295503798272000000t^{13}\\
&+ 27466592693395338362880000000t^{12}\\
&+ 11329969486025577074688000000t^{11}\\
&+ 4385794639751836286976000000t^{10}\\
&+ 1583759175465940881408000000t^{9}\\
&- 1119133804626330638745600000t^{8}\\
&- 779936879285917684531200000t^{7}\\
&- 321512914842599492812800000t^{6}\\
&- 101727685867512383078400000t^{5}\\
&- 25852016738884976640000000t^{4}\\
&- 5226416050711246110720000t^{3} - 801412518905434275840000t^{2}\\
&- 83738054219431772160000t - 4505734519143137280000,\\
H^{(21)}(t) = {}& 68750254841203201689206784000000t^{20}\\
&+ 67073419357271416282152960000000t^{19}\\
&+ 47789811292055884101033984000000t^{18}\\
&+ 29409114641265159446790144000000t^{17}\\
&+ 16445886477023279953797120000000t^{16}\\
&+ 8534081631320188516564992000000t^{15}\\
&+ 4148511904113980528885760000000t^{14}\\
&+ 1896462584737819670347776000000t^{13}\\
&+ 815757802993841549377536000000t^{12}\\
&+ 329599112320744060354560000000t^{11}\\
&+ 124629664346281347821568000000t^{10}\\
&+ 43857946397518362869760000000t^{9}\\
&+ 14253832579193467932672000000t^{8}\\
&- 8953070437010645109964800000t^{7}\\
&- 5459558155001423791718400000t^{6}\\
&- 1929077489055596956876800000t^{5}
\end{aligned}
$$

$$
\begin{aligned}
& - 5086384293375619153920000000t^4 \\
& - 1034080669555399065600000000t^3 \\
& - 156792481521337383321600000t^2 \\
& - 16028250378108685516800000t - 837380542194317721600000, \\
H^{(22)}(t) = {} & 137500509682406403378413568000000000t^{19} \\
& + 1274394967788156909360906240000000000t^{18} \\
& + 860216603257005913818611712000000t^{17} \\
& + 4999549489015077105954324480000000t^{16} \\
& + 2631341836323724792607539200000000t^{15} \\
& + 12801122446980282774847488000000000t^{14} \\
& + 5807916665759572740440064000000000t^{13} \\
& + 2465401360159165571452108800000000t^{12} \\
& + 978909363592609859253043200000000t^{11} \\
& + 362559023552818466390016000000000t^{10} \\
& + 124629664346281347821568000000000t^9 \\
& + 39472151757766526582784000000000t^8 \\
& + 11403066063354774346137600000000t^7 \\
& - 626714930590745157697536000000t^6 \\
& - 327573489300085427503104000000t^5 \\
& - 96453874452779847843840000000t^4 \\
& - 20345537173502476615680000000t^3 \\
& - 3102242008666197196800000000t^2 \\
& - 313584963042674766643200000t \\
& - 16028250378108685516800000, \\
H^{(23)}(t) = {} & 2612509683965721664189857792000000000t^{18} \\
& + 22939109420186824368496312320000000000t^{17} \\
& + 146236822553691005349163991040000000t^{16} \\
& + 79992791824241233695269191680000000t^{15} \\
& + 394701275448558718891130880000000000t^{14} \\
& + 179215714257723958847864832000000000t^{13} \\
& + 75502916654874445625720832000000000t^{12} \\
& + 29584816321909986857425305600000000t^{11} \\
& + 10768002999518708451783475200000000t^{10} \\
& + 3625590235528184663900160000000000t^9 \\
& + 1121666979116532130394112000000000t^8 \\
& + 315777214062132212662272000000000t^7 \\
& + 79821462443483420422963200000000t^6 \\
& - 3760289583544470946185216000000t^5 \\
& - 1637867446500427137515520000000t^4 \\
& - 385815497811119391375360000000t^3 \\
& - 61036611520507429847040000000t^2 \\
& - 6204484017332394393600000000t \\
& - 313584963042674766643200000, \\
H^{(24)}(t) = {} & 47025174311382989955417440256000000000t^{17} \\
& + 389964860143176014264437309440000000000t^{16} \\
& + 23397891608590560855866238566400000000t^{15} \\
& + 119989187736361850542903787520000000000t^{14} \\
& + 55258178562798220644758323200000000000t^{13} \\
& + 23298042853504114650222428160000000000t^{12}
\end{aligned}
$$

$$
\begin{aligned}
&+ 90603499985849334750864998400000000t^{11} \\
&+ 3254329795410098554316783616000000t^{10} \\
&+ 1076800299951870845178347520000000t^{9} \\
&+ 326303121197536619751014400000000t^{8} \\
&+ 89733358329322570431528960000000t^{7} \\
&+ 22104404984349254886359040000000t^{6} \\
&+ 4789287746609005225377792000000t^{5} \\
&- 188014479177223547309260800000t^{4} \\
&- 655146978600170855006208000000t^{3} \\
&- 115744649343335817412608000000t^{2} \\
&- 12207322304101485969408000000t \\
&- 620448401733239439360000000, \\
H^{(25)}(t) = &\ 7994279632935108292420964843520000000t^{16} \\
&+ 623943776229081622823099695104000000t^{15} \\
&+ 350968374128858412837993578496000000t^{14} \\
&+ 167984862830906590760065302528000000t^{13} \\
&+ 71835632131637686838185820160000000t^{12} \\
&+ 27957651424204937580266913792000000t^{11} \\
&+ 9966384998443426822595149824000000t^{10} \\
&+ 3254329795410098554316783616000000t^{9} \\
&+ 969120269956683760660512768000000t^{8} \\
&+ 261042496958029295800811520000000t^{7} \\
&+ 62813350830525799302070272000000t^{6} \\
&+ 13262642990609552931815424000000t^{5} \\
&+ 2394643873304502612688896000000t^{4} \\
&- 752057916708894189237043200000t^{3} \\
&- 196544093580051256501862400000t^{2} \\
&- 23148929868667163482521600000t \\
&- 1220732230410148596940800000, \\
H^{(26)}(t) = &\ 127908474126961732678735437496320000000t^{15} \\
&+ 9359156643436224342346495426560000000t^{14} \\
&+ 4913557237804017779731910098944000000t^{13} \\
&+ 2183803216801785679880848932864000000t^{12} \\
&+ 862027585579652242058229841920000000t^{11} \\
&+ 307534165666254313382936051712000000t^{10} \\
&+ 99663849984434268225951498240000000t^{9} \\
&+ 29288968158690886988851052544000000t^{8} \\
&+ 7752962159653470085284102144000000t^{7} \\
&+ 1827297478706205070605680640000000t^{6} \\
&+ 376880104983154795812421632000000t^{5} \\
&+ 66313214953047764659077120000000t^{4} \\
&+ 9578575493218010450755584000000t^{3} \\
&- 2256173750126682567711129600000t^{2} \\
&- 393088187160102513003724800000t \\
&- 23148929868667163482521600000, \\
H^{(27)}(t) = &\ 1918627111904425990181031562444800000000t^{14} \\
&+ 131028193008107140792850935971840000000t^{13} \\
&+ 63876244091452231136514831286272000000t^{12} \\
&+ 26205638601621428158570187194368000000t^{11} \\
&+ 9482303441376174666264052826112000000t^{10}
\end{aligned}
$$

$$
\begin{aligned}
&+ 30753416566625431338293605171200000000t^9 \\
&+ 8969746498599084140335634841600000000t^8 \\
&+ 2343117452695270959108084203520000000t^7 \\
&+ 542707351175742905969887150080000000t^6 \\
&+ 109637848722372304236340838400000000t^5 \\
&+ 18844005249157739790621081600000000t^4 \\
&+ 2652528598121910586363084800000000t^3 \\
&+ 287357264796540313522667520000000t^2 \\
&- 45123475002533651354222592000000t \\
&- 3930881871601025130037248000000, \\
H^{(28)}(t) = &\ 2686077956666196386253444187422720000000t^{13} \\
&+ 1703366509105392830307062167633920000000t^{12} \\
&+ 766514929097426773638177975435264000000t^{11} \\
&+ 288262024617835709744272059138048000000t^{10} \\
&+ 94823034413761746626405282611200000000t^9 \\
&+ 27678074909962888204464244654080000000t^8 \\
&+ 7175797198879267312268507873280000000t^7 \\
&+ 1640182216886689671375658942464000000t^6 \\
&+ 325624410705445743581932290048000000t^5 \\
&+ 54818924361186152118170419200000000t^4 \\
&+ 7537602099663095916248432640000000t^3 \\
&+ 795758579436573175908925440000000t^2 \\
&+ 57471452959308062704533504000000t \\
&- 45123475002533651354222592000000, \\
H^{(29)}(t) = &\ 34919013436660553021294774436495360000000t^{12} \\
&+ 20440398109264713963684746011607040000000t^{11} \\
&+ 8431664220071694510019957729787904000000t^{10} \\
&+ 2882620246178357097442720591380480000000t^9 \\
&+ 853407309723855719637647543500800000000t^8 \\
&+ 221424599279703105635713957232640000000t^7 \\
&+ 50230580392154871185879555112960000000t^6 \\
&+ 9841093301320138028253953654784000000t^5 \\
&+ 1628122053527228717909661450240000000t^4 \\
&+ 219275697444744608472681676800000000t^3 \\
&+ 22612806298989287748745297920000000t^2 \\
&+ 1591517158873146351817850880000000t \\
&+ 57471452959308062704533504000000 \\
&> 0
\end{aligned}
$$

on $[0,1]$. The values at $t = 0, 1$ of these derivatives are


doi.org/10.32388/ID6RWM

$$
\begin{gathered}
H'(0) = -104, \quad H''(0) = -312, \quad H'''(0) = -1248, \quad H^{(4)}(0) = -6240, \\
H^{(5)}(0) = -37440, \quad H^{(6)}(0) = -262080, \quad H^{(7)}(0) = -2096640, \\
H^{(8)}(0) = -18869760, \quad H^{(9)}(0) = -188697600, \\
H^{(10)}(0) = -2075673600, \quad H^{(11)}(0) = -24908083200, \\
H^{(12)}(0) = -323805081600, \quad H^{(13)}(0) = -7366565606400, \\
H^{(14)}(0) = -147331312128000, \quad H^{(15)}(0) = -2872960586496000, \\
H^{(16)}(0) = -56575223857152000, \quad H^{(17)}(0) = -886017383387136000, \\
H^{(18)}(0) = -14584607301648384000, \\
H^{(19)}(0) = -251197132344238080000, \\
H^{(20)}(0) = -4505734519143137280000, \\
H^{(21)}(0) = -83738054219431772160000, \\
H^{(22)}(0) = -1602825037810868551680000, \\
H^{(23)}(0) = -31358496304267476664320000, \\
H^{(24)}(0) = -620448401733239439360000000, \\
H^{(25)}(0) = -12207322304101485969408000000, \\
H^{(26)}(0) = -231489298686671634825216000000, \\
H^{(27)}(0) = -3930881871601025130037248000000, \\
H^{(28)}(0) = -45123475002533651354222592000000,
\end{gathered}
$$

and

$$
\begin{gathered}
H'(1) = -463905, \quad H''(1) = -7937930, \quad H'''(1) = -134301258, \\
H^{(4)}(1) = -2179937760, \quad H^{(5)}(1) = -32335446000, \\
H^{(6)}(1) = -384463778400, \quad H^{(7)}(1) = -1422141084000, \\
H^{(8)}(1) = 123005557584000, \quad H^{(9)}(1) = 6118209626256000, \\
H^{(10)}(1) = 209898202524192000, \quad H^{(11)}(1) = 6258088312283808000, \\
H^{(12)}(1) = 172533094320787200000, \\
H^{(13)}(1) = 4507415070256530432000, \\
H^{(14)}(1) = 112826895527710780416000, \\
H^{(15)}(1) = 2719636547804209313280000, \\
H^{(16)}(1) = 63248332563385515786240000, \\
H^{(17)}(1) = 1419354109575085036953600000, \\
H^{(18)}(1) = 30707607546762254278410240000, \\
H^{(19)}(1) = 639503918235364398715699200000, \\
H^{(20)}(1) = 12794562254320944793952256000000, \\
H^{(21)}(1) = 245358669969239904045416448000000, \\
H^{(22)}(1) = 4498512485856551647442141184000000, \\
H^{(23)}(1) = 78635738360653790735853848494080000, \\
H^{(24)}(1) = 1306572674897590006963163627520000000, \\
H^{(25)}(1) = 20566466352649903703471698083840000000, \\
H^{(26)}(1) = 305559094392501547207633344921600000000, \\
H^{(27)}(1) = 4267291263884568841390753964359680000000, \\
H^{(28)}(1) = 55759273378811934823769599128895488000000.
\end{gathered}
$$

These datum imply that

1. $H^{(28)}(t)$ is increasing and has only one zero on $(0,1)$,
2. $H^{(k)}(t)$ for $8 \le k \le 27$ have only one minimum and only one zero on $(0,1)$,
3. $H^{(k)}(t)$ for $1 \le k \le 7$ are all negative on $[0,1]$,
4. $H(t)$ is decreasing on $[0,1]$.

From $H(0) = -52$ and $H(1) = -27885$, it follows that $H(t)$ is negative on $[0,1]$. Hence, the function $G(t) = (t-1)^2 H(t)$ is negative on $[0,1)$ and $G(1) = 0$ clearly.

The first proof is complete. □

*Second proof.* Descartes' rule of signs [[1], p. 22] states that,

1. if the nonzero terms of a single-variable polynomial with real coefficients are ordered by descending variable exponent, then the number of positive zeros of the polynomial is either equal to the number of sign changes between consecutive (nonzero) coefficients, or is less than it by an even number. A zero of multiplicity $k$ is counted as $k$ zeros.
2. the number of negative zeros is the number of sign changes after multiplying the coefficients of odd-power terms by $-1$, or fewer than it by an even number.

Consequently, the function $H(t)$ defined by (1) has at most one positive zero and at most one negative zero. From

$$H(0) = -52, \quad H(1) = -27885, \quad H(2) = 43746480037836,$$

we easily see that there exists a positive zero on $(1,2)$. Therefore, the function $H(t)$ is negative on $[0,1]$. Accordingly, the function $G(t) = (t-1)^2 H(t)$ is negative on $[0,1)$ and $G(1) = 0$. The second proof is complete. □

## 3. Remarks

Finally, we list several remarks on our main results.

*Remark* 1. The first proof is long, but it is elementary. The second proof is short, however, it is advanced.

*Remark 2.* The negativity verified in Theorem 1 has been applied in [2] to refine the Shafer--Fink type inequalities for arcsin x, arctan x, and arctanh x. This type of inequalities have been investigated in the papers [3][4][5][6][7][8], for example.

## References


1. ^*F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, New York, 2010; available online at http://dlmf.nist.gov/.*
2. ^*J.-L. Sun and C.-P. Chen, Refinements of Shafer–Fink type inequalities, submitted in 2023.*
3. ^*B.-N. Guo, Q.-M. Luo, and F. Qi, Monotonicity results and inequalities for the inverse hyperbolic sine functio n, J. Inequal. Appl. 2013, 2013:536, 6 pages; available online at https://doi.org/10.1186/1029- 242X- 2013- 536.*
4. ^*B.-N. Guo, Q.-M. Luo, and F. Qi, Sharpening and generalizations of Shafer-Fink's double inequality for the a rc sine function, Filomat 27 (2013), no. 2, 261–265; available online at https://doi.org/10.2298/FIL1302261G.*
5. ^*B.-N. Guo and F. Qi, Sharpening and generalizations of Carlson's inequality for the arc cosine function, Hac et. J. Math. Stat. 39 (2010), no. 3, 403–409.*
6. ^*F. Qi and B.-N. Guo, Sharpening and generalizations of Shafer's inequality for the arc sine function, Integra l Transforms Spec. Funct. 23 (2012), no. 2, 129–134; available online at https://doi.org/10.1080/10652469.201 1.564578.*
7. ^*F. Qi, S.-Q. Zhang, and B.-N. Guo, Sharpening and generalizations of Shafer's inequality for the arc tangent function, J. Inequal. Appl. 2009, Art. ID 930294, 9 pages; available online at https://doi.org/10.1155/2009/930 294.*
8. ^*J.-L. Zhao, C.-F. Wei, B.-N. Guo, and F. Qi, Sharpening and generalizations of Carlson's double inequality for the arc cosine function, Hacet. J. Math. Stat. 41 (2012), no. 2, 201–209.*


## Declarations


**Funding:** No specific funding was received for this work.

**Potential competing interests:** No potential competing interests to declare.