Research Article

The edge rings of compact graphs

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We define a simple graph as compact if it lacks even cycles and satisfies the odd-cycle condition. Our focus is on classifying all compact graphs and examining the characteristics of their edge rings. Let G be a compact graph and $\mathbb{K}[G]$ be its edge ring. Specifically, we demonstrate that the Cohen-Macaulay type and the projective dimension of $\mathbb{K}[G]$ are both equal to the number of induced cycles of G minus one and that the regularity of $\mathbb{K}[G]$ is equal to the matching number of G_0. Here, G_0 is obtained from G by removing the vertices of degree one successively, resulting in a graph where every vertex has a degree greater than 1.

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Introduction

Recently, many authors have investigated the algebraic properties of edge rings of simple graphs. Consider a simple graph $G=(V,E)$ with vertex set $V=\{x_1,\dots,x_n\}$ and edge set $E=\{e_1,\dots,e_r\}$. The *edge ring* $\mathbb{K}[G]$ is defined to be the toric ring $\mathbb{K}[x_e:\ e\in E(G)]\subset \mathbb{K}[x_1,\dots,x_n]$, where $x_e=\prod_{x_i\in e}x_i$ for all $e\in E(G)$. Let $\mathbb{K}[E(G)]$ (or $\mathbb{K}[E]$ for short) denote the polynomial ring $\mathbb{K}[e_1,\dots,e_r]$ in variables $e_1,\dots,e_r$. Then, there is exactly one ring homomorphism $\phi:\mathbb{K}[E(G)]\to\mathbb{K}[V]$ such that $e_i\mapsto x_{e_i}\ i=1,\dots,r.$ The kernel of the homomorphism map ϕ is called the *toric ideal* or the *defining ideal* of $\mathbb{K}[G]$ or G, which is denoted by I_G. It follows that $\mathbb{K}[G]\cong\mathbb{K}[E(G)]/I_G$. The main focus of these studies is to establish connections between the combinatorial properties of simple graphs and the algebraic properties of their edge rings, see e.g. [2, 3, 6, 7, 8, 9, 13, 14] for some developments in this area.

In 1999, Ohsugi and Hibi demonstrated in [14] that $\mathbb{K}[G]$ is a normal domain if and only if G satisfies the odd-cycle condition. Recall a simple graph is said to satisfy the *odd-cycle* condition if, for every pair of cycles C_1 and C_2, either C_1 and C_2 have at least one vertex in common or there is an edge that connects a vertex of C_1 to a vertex of C_2. We call a simple graph to be *compact* if it not only satisfies the odd-cycle condition but also contains no even cycles. In this paper, we devote to investigating the properties of the edge rings of compact graphs.

Let G be a compact graph. The main results of this paper can be summarized as follows. Firstly, we demonstrate that the projective dimension and Cohen-Macaulay type of $\mathbb{K}[G]$ are both equal to the number of the induced cycles of G minus one.

Additionally, we show that the regularity of $\mathbb{K}[G]$ coincides with the matching number of G_0. Here, G_0 refers to the graph derived from G by successively removing all vertices of degree one. This finding serves as an interesting complement to the result presented in [11, Theorem 1 (a)], which states if G is a non-bipartite graph satisfying the odd-cycle condition, the regularity of $\mathbb{K}[G]$ does not exceed the matching number of G. Finally, we determine the top graded Betti numbers of $\mathbb{K}[G]$. Here, for a simple graph G, a *matching* of G is a subset $M\subset E(G)$ where $e\cap e'=\emptyset$ for any distinct edges $e,e'\in M$, and the *matching number* of G, denoted by $\mathrm{mat}(G)$, is the maximal cardinality of matchings of G.

The paper is organized as follows. Let G be a compact graph. Section 1 provides a brief overview of toric ideals of graphs and canonical modules. Section 2 classifies the compact graphs up to the (essentially) same edge rings. In Section 3 we compute the universal Gröbner bases for the toric ideals of compact graphs and then obtain their initial ideals with respect to some suitable monomial order. In Section 4, we show that all initial ideals obtained in Section 3 possess a "good" E-K splitting, enabling us to present a simple formula for the total Betti numbers of such ideals. Consequently, the regularity, projection dimension, and an upper bound for the Cohen-Macaulay type of $\mathbb{K}[G]$ are derived. Section 5 provides the top graded Betti numbers for $\mathbb{K}[G]$ by computing the minimal generators of its canonical module. In Section 6, a question regarding the Betti numbers for $\mathbb{K}[G]$ is posed.

1. Preliminaries

In this section, we provide a brief review of the notation and fundamental facts that will be utilized later on.

1.1 Betti numbers and Canonical modules

Let $R := \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in variables $x_1, \ldots, x_n$, which is standard graded. For a finitely generated graded $R-$module M, there exists the minimal graded free resolution of M that has the form:

$$(\S)$$

$$0 \to \bigoplus_{j \in \mathbb{Z}} R[-j]^{\beta_{p,j}(M)} \to \cdots \to \bigoplus_{j \in \mathbb{Z}} R[-j]^{\beta_{1,j}(M)} \to \bigoplus_{j \in \mathbb{Z}} R[-j]^{\beta_{0,j}(M)} \to M \to 0$$

. Here, $R[-j]$ is the cyclic free R-module generated in degree j. The number $\beta_{i,j}(M) := \dim_{\mathbb{K}} \mathrm{Tor}_i^R(M, \mathbb{K})_j$ is called the (i,j)-th *graded Betti number* of M and $\beta_i(M) := \sum_{j \in Z} \beta_{i,j}$ is called the i-th *total Betti number* of M. Many homological invariants of M can be defined in terms of its minimal graded free resolution. The *Castelnuovo-Mumford regularity* and *projective dimension* of M are defined to be

$$\mathrm{reg}(M) := \max \left\{ j - i \mid \beta_{i,j}(M) \neq 0 \right\}$$

and

$$\mathrm{pdim}(M) := \max \left\{ i \mid \beta_{i,j}(M) \neq 0 \text{ for some } j \right\}.$$

Denote $\mathrm{pdim}(M)$ by p. Then, $\beta_p(M)$ and $\beta_{p,j}(M), j \in Z$ are referred to as the t*op total Betti number* and the *top graded Betti numbers* of M, respectively.

By applying the functor $\mathrm{Hom}_R(-, R[-n])$ to the sequence (SS), we obtain the following complex:

$$0 \to \mathrm{Hom}_R(F_0, R[-n]) \to \mathrm{Hom}_R(F_1, R[-n]) \to \cdots$$

$$\to \mathrm{Hom}_R(F_p, R[-n]) \to \mathrm{Ext}_R^p(M, R[-n]) \to 0.$$

Here, F_i denotes the free module $\bigoplus_{j \in \mathbb{Z}} R[-j]^{\beta_{i,j}(M)}$. Assume further that M is Cohen-Macaulay. Then, it follows from the local duality (see [1]) that the above complex is exact and so it is a minimal free resolution of $\mathrm{Ext}_R^p(M, R[-n])$. The module $\mathrm{Ext}_R^p(M, R[-n])$, also denoted by ω_M, is called the *canonical module* of M. Note that

$$\mathrm{Hom}_R(F_i, R[-n]) = \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_R(R[-j]^{\beta_{i,j}(M)}, R[-n]) = \bigoplus_{j \in \mathbb{Z}} R[-n+j]^{\beta_{i,j}(M)}.$$

Based on these discussions, we can derive the following well-known result.

Lemma 1.1. *Let M be a Cohen-Macaulay graded $R = \mathbb{K}[x_1, \ldots, x_n]$-module, and ω_M its canonical module. Assume $p = \mathrm{pdim}(M)$. Then $\beta_{i,j}(\omega_M) = \beta_{p-i,n-j}(M)$ for all i, j.*

The *Cohen-Macaulay type* of a finitely generated Cohen-Macaulay R-module M is defined to be the number

$$\mathrm{type}(M) := \beta_p(M) = \beta_0(\omega_M),$$

where p is the projective dimension of M. In the following, we will consider the case when $M = \mathbb{K}[G]$ as a $\mathbb{K}[E(G)]$-module.

1.2. **Toric ideals of graphs.** Let G be a simple graph, i.e., a finite graph without loops and multiple edges, with vertex set $V(G)$ and edge set $E(G)$. A *matching* of G is a subset $M \subset E(G)$ for which $e \cap e' = \emptyset$ for $e \neq e'$ belonging to M. The *matching number*, denoted by $\mathrm{mat}(G)$, is the maximal cardinality of matchings of G. Recall that a walk of G of length q is a subgraph W of G such that $E(W) = \{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{q-1}, v_q\}\}$, where $v_0, v_1, \ldots, v_q$ are vertices of G. A walk W of G is even if q is even, and it is closed if $v_0 = v_q$. A cycle is a special closed walk with edge set $\{\{v_0, v_1\}, \{v_1, v_2\}, \{v_{q-1}, v_q = v_0\}\}$ such that $v_1, \ldots, v_q$ are pairwise distinct and $q \geq 3$. A cycle is called *even* (resp. *odd*) if q is even (resp. odd). For a subset W of $V(G)$, the *induced subgraph*G_W is the graph with vertex set W and for every pair $x, y \in W$, they are adjacent in G_W if and only if they are adjacent in G.

The generators of the toric ideal of I_G are binomials which are tightly related to even closed walks in G. Given an even closed walk W of G with

$$E(W) = \{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{2q-2}, v_{2q-1}\}, \{v_{2q-1}, v_0\}\},$$

we associate W with the binomial defined by

$$f_W := \prod_{j=1}^{q} e_{2j-1} - \prod_{j=1}^{q} e_{2j},$$

where $e_j = \{v_{j-1}, v_j\}$ for $1 \leq j \leq 2q-1$ and $e_{2q} = \{v_{2q-1}, v_0\}$. A binomial $f = u - v \in I_G$ is called a *primitive binomial* if there is no binomial $g = u' - v' \in I_G$ such that $u' \mid u$ and $v' \mid v$. An even closed walk W of G is a *primitive even closed walk* if its associated binomial f_W is a primitive binomial in I_G. It is known that the set

$$\{f_W \colon W \text{ is a primitive even closed walks of } G\}$$

is the universal Gröbner base of I_G by e.g. [16, Proposition 10.1.10] or [4, Proposition 5.19]. In particular, it is a Gröbner base of I_G with respect to any monomial order. The set of primitive even walks of a graph G was described in [12] explicitly.

Lemma 1.2. [12, Lemma 5.11] *A primitive even closed walk Γ of G is one of the following:*

i. Γ *is an even cycle of* G;

ii. $\Gamma = (C_1, C_2)$, *where each of* C_1 *and* C_2 *is an odd cycle of* G *having exactly one common vertex;*

iii. $\Gamma = (C_1, \Gamma_1, C_2, \Gamma_2)$, *where each of* C_1 *and* C_2 *is an odd cycle of* G *with* $V(C_1) \cap V(C_2) = \emptyset$ *and where* Γ_1 *and* Γ_2 *are walks of* G *of the forms* $\Gamma_1 = (e_{i_1}, \ldots, e_{i_r})$ *and* $\Gamma_1 = (e_{i_1}^{'}, \ldots, e_{i_r}^{'})$ *such that* Γ_1 *combines* $j \in e_{i_1} \cap e_{i_r}^{'} \cap V(C_1)$ *with* $j^{'} \in e_{i_r} \cap e_{i_1}^{'} \cap V(C_2)$ *and* Γ_2 *combines* $j^{'}$ *with* j. *Furthermore, none of the vertices belonging to* $V(C_1) \cup V(C_2)$ *appears in each of* $e_{i_1} \setminus \{j\}, e_{i_2}, \ldots, e_{i_{r-1}}, e_{i_r} \setminus \{j^{'}\}, e_{i_1}^{'} \setminus \{j\}, e_{i_2}^{'}, \ldots, e_{i_{r-1}}^{'}, e_{i_r}^{'} \setminus \{j^{'}\}$.

We would like to note that in (iii) the sum of lengths of Γ_1 and Γ_2 must be even in order to ensure it is indeed an even closed walk.

1.3. **Edge Cones and Canonical modules.** Let G be a simple graph with vertex set $V(G) = \{1, \ldots, n\}$ and edge set $E(G)$. For any $f = \{i, j\} \in E(G)$ denote $v_f = \mathbf{e}_i + \mathbf{e}_j$, where $\mathbf{e}_i$ is the ith unit vector of $\mathbb{R}^n$. The edge cone of G, denoted by $\mathbb{R}_+(G)$, is defined to be the cone of $\mathbb{R}^n$ generated by $\{v_f \mid f \in E(G)\}$. In other words,

$$\mathbb{R}_+(G) = \{ \sum_{f \in E(G)} a_f v_f \mid a_f \in \mathbb{R}_+ \text{ for all } f \in E(G)\}.$$

If G satisfies the odd-cycle condition, then the edge ring $\mathbb{K}[G]$ is normal, see [14], and particularly, $\mathbb{K}[G]$ is Cohen-Macaulay, see [1, Theorem 6.3.5]. It follows that the ideal of $\mathbb{K}[G]$ generated all the monomials x^α with $\alpha \in \mathbb{Z}^n \cap \mathrm{relint}(\mathbb{R}_+(G))$ is the canonical module of $\mathbb{K}[G]$, see e.g. [1, section 6.3] for the details.

Let us describe the cone $\mathbb{R}_+(G)$ in terms of linear inequalities. For the description, we need to introduce some more notions on graphs.

- For a subset $W \subset V(G)$, let $G \setminus W$ be the subgraph induced on $V(G) \setminus W$. If $W = \{k\}$, then we write $G \setminus k$ instead of $G \setminus \{k\}$.
- For $j \in V(G)$, let $N_G(j) = \{i \in V(G) \mid \{i, j\} \in E(G)\}$, and for any subset $W \subset V(G)$, let $N_G(W) = \bigcup_{k \in W} N_G(k)$.
- A non-empty subset $T \subset V(G)$ is called an *independent set* if $\{j, k\} \notin E(G)$ for any $j, k \in T$.
- We call a vertex j of G *regular* if each connected component of $G \setminus j$ contains an odd cycle.
- We say that an independent set T of $V(G)$ is a *fundamental* set if
  - the bipartite graph on the vertex set $T \cup N_G(T)$ with the edge set $E(G) \cap \{\{j, k\} \mid j \in T, k \in N_G(T)\}$ is connected, and
  - either $T \cup N_G(T) = V(G)$ or each of the connected components of the graph $G \setminus (T \cup N_G(T))$ contains an odd cycle.

It follows from [15, Theorem 3.2] or ([14, Theorem 1.7 (a)]) that $\mathbb{R}_+(G)$ consists of the elements $(x_1, \ldots, x_n) \in \mathbb{R}^n$ satisfying all the following inequalities:

$$(\Delta)$$

$$x_u \geq 0 \text{ for any regular vertex } u;$$

$$\sum_{v \in N_G(T)} x_v \geq \sum_{u \in T} x_u \quad \text{for any fundamental set } T.$$

1.4. **E-K splitting.** Based on the approach in [5], Eliahou and Kervaire introduced the notion of splitting a monomial ideal.

Definition 1.3. Let I, J and K be monomial ideals such that $G(I)$, the unique set of minimal generators of I, is the disjoint union of $G(J)$ and $G(K)$. Then $I = J + K$ is an **Eliahou-Kervaire splitting** (abbreviated as "E-K splitting") if there exists a splitting function

$$G(J \cap K) \to G(J) \times G(K)$$

sending $w \mapsto (\phi(w), \psi(w))$ such that

1. $w = \mathrm{lcm}(\phi(w), \psi(w))$ for all $w \in G(J \cap K)$, and
2. for every subset $\emptyset \neq S \subset G(J \cap K)$, $\mathrm{lcm}(\phi(S))$ and $\mathrm{lcm}(\psi(S))$ strictly divide $\mathrm{lcm}(S)$.

Lemma 1.4. [5, Proposition 3.1] Let $I = J + K$ be an E-K splitting. Then, for all $i \geq 0$,

$$(*)$$

$$\beta_i(I) = \beta_i(J) + \beta_i(K) + \beta_{i-1}(J \cap K), \beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K),$$

where $\beta_{-1,j}(J \cap K) = 0$ for all j by convention.

2. A classification of compact graphs

In this section, we aim to classify all the compact graphs up to the essentially same edge rings. We start by presenting the following straightforward observation, which we will not provide a proof for.

Lemma 2.1. *Let* x_1 *be a vertex of degree one in a simple graph* G *and let* $G^{'}$ *be the graph obtained from* G *by removing* x_1. *Then* I_G *and* $I_{G^{'}}$ *have the same set of minimal binomial generators. More precisely, if* G *has edge set* $\{e_1, \ldots, e_r\}$ *with* $x_1 \in e_r$, *then* $I_G = I_{G^{'}} \cdot \mathbb{K}[e_1, \ldots, e_r]$ *and* $\mathbb{K}[G] \cong \mathbb{K}[G^{'}] \otimes_{\mathbb{K}} \mathbb{K}[e_r]$. *Here, both* $\mathbb{K}[e_1, \ldots, e_r]$ *and* $\mathbb{K}[e_r]$ *are polynomial rings by definitions.*

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This observation indicates that the removal of vertices with a degree of one does not essentially alter the edge ring. Given a simple graph G, by iteratively removing all vertices of degree one, we obtain a new graph, denoted as G_0, where every remaining vertex has a degree greater than one. It is evident that G and G_0 essentially share the same edge ring by Lemma 2.1. From this point forward, we will solely focus on simple graphs in which every vertex has a degree greater than one.

Definition 2.2. Let G be a connected simple graph where every vertex has a degree greater than one. We call G to be a *compact* graph if it does not contain any even cycles and satisfies the odd-cycle condition.

Proposition 2.3. *Let G be a compact graph. Then there exist at most three vertices of degree ≥ 3 in G.*

Proof: First, we observe the following easy but useful fact: Given distinct cycles C_1, C_2 of G with $V(C_1) \cap V(C_2) \neq \emptyset$, one has $V(C_1) \cap V(C_2)$ is a singleton. This is because if $V(C_1) \cap V(C_2)$ contains more than one vertex then G must contain an even cycle. As a result, we see that every cycle of G is an induced cycle of G and every edge of G belongs to at most one cycle of G. We label this as the first assertion.

Next, we will prove the second assertion, which states that every vertex of G belongs to at least one cycle. Assume on the contrary that there is a vertex v which does not belong to any cycle. Then, since $\deg(v) \geq 2$ for each $v \in V(G)$, there is a path $v_1 - - \cdots - - v_s = v - - - \cdots - - v_t$ such that v_1 and v_t belong to the cycles C_1 and C_t respectively. It is clear that C_1 and C_k are disjoint, for otherwise v belongs to a cycle. This implies there is an edge connecting C_1 and C_t by the odd-cycle condition, which is also a contradiction. This proves the second assertion.

For convenience, we say a cycle of G to be *almost-isolated* if it has exactly one vertex of degree ≥ 3. We can then prove the third assertion, which states that if v is a vertex with $\deg(v) \geq 3$, then v belongs to at least one almost-isolated cycle C. For this, we let $v_1, v_2, \ldots, v_k$ be all the vertices which are adjacent to v. Note that $k \geq 3$. In view of the second assertion we have proved, we may assume

$$C: v_1 - - v - - v_2 - - u_1 - - - \cdots - - u_{2\ell} - - v_1$$

is an odd cycle. If C is almost-isolated, we are done. Suppose now that C is not almost-isolated. In this case, we may assume C_k is a cycle containing v_k as a vertex by the second assertion. We consider the following cases:

Case 1: $\deg(u_i) \geq 3$ for some $i \in \{1, 2, \ldots, 2\ell\}$. Say $i = 1$ and that $u \notin V(C)$ is a vertex adjacent to u_1. Let C_1 be a cycle containing u. Then either there is an edge connecting C_1 and C_k, or C_1 and C_k shares a common vertex. In both cases, there is a path that connects v_1 and v_k, but does not pass through v. Thus, the edge $e = \{v, v_1\}$ not only belongs to C, but also to a cycle containing the vertex v_k. This is impossible by the first assertion.

Case 2: $\deg(v_i) \geq 3$ *for some* $i \in \{1, 2\}$. Say $i = 1$ and that $u \notin V(C)$ is a vertex adjacent to v_1. Let C_1 be a cycle containing u. If $v \notin C_k$, then $e = \{v, v_1\}$ also belongs to a cycle containing v_k by the odd-cycle condition, a contradiction. So we only need to consider the case that $v \in C_k$. If C_k is almost-isolated, we are also done. If C_k is not almost-isolated, we let $w \in V(C_k)$ other than v with degree ≥ 3 and let w_1 be a vertex adjacent to w with $w_1 \notin V(C_k)$. By the second assertion, there is a cycle, denoted by C_2, which contains w_1. Since C_1 and C_2 are connected by an edge, we see that $e = \{v, v_1\}$ belongs to an cycle containing w. This is impossible according to the first assertion.

Thus, the third assertion has been proved. From this, it follows that if $v_1, \ldots, v_k$ are vertices of degree ≥ 3, then the induced graph on the set $\{v_1, \ldots, v_k\}$ is a complete graph. Hence $k \leq 3$. □

Classification and Notation: For the sake of simplicity, a vertex of a compact graph is called a *big* vertex if it has a degree greater than 2. According to Proposition 2.3, compact graphs can be categorized into four classes, each determined by the number of big vertices. More specifically, we say a compact graph falls into type i if it possesses i big vertices for $i = 0, 1, 2, 3$.

A compact graph of type 0 is simply an odd cycle.

A compact graph of type 1 is a finite collection of odd cycles that share a vertex.

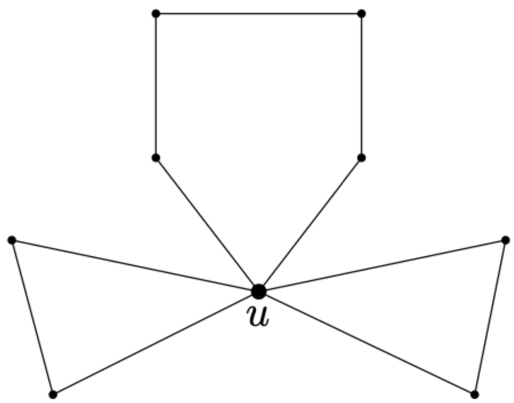

Figure 1. The graph $A_{(1,2,1)}$

A compact graph of type 2 consists of two disjoint compact graphs of type 1, where the two big vertices are connected either by an edge or by an edge as well as a path of even length.

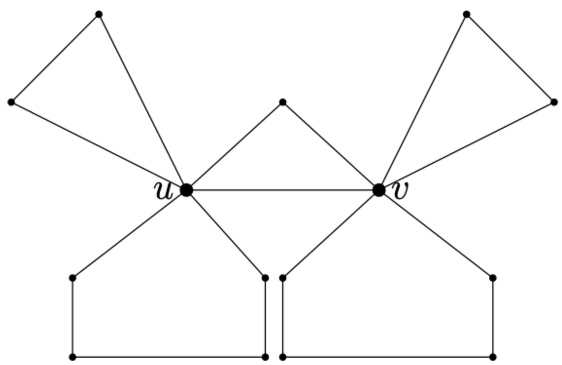

Figure 2. The graph $B^2_{(2,1):(2,1)}$

A compact graph of type 3 consists of three disjoint compact graphs of type 1, where every pair of big vertices is connected by an edge.

Suppose $\underline{p} = (p_1, \dots, p_m)$, $\underline{q} = (q_1, \dots, q_n)$ and $\underline{r} = (r_1, \dots, r_k)$ are positive integral vectors with dimensions m, n and k respectively. We denote a compact graph of type 1, where the odd cycles have lengths $2p_1 + 1, \dots 2p_m + 1$ respectively, as $A_{\underline{p}}$ or $A_{p_1, \dots, p_m}$.

By $B^0_{\underline{p}, \underline{q}}$ we mean a compact graph of type 2 where the two disjoint compact graphs of type 1 that compose it are $A_{\underline{p}}$ and $A_{\underline{q}}$ and where the two big vertices are

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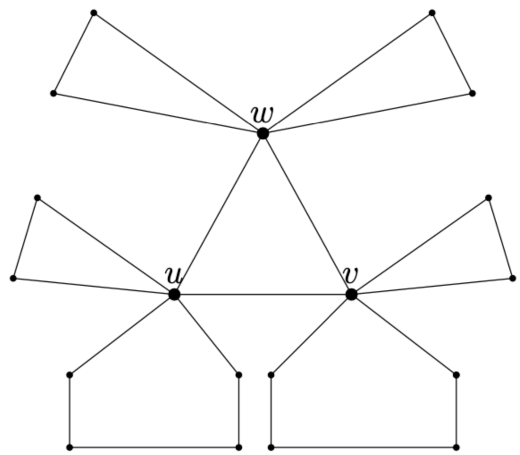

Figure 3. The graph $C_{(2,1):(1,1):(2,1)}$

connected by an edge. Furthermore, if $s > 0$ is an even number, then $B^s_{\underline{p};\underline{q}}$ represents the graph obtained by appending a path of length s connecting two big vertices to $B^0_{\underline{p}:\underline{q}}$.

A compact graph of type 3 is denoted by $C_{\underline{p};\underline{q};\underline{r}}$ if the three disjoint compact graphs of type 1 that make up it are $A_{\underline{p}}$, $A_{\underline{q}}$ and $A_{\underline{r}}$ respectively.

3. Universal gröbner bases and initial ideals.

In this section, we will discuss the universal Gröbner bases and initial ideals of toric ideals of compact graphs, with respect to a specific monomial order. The main objective of this section is to identify suitable monomial orders that yield initial ideals with a favorable E-K splitting, as demonstrated in the subsequent section.

3.1 **Compact graphs of type 1.** Given positive integers $m \geq 2$ and $p_1, \ldots, p_m$, we use A to denote the graph $A_{p_1, \ldots, p_m}$ for short. Thus A has vertex set

$$V(A) = \{u\} \cup \{u_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq 2p_i\}$$

and edge set

$$E(A) = \{\{u_{i,j}, u_{i,j+1}\} \mid 1 \leq i \leq m, 1 \leq j \leq 2p_i - 1\}$$
$$\cup \{\{u, u_{i,1}\}, \{u, u_{i,2p_i}\} \mid 1 \leq i \leq m\}.$$

We label the edges of A as follows. For $i \in \{1, \ldots, m\}$, we let $e_{i,1} = \{u, u_{i,1}\}$ and $e_{i,2p_i+1} = \{u, u_{i,2p_i}\}$. For $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, 2p_i - 1\}$ let $e_{i,j+1} = \{u_{i,j}, u_{i,j+1}\}$. For $1 \leq i, j \leq m$, we put

$$e_i^{'} = e_{i,1}e_{i,3}\cdots e_{i,2p_i+1} \text{ and } e_j^{''} = e_{j,2}e_{j,4}\cdots e_{j,2p_j}.$$

Lemma 3.1. *For any integers $m \geq 2$ and positive integers $p_1, \ldots, p_m$, the universal Gröbner basis for the toric ideal I_A is given by*

$$\mathcal{G} = \{e_i^{'}e_j^{''} - e_i^{''}e_j^{'} \mid 1 \leq i < j \leq m\}.$$

Proof: It follows from [14, Lemma 3.2] together with [16, Proposition 10.1.10]. □

Going forward, we work in the standard graded polynomial ring

$$\mathbb{K}[E(A)] = \mathbb{K}[e_{1,1}, \ldots, e_{1,2p_1+1}, \ldots\ldots, e_{m,1}, \ldots, e_{m,2p_m+1}].$$

Let $<$ denote the lexicographic monomial order on $\mathbb{K}[E(A)]$ satisfying

$$e_{1,1} < \cdots < e_{1,2p_1+1} < \cdots\cdots < e_{m,1} < \cdots < e_{m,2p_m+1},$$

and let J_A denote the initial ideal of I_A with respect to the monomial order $<$.

Proposition 3.2. *The minimal set of monomial generators of J_A is given by*

$$\mathcal{M} = \left\{ e_i'' e_j' \mid 1 \le i < j \le m \right\}.$$

Proof: Note that $e_i' e_j'' < e_i'' e_j'$ for $1 \le i < j \le m$, we can deduce from Lemma 3.1 that J_A is generated by $\mathcal{M}$. The minimality of $\mathcal{M}$ can be checked directly. □

3.2. **Compact graphs of type 2.** Assume that n, m and $p_1, \dots, p_m, q_1, \dots, q_n$ are given positive integers. Let $s \ge 0$ be an even number. We use B denote the graph $B^s_{p_1, \dots, p_m : q_1, \dots, q_n}$ for short. Then, we may assume that B has vertex set

$$V(B) = \{u, v\} \cup \{w_1, \dots, w_{s-1}\}$$
$$\cup \{u_{i,j} \mid 1 \le i \le m, 1 \le j \le 2p_i\} \cup \{v_{i,j} \mid 1 \le i \le n, 1 \le j \le 2q_i\}$$

and edge set

$$E(B) = \{\{u_{i,j}, u_{i,j+1}\} \mid 1 \le i \le m, 1 \le j \le 2p_i - 1\}$$
$$\cup \{\{u, u_{i,1}\}, \{u, u_{i,2p_i}\} \mid 1 \le i \le m\}$$
$$\cup \{\{u, v\}, \{u, w_1\}, \{v, w_{s-1}\}\} \cup \{\{w_i, w_{i+1}\} \mid 1 \le i \le s-2\}$$
$$\cup \{\{v_{i,j}, v_{i,j+1}\} \mid 1 \le i \le n, 1 \le j \le 2q_i - 1\}$$
$$\cup \{\{v, v_{i,1}\}, \{v, v_{i,2q_i}\} \mid 1 \le i \le n\}.$$

The edges of B are labeled as follows. For $i \in \{1, \dots, m\}$ let $e_{i,1} = \{u, u_{i,1}\}$ and $e_{i,2p_i+1} = \{u, u_{i,2p_i}\}$. For $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, 2p_i - 1\}$ let $e_{i,j+1} = \left\{u_{i,j}, u_{i,j+1}\right\}$. Let $x = \{u, v\}$, $x_1 = \{u, w_1\}$ and $x_s = \{v, w_{s-1}\}$. For $i \in \{1, \dots, s-2\}$ let $x_{i+1} = \{w_i, w_{i+1}\}$. For $i \in \{1, \dots, n\}$ let $f_{i,1} = \{v, v_{i,1}\}$ and $f_{i,2q_i+1} = \{v, v_{i,2q_i+1}\}$. For $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, 2q_i - 1\}$ let $f_{i,j+1} = \{v_{i,j}, v_{i,j+1}\}$.

We put

$$e_i' = e_{i,1} e_{i,3} \cdots e_{i,2p_i+1} \text{ and } e_i'' = e_{i,2} e_{i,4} \cdots e_{i,2p_i},$$
$$f_i' = f_{i,1} f_{i,3} \cdots f_{i,2q_i+1} \text{ and } f_i'' = f_{i,2} f_{i,4} \cdots f_{i,2q_i},$$

and put

$$x' = x_1 x_3 \cdots x_{s-1}, \text{ and } x'' = x_2 x_4 \cdots x_s.$$

Note that if $s = 0$ then both x' and x'' vanish.

Lemma 3.3. *For any positive integers m, n and $p_1, \dots, p_m, q_1, \dots, q_n$ and an integer $s \ge 0$, the universal Gröbner basis of I_B is given by $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4 \cup \mathcal{G}_5 \cup \mathcal{G}_6$, where*

1. $\mathcal{G}_1 = \{e_i' e_j'' - e_i'' e_j' \mid 1 \le i < j \le m\}$,
2. $\mathcal{G}_2 = \{f_i' f_j'' - f_i'' f_j' \mid 1 \le i < j \le n\}$,
3. $\mathcal{G}_3 = \{e_i' f_j' - e_i'' x^2 f_j'' \mid 1 \le i \le m, 1 \le j \le n\}$,
4. $\mathcal{G}_4 = \{e_i' x''^2 f_j' - e_i' x' 2 f_j' \mid 1 \le i \le m, 1 \le j \le n\}$,
5. $\mathcal{G}_5 = \{e_i' x'' - e_i' x' x \mid 1 \le i \le m\}$,and
6. $\mathcal{G}_6 = \{f_i' x' - f_i' x'' x \mid 1 \le i \le n\}$.

It should be noted that $\mathcal{G}_4, \mathcal{G}_5$ and $\mathcal{G}_6$ vanish if $s = 0$.

Proof: By [12, Lemma 5.11], every primitive even closed walk of B is one of the followings:

- $(e_{i,1}, \dots, e_{i,2p_i+1}, e_{j,1}, \dots, e_{j,2p_j+1})$, where $1 \le i < j \le m$,
- $(f_{i,1}, \dots, f_{i,2q_i+1}, f_{j,1}, \dots, f_{j,2q_j+1})$, where $1 \le i < j \le n$,
- $(e_{i,1}, \dots, e_{i,2p_i+1}, x, f_{j,1}, \dots, f_{j,2q_j+1}, x)$, where $1 \le i \le m$, $1 \le j \le n$,
- $(e_{i,1}, \dots, e_{i,2p_i+1}, x_1, \dots, x_s, f_{j,1}, \dots, f_{j,2q_j+1}, x_s, \dots, x_1)$, where $1 \le i \le m$, $1 \le j \le n$,
- $(e_{i,1}, \dots, e_{i,2p_i+1}, x_1, \dots, x_s, x)$, where $1 \le i \le m$, and
- $(f_{i,1}, \dots, f_{i,2q_i+1}, x, x_1, \dots, x_s)$, where $1 \le i \le n$.

The result now follows from [16, Proposition 10.1.10]. □

Let $<$ denote the lexicographic monomial ordering on the polynomial ring $\mathbb{K}[E(B)]$ satisfying

$$e_{1,1} < \cdots < e_{1,2p_1+1} < \cdots\cdots < e_{m,1} < \cdots < e_{m,2p_m+1} < x < x_1 < \cdots < x_s < f_{1,1} < \cdots < f_{1,2q_1+1} < \cdots\cdots < f_{n,1} < \cdots < f_{n,2q_n+1}.$$

and let J_B be the initial ideal of I_B with respect to this order.

Proposition 3.4. *The minimal set of monomial generators of J_B is given by $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{M}_4 \cup \mathcal{M}_5$, where*

1. $\mathcal{M}_1 = \left\{ e_i'' e_j' \mid 1 \le i < j \le m \right\}$,

2. $\mathcal{M}_2 = \left\{ f_i'' f_j' \mid 1 \le i < j \le n \right\}$,

3. $\mathcal{M}_3 = \left\{ e_i' f_j' \mid 1 \le i \le m, 1 \le j \le n \right\}$,

4. $\mathcal{M}_4 = \left\{ e_i' x'' \mid 1 \le i \le m \right\}$, and

5. $\mathcal{M}_5 = \{ f_i' x' \mid 1 \le i \le n \}$.

It should be noted that $\mathcal{M}_4$ and $\mathcal{M}_5$ vanish if $s = 0$.

Proof: That $\mathcal{M}$ is a generating set with respect to the given order follows from Lemma 3.3. That it is minimal follows from the fact that none of the monomials are divided by any of the others. □

3.3. **Compact graphs of type 3.** Given positive integers m, n, k, as well as the tuples $\underline{p} = (p_1, \dots, p_m)$, $\underline{q} = (q_1, \dots, q_n)$ and $\underline{r} = (r_1, \dots, r_k)$, we denote the graph $C_{\underline{p} \cdot \underline{r}}$ as C for brevity. Here, p_i, q_i, r_i are all positive integers. By definition, we may assume C has vertex set

$$V(C) = \{\{u, v, w\} \cup \{u_{i,j} \mid 1 \le i \le m, 1 \le j \le 2p_i\}$$
$$\cup \{v_{i,j} \mid 1 \le i \le n, 1 \le j \le 2q_i\} \cup \{w_{i,j} \mid 1 \le i \le k, 1 \le j \le 2r_i\},$$

and edge set

$$E(C) = \{\{u_{i,j}, u_{i,j+1}\} \mid 1 \le i \le m, 1 \le j \le 2p_i - 1\}$$
$$\cup \{\{u, u_{i,1}\}, \{u, u_{i,2p_i}\} \mid 1 \le i \le m\}$$
$$\cup \{\{v_{i,j}, v_{i,j+1}\} \mid 1 \le i \le n, 1 \le j \le 2q_i - 1\}$$
$$\cup \{\{v, v_{i,1}\}, \{v, v_{i,2q_i}\} \mid 1 \le i \le n\}$$
$$\cup \{\{w_{i,j}, w_{i,j+1}\} \mid 1 \le i \le k, 1 \le j \le 2r_i - 1\}$$
$$\cup \{\{w, w_{i,1}\}, \{w, w_{i,2r_i}\} \mid 1 \le i \le k\}$$
$$\cup \{\{u, v\}, \{v, w\}, \{w, u\}\}.$$

We assign labels to the edges of C as follows: For $i \in \{1, \dots, m\}$, let $e_{i,1} = \{u, u_{i,1}\}$ and $e_{i,2p_i+1} = \{u, u_{i,2p_i+1}\}$. For $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, 2p_i - 1\}$, let $e_{i,j+1} = \{u_{i,j}, u_{i,j+1}\}$.

For $i \in \{1, \dots, n\}$, let $f_{i,1} = \{v, v_{i,1}\}$ and $f_{i,2q_i+1} = \{v, v_{i,2q_i+1}\}$. For $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, 2q_i - 1\}$, let $f_{i,j+1} = \{v_{i,j}, v_{i,j+1}\}$.

For $i \in \{1, \dots, k\}$, let $g_{i,1} = \{w, w_{i,1}\}$ and $g_{i,2r_i+1} = \{w, w_{i,2r_i+1}\}$. For $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, 2r_i - 1\}$, let $g_{i,j+1} = \{w_{i,j}, w_{i,j+1}\}$.

Furthermore, we define $x = \{u, v\}$, $y = \{v, w\}$, and $z = \{w, u\}$.

Lemma 3.5. *For any integers m, n, k and $p_1, \dots, p_m, q_1, \dots, q_n, r_1, \dots, r_k$, the universal Gröbner basis of I_C is given by in $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4 \cup \mathcal{G}_5 \cup \mathcal{G}_6 \cup \mathcal{G}_7 \cup \mathcal{G}_8 \cup \mathcal{G}_9 \cup \mathcal{G}_{10} \cup \mathcal{G}_{11} \cup \mathcal{G}_{12}$, where*

1. $\mathcal{G}_1 = \{e_i' e_j'' - e_i'' e_j' \mid 1 \le i < j \le m\}$,

2. $\mathcal{G}_2 = \{f_i' f_j'' - f_i'' f_j' \mid 1 \le i < j \le n\}$,

3. $\mathcal{G}_3 = \{g_i' g_j'' - g_i'' g_j' \mid 1 \le i < j \le k\}$,

4. $\mathcal{G}_4 = \{e_i' f_j' - e_i'' x^2 f_j'' \mid 1 \le i \le m, 1 \le j \le n\}$,

5. $\mathcal{G}_5 = \{f_i' g_j' - f_i'' y^2 g_j'' \mid 1 \le i \le n, 1 \le j \le k\}$,

6. $\mathcal{G}_6 = \{g_i' e_j' - g_i'' z^2 e_j'' \mid 1 \le i \le k, 1 \le j \le m\}$,

7. $\mathcal{G}_7 = \{e_i' y^2 f_j'' - e_i'' z^2 f_j' \mid 1 \le i \le m, 1 \le j \le n\}$,

8. $\mathcal{G}_8 = \{f_i' z^2 g_j'' - f_i'' x^2 g_j' \mid 1 \le i \le n, 1 \le j \le k\}$,

9. $\mathcal{G}_9 = \{g_i' x^2 e_j'' - g_i'' y^2 e_j' \mid 1 \le i \le k, 1 \le j \le m\}$,

10. $\mathcal{G}_{10} = \{e_i' y - e_i'' zx \mid 1 \le i \le m\}$,

11. $\mathcal{G}_{11} = \{f_i' z - f_i'' xy \mid 1 \le i \le n\}$, and

12. $\mathcal{G}_{12} = \{g_i' x - g_i'' yz \mid 1 \le i \le k\}$.

Proof: In view of [12, Lemma 5.11], every primitive even closed walk of C is one of the followings:

- $(e_{i,1}, \dots, e_{i,2p_i+1}, e_{j,1}, \dots, e_{j,2p_j+1})$, where $1 \le i < j \le m$,
- $(f_{i,1}, \dots, f_{i,2q_i+1}, f_{j,1}, \dots, f_{j,2q_j+1})$, where $1 \le i < j \le n$,
- $(g_{i,1}, \dots, g_{i,2r_i+1}, g_{j,1}, \dots, g_{j,2r_j+1})$, where $1 \le i < j \le k$,
- $(e_{i,1}, \dots, e_{i,2p_i+1}, x, f_{j,1}, \dots, f_{j,2q_j+1}, x)$, where $1 \le i \le m, 1 \le j \le n$,

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- $(f_{i,1}, \ldots, f_{i,2q_i+1}, y, g_{j,1}, \ldots, g_{j,2r_j+1}, y)$, where $1 \le i \le n$, $1 \le j \le k$,
- $(g_{i,1}, \ldots, g_{i,2r_i+1}, z, e_{j,1}, \ldots, e_{j,2p_j+1}, z)$, where $1 \le i \le k$, $1 \le j \le m$,
- $(e_{i,1}, \ldots, e_{i,2p_i+1}, z, y, f_{j,1}, \ldots, f_{j,2q_j+1}, y, z)$, where $1 \le i \le m$, $1 \le j \le n$,
- $\left(f_{i,1}, \ldots, f_{i,2q_i+1}, x, z, g_{j,1}, \ldots, g_{j,2r_j+1}, z, x\right)$, where $1 \le i \le n$, $1 \le j \le k$,
- $\left(g_{i,1}, \ldots, g_{i,2r_i+1}, z, x, e_{j,1}, \ldots, e_{j,2p_j+1}, x, y\right)$, where $1 \le i \le k$, $1 \le j \le m$,
- $\left(e_{i,1}, \ldots, e_{i,2p_i+1}, z, y, x\right)$, where $1 \le i \le m$,
- $\left(f_{i,1}, \ldots, f_{i,2q_i+1}, x, z, y\right)$, where $1 \le i \le n$, and
- $\left(g_{i,1}, \ldots, g_{i,2r_i+1}, y, x, z\right)$, where $1 \le i \le k$.

Now the result follows from [16, Proposition 10.1.10]. $\square$

Going forward, we work in the standard graded polynomial ring $\mathbb{K}[E(C)]$, where the variables (i.e., the edges of C) is ordered as follows:

$$e_{1,1} < \cdots < e_{1,2p_1+1} < \cdots\cdots < e_{m,1} < \cdots < e_{m,2p_m+1} < x < z < y < f_{1,1} < \cdots < f_{1,2q_1+1} < \cdots\cdots < f_{n,1} < \cdots < f_{n,2q_n+1} < g_{1,1} < \cdots < g_{1,2r_1+1} < \cdots\cdots < g_{k,1} < \cdots < g_{k,2r_k+1}.$$

Let J_C denote the initial ideal of I_C with respect to the lexicographic monomial ordering $<$ on $\mathbb{K}[E(C)]$ induced by the above order of variables.

By putting:

$$e_i' = e_{i,1}e_{i,3}\cdots e_{i,2p_i+1} \text{ and } e_i'' = e_{i,2}e_{i,4}\cdots e_{i,2p_i},$$

$$f_j' = f_{j,1}f_{j,3}\cdots f_{j,2q_j+1} \text{ and } f_j'' = f_{j,2}f_{j,4}\cdots f_{j,2q_j},$$

and

$$g_\ell' = g_{\ell,1}g_{\ell,3}\cdots g_{\ell,2r_\ell+1} \text{ and } g_\ell'' = g_{\ell,2}g_{\ell,4}\cdots g_{\ell,2r_\ell},$$

where $1 \le i \le m$, $1 \le j \le n$ and $1 \le \ell \le k$, we obtain the following result.

Proposition 3.6. *The minimal set of monomial generators of J_C is given by $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{M}_4 \cup \mathcal{M}_5 \cup \mathcal{M}_6 \cup \mathcal{M}_7 \cup \mathcal{M}_8 \cup \mathcal{M}_9$, where*

i. $\mathcal{M}_1 = \left\{e_i''e_j' \mid 1 \le i < j \le m\right\}$,
ii. $\mathcal{M}_2 = \left\{f_i''f_j' \mid 1 \le i < j \le n\right\}$,
iii. $\mathcal{M}_3 = \left\{g_i''g_j' \mid 1 \le i < j \le k\right\}$,
iv. $\mathcal{M}_4 = \left\{e_i'f_j' \mid 1 \le i \le m, 1 \le j \le n\right\}$,
v. $\mathcal{M}_5 = \left\{f_i'g_j' \mid 1 \le i \le n, 1 \le j \le k\right\}$,
vi. $\mathcal{M}_6 = \left\{g_i'e_j' \mid 1 \le i \le k, 1 \le j \le m\right\}$,
vii. $\mathcal{M}_7 = \left\{e_i'y \mid 1 \le i \le m\right\}$,
viii. $\mathcal{M}_8 = \left\{f_i'z \mid 1 \le i \le n\right\}$, and
ix. $\mathcal{M}_9 = \left\{g_i'x \mid 1 \le i \le k\right\}$.

Proof: That $\mathcal{M}$ is a generating set with respect to the given order follows from Lemma 3.5. That it is minimal follows from the fact that none of the monomials are divided by any of the others. $\square$

4. Projective dimension and regularity

In this section, we aim to establish the following results. For convenience, we denote by $t(G)$ the number of induced cycles of G.

Theorem 4.1. *Let G be a compact graph. Then there is a monomial order $<$ such that*

1. $\beta_i(\mathrm{in}_<(I_G)) = (i+1)\binom{t(G)}{i+2}$ for all $i \ge 0$;
2. $\mathrm{pdim}(\mathbb{K}[E(G)]/\mathrm{in}_<(I_G)) = t(G) - 1$;
3. $\mathbb{K}[E(G)]/\mathrm{in}_<(I_G)$ is a Cohen-Macaulay ring;
4. $\mathrm{reg}(\mathbb{K}[E(G)]/\mathrm{in}_<(I_G)) = \mathrm{mat}(G)$.

Corollary 4.2. *Let Let G be a compact graph. Then*

1. $\mathrm{pdim}(\mathbb{K}[G]) = t(G) - 1$;
2. $\mathrm{reg}(\mathbb{K}[G]) = \mathrm{mat}(G)$.

It is known that if I is a graded ideal of a polynomial ring R such that $R/\mathrm{in}_<(I)$ is Cohen-Macaulay for some monomial order $<$, then R/I is also Cohen-Macaulay. Furthermore, we have $\mathrm{reg}(R/I) = \mathrm{reg}(R/\mathrm{in}_<(I))$ and $\mathrm{pdim}(R/I) = \mathrm{pdim}(R/\mathrm{in}_<(I))$. Based on these facts we could see that Corollary 4.2 follows immediately from Theorem 4.1. Regarding the proof of Theorem 4.1, we will provide it at the end of this section.

Assume through this section that m, n, k are positive integers and $\underline{p} = (p_1, \ldots, p_m)$, $\underline{q} = (q_1, \ldots, q_n)$ and $\underline{r} = (r_1, \ldots, r_k)$ are integral tuples with positive entries. Also, we write $\underline{p}' = (p_1, \ldots, p_{m-1})$, $\underline{q}' = (q_1, \ldots, q_{n-1})$, and $\underline{r}' = (r_1, \ldots, r_{k-1})$.

4.1. **type one.** In this subsection, we always use the monomial order given in Subsection 3.1, and denote the toric ideal of $A_{\underline{p}}$ and its initial ideal as I_m and J_m, respectively. Similarly, I_{m-1} and J_{m-1} represent the toric ideal of $A_{\underline{p}'}$ and its initial ideal respectively. Recall from Subsection 3.1 that $G(J_m) = \left\{e_i''e_j' \mid 1 \le i < j \le m\right\}$.

Proposition 4.3. *Denote by H_m the monomial ideal $(e_1'', \ldots, e_{m-1}'')$. Then $J_m = J_{m-1} + e_m'H_m$ is an E-K splitting. Furthermore $J_{m-1} \cap e_m'H_m = e_m'J_{m-1}$.*

Proof: First of all, it is easy to see that $G(J_m) = G(J_{m-1}) \sqcup G(e_m'H_m)$. Let us check $J_{m-1} \cap e_m'H_m = e_m'J_{m-1}$. Take any $e_i''e_j' \in G(J_{m-1})$, since $1 \le i < j \le m-1$, we have $e_m''e_i''e_j' \in e_m'H_m \cap J_{m-1}$. For the converse, take $e_m'e_{i_1}'' \in G(e_m'H_m)$ and $e_{i_2}''e_j' \in G(J_{m-1})$. Here, $1 \le i_1 \le m-1$ and $1 \le i_2 < j \le m-1$. Then,

$$\mathrm{lcm}(e_m'e_{i_1}'', e_{i_2}''e_j') \in (e_m'e_{i_2}''e_j') \subseteq e_m'J_{m-1}.$$

This shows $J_{m-1} \cap e_m'H_m = e_m'J_{m-1}$.

Next, we define functions ϕ and ψ as follows:

$$\phi : G(e_m'J_{m-1}) \to G(J_{m-1}), \quad e_m'e_i''e_j' \mapsto e_i''e_j', \quad 1 \le i < j \le m-1,$$

$$\psi : G(e_m'J_{m-1}) \to G(e_m'H_m), \quad e_m'e_i''e_j' \mapsto e_m'e_i'', \quad 1 \le i < j \le m-1.$$

(1) Let u be a minimal generator of $e_m'J_{m-1}$. Then $u = e_m'e_i''e_j'$ for some $1 \le i < j \le m-1$. It follows that

$$\mathrm{lcm}(\phi(u), \psi(u)) = \mathrm{lcm}(e_i''e_j', e_m'e_i'') = e_m'e_i''e_j' = u.$$

(2) Let $C = e_m'(e_{i_1}''e_{j_1}', \ldots, e_{i_k}''e_{j_k}')$ be a non-empty subset of $G(e_m'J_{m-1})$, where $1 \le i_q < j_q \le m-1$ for $q = 1, \ldots, k$. Then

$$\phi(C) = (e_{i_1}''e_{j_1}', \ldots, e_{i_k}''e_{j_k}') \text{ and } \psi(C) = e_m'(e_{i_1}'', \ldots, e_{i_k}'').$$

Since e_i'' and e_j' are co-prime for all $1 \le i, j \le m$, we have

$$\mathrm{lcm}(C) = e_m'\mathrm{lcm}(\phi(C)) \text{ and } \mathrm{lcm}(C) = \mathrm{lcm}(\psi(C))\mathrm{lcm}(e_{j_1}', \ldots, e_{j_k}').$$

This completes the proof. □

In the following, we will utilize the following formula without explicitly referencing it: For a finitely generated graded module M over a standard graded polynomial ring, one has

$$\max\ \{j - i \mid \beta_{i-a, j-b}(M) \ne 0\} = \max\ \{\ell + b - (k + a) \mid \beta_{k,\ell}(M) \ne 0\} = \mathrm{reg}(M) + b - a.$$

Proposition 4.4. *Let $m \ge 2$. Then $\mathrm{reg}(J_m) = \mathrm{mat}(A_{\underline{p}}) + 1$.*

Proof: It is easy to check that $\mathrm{mat}(A_{\underline{p}}) = \sum_{i=1}^{m} p_i$. We proceed with the induction on m. If $m = 2$, since J_2 is generated by a single monomial of degree $p_1 + p_2 + 1$, we obtain $\mathrm{reg}(J_2) = p_1 + p_2 + 1$.

Suppose that $m > 2$. Then, by Lemma 4.3, we have

$$(\clubsuit)$$

$$\beta_{i,j}(J_m) = \beta_{i,j}(J_{m-1}) + \beta_{i,j-p_m-1}(H_m) + \beta_{i-1,j-p_m-1}(J_{m-1}).$$

It follows that

$$\begin{aligned}\mathrm{reg}(J_m) &= \max\ \{j - i \mid \beta_{i,j}(J_{m-1}) + \beta_{i-1,j-p_m-1}(J_{m-1}) + \beta_{i,j-p_m-1}(H_m) \ne 0\} \\ &= \max\ \{\mathrm{reg}(J_{m-1}), \mathrm{reg}(J_{m-1}) + p_m, \mathrm{reg}(H_m) + p_m + 1\}.\end{aligned}$$

Note that H_m is generated by a regular sequence of degrees $p_1, \ldots, p_{m-1}$. By using the Koszul theory, we obtain $\mathrm{reg}(H_m) = \sum_{i=1}^{m-1} p_i - m + 2$. Hence,

$$\begin{aligned}\mathrm{reg}(J_m) &= \max\ \{\sum_{i=1}^{m} p_i + 1, \sum_{i=1}^{m} p_i - m + 3\} \\ &= \sum_{i=1}^{m} p_i + 1,\end{aligned}$$

as desired. □

Proposition 4.5. *Let $m \geq 2$. Then $\beta_i(J_m) = (i+1)\binom{m}{i+2}$ for all $i \geq 0$. In particular,* $\operatorname{pdim}(J_m) = m-2$.

Proof: We also employ the induction on m. The case that $m = 2$ or $i = 0$ is straightforward. If $m \geq 3$ and $i \geq 1$, then, by noting that the formula $\binom{m}{i} = \binom{m-1}{i-1} + \binom{m-1}{i}$ holds for all $m \geq 1$ and $i \geq 1$, we have

$$\beta_i(J_m) = \beta_i(J_{m-1}) + \beta_i(H_m) + \beta_{i-1}(J_{m-1})$$
$$= (i+1)\binom{m-1}{i+2} + \binom{m-1}{i+1} + i\binom{m-1}{i+1}$$
$$= (i+1)\binom{m}{i+2},$$

as desired. □

We may compute the graded Betti numbers of J_m in a special case.

Proposition 4.6. *If $p_1 = \cdots = p_m = p$, then for all $i \geq 0$, we have*

$$\beta_{i,j}(J_m) = \begin{cases} \binom{m}{i+2}, & j = (i+2)p + \ell,\ \ell = 1, \ldots, i+1; \\ 0, & \text{otherwise.} \end{cases}$$

Proof: We use the induction on m. The case that $m = 2$ or $i = 0$ are obvious. So we suppose $m \geq 3$ and $i \geq 1$. By the induction hypothesis we have

$$\beta_{i,j}(J_{m-1}) = \begin{cases} \binom{m-1}{i+2}, & j = (i+2)p + \ell,\ \ell = 1, \ldots, i+1; \\ 0, & \text{otherwise} \end{cases}$$

and so

$$\beta_{i-1,j-p-1}(J_{m-1}) = \begin{cases} \binom{m-1}{i+1}, & j = (i+2)p + \ell + 1,\ \ell = 1, \ldots, i; \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, by the theory of Koszul complex, we have

$$\beta_{i,j-p-1}(H_m) = \begin{cases} \binom{m-1}{i+1}, & j = (i+2)p + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Now, the result follows by applying the equality (♣). □

4.2. **Type two.** In this subsection, we denote the toric ideal of $B^s_{\underline{p}\cdot\underline{q}}$ and its initial ideal as $I^s_{m:n}$ and $J^s_{m:n}$ respectively. Similarly, $I^s_{m:n-1}$ and $J^s_{m:n-1}$ represent the toric ideal and its initial ideal of $B^s_{\underline{p}\cdot\underline{q}'}$ respectively. Here, we use the monomial order given in Subsection 3.2. The distinction between the case when $s > 0$ and the case when $s = 0$ is significant. Let us first consider the case when $s > 0$. Recall from Subsection 3.2 that $G(J^s_{m:n})$ is the set

$$\left\{e''_i e'_j \mid 1 \leq i < j \leq m\right\} \sqcup \left\{f''_i f'_j \mid 1 \leq i < j \leq n\right\} \sqcup \left\{e'_i f'_j \mid 1 \leq i \leq m, 1 \leq j \leq n\right\} \sqcup \left\{e'_i x'' \mid 1 \leq i \leq m\right\} \sqcup \left\{f'_i x' \mid 1 \leq i \leq n\right\}.$$

Proposition 4.7. *Denote by $H^s_{m:n}$ the monomial ideal $(e'_1, \ldots, e'_m, f''_1, \ldots, f''_{n-1}, x')$. Then $J^s_{m:n} = J^s_{m:n-1} + f'_n H^s_{m:n}$ is an E-K splitting. Furthermore, $J^s_{m:n-1} \cap f'_n H^s_{m:n} = f'_n J^s_{m:n-1}$.*

Proof: First of all, it is routine to see that $G(J^s_{m:n}) = G(J^s_{m:n-1}) \sqcup G(f'_n H^s_{m:n})$ and $J^s_{m:n-1} \cap f'_n H^s_{m:n} = f'_n J^s_{m:n-1}$.

Let us define a function $\phi\colon G(f'_n J^s_{m:n-1}) \to G(J^s_{m:n-1})$ that sends $f'_n u$ to u for all $u \in G(J^s_{m:n-1})$.

Similarly, we define a function $\psi\colon G(f'_n J^s_{m:n-1}) \to G(f'_n H^s_{m:n})$ using the following rules:

- $f'_n e''_i e'_j \mapsto f'_n e'_j$ for all $1 \leq i < j \leq m$ and $f'_n f''_i f'_j \mapsto f'_n f''_i$ for all $1 \leq i < j \leq n-1$;
- $f'_n e'_i f'_j \mapsto f'_n e'_i$ for all $1 \leq i \leq m$ and $1 \leq j \leq n-1$;
- $f'_n e'_i x'' \mapsto f'_n e'_i$ for all $1 \leq i \leq m$ and $f'_n f'_i x' \mapsto f'_n x'$ for all $1 \leq i \leq n-1$.

It is routine to check that conditions (1) and (2) of Definition 1.3 are satisfied, thus confirming that it is indeed an E-K splitting. □

If $s = 0$, then $G(J^0_{m:n})$ is the disjoint union

$$\left\{e'_i f'_j \mid 1 \leq i \leq m, 1 \leq j \leq n\right\} \cup \left\{e''_i e'_j \mid 1 \leq i < j \leq m\right\} \cup \left\{f''_i f'_j \mid 1 \leq i < j \leq n\right\}$$

. Similarly, we obtain the following.

Proposition 4.8. *Denote by $H^0_{m:n}$ the monomial ideal $(e'_1, \ldots, e'_m, f''_1, \ldots, f''_{n-1})$. Then $J^0_{m:n} = J^0_{m:n-1} + f'_n H^0_{m:n}$ is an E-K splitting, and $J^0_{m:n-1} \cap f'_n H^0_{m:n} = f'_n J^0_{m:n}$.*

Proposition 4.9. *Let* $s \geq 0$ *be an even number,* $m, n \geq 1$. *Then for all* $i \geq 0$, *we have*

$$\beta_i(J^s_{m:n}) = \begin{cases} (i+1)\binom{m+n}{i+2}, & s = 0; \\ (i+1)\binom{m+n+1}{i+2}, & s > 0. \end{cases}$$

Proof: We consider the following two cases.

Case s=0: We also employ the induction on n. The case that $m = n = 1$ or $i = 0$ is straightforward. If $m + n \geq 3$ and $i \geq 1$, then, we have

$$\beta_i(J^0_{m:n}) = \beta_i(J^0_{m:n-1}) + \beta_i(H^0_{m:n}) + \beta_{i-1}(J^0_{m:n-1})$$

$$= (i+1)\binom{m+n-1}{i+2} + i\binom{m+n-1}{i+1} + \binom{m+n-1}{i+1}$$

$$= (i+1)\binom{m+n}{i+2},$$

as desired.

Case $s > 0$: We also employ the induction on n. The case that $i = 0$ is straightforward. If $i \geq 1$, we have

$$\beta_i(J^s_{m:n}) = \beta_i(J^s_{m:n-1}) + \beta_i(H^s_{m:n}) + \beta_{i-1}(J^s_{m:n-1})$$

$$= (i+1)\binom{m+n}{i+2} + i\binom{m+n}{i+1} + \binom{m+n}{i+1}$$

$$= (i+1)\binom{m+n+1}{i+2}.$$

This completes the proof. □

Proposition 4.10. *Let* $s \geq 0$ *be an even number and* $m, n \geq 1$. *Then*

$$\operatorname{reg}(J^s_{m:n}) = \operatorname{mat}(B^s_{\underline{p}:\underline{q}}) + 1.$$

Proof: *Case s=0:* In this case, $\operatorname{mat}(B^0_{\underline{p}:\underline{q}}) = \sum_{i=1}^m p_i + \sum_{i=1}^n q_i + 1$. We proceed with the induction on n. By Proposition 4.8, we have

$$\beta_{i,j}(J^0_{m:n}) = \beta_{i,j}(J^0_{m:n-1}) + \beta_{i,j-q_n-1}(H^0_{m:n}) + \beta_{i-1,j-q_n-1}(J^0_{m:n-1}).$$

It follows that

$$\operatorname{reg}(J^0_{m:n}) = \max\{j - i \mid \beta_{i,j}(J^0_{m:n-1}) + \beta_{i-1,j-q_n-1}(J^0_{m:n-1}) + \beta_{i,j-q_n-1}(H^0_{m:n}) \neq 0\}$$

$$= \max\{\operatorname{reg}(J^0_{m:n-1}), \operatorname{reg}(J^0_{m:n-1}) + q_n, \operatorname{reg}(H^0_{m:n}) + q_n + 1\}.$$

Note that $H^0_{m:n}$ is generated by a regular sequence of degrees $p_1 + 1, \ldots, p_m + 1, q_1, \ldots, q_{n-1}$. By using the Koszul theory, we obtain

$$\operatorname{reg}(H^0_{m:n}) = \sum_{i=1}^m p_i + \sum_{i=1}^{n-1} q_i - n + 2.$$

Hence, if $n = 1$, since $\operatorname{reg}(J^0_{m:0}) = \operatorname{reg}(J_m) = \sum_{i=1}^m p_i + 1$, we have

$$\operatorname{reg}(J^0_{m:1}) = \max\{\sum_{i=1}^m p_i + q_1 + 1, \sum_{i=1}^m p_i + q_1 + 2\} = \sum_{i=1}^m p_i + q_1 + 2.$$

This proves the case when $n = 1$.

If $n > 1$, then

$$\operatorname{reg}(J^0_{m:n}) = \max\{\sum_{i=1}^m p_i + \sum_{i=1}^n q_i + 2, \sum_{i=1}^m p_i + \sum_{i=1}^n q_i - n + 3\}$$

$$= \sum_{i=1}^m p_i + \sum_{i=1}^n q_i + 2.$$

Case $s > 0$: In this case, we have $\operatorname{mat}(B^s_{\underline{p}:\underline{q}}) = \sum_{i=1}^m p_i + \sum_{i=1}^n q_i + \frac{s}{2}$. We proceed with the induction on n again. First, note that

$$\operatorname{reg}(H^s_{m:n}) = \sum_{i=1}^m p_i + \sum_{i=1}^{n-1} q_i + \frac{s}{2} - n + 1.$$

If $n = 1$, then

$$\beta_{i,j}(J^s_{m:1}) = \beta_{i,j}(J^s_{m:0}) + \beta_{i,j-q_1-1}(H^s_{m:1}) + \beta_{i-1,j-q_1-1}(J^s_{m:0}).$$

Note that $G(J^s_{m:0}) = \{e^{''}_i e^{'}_j \mid 1 \le i < j \le m\} \cup \{e^{'}_i x^{''} \mid 1 \le i \le m\}$. By putting $e^{''}_0 = x^{''}$, we may write $G(J^s_{m:0}) = \{e^{'}_i e^{'}_j \mid 0 \le i < j \le m\}$. This is exactly the ideal studied in Subsection 4.1, and so it follows from Proposition 4.4 that

$$\mathrm{reg}(J^s_{m:0}) = \sum_{i=1}^{m} p_i + \frac{s}{2} + 1.$$

Hence, $\mathrm{reg}(J^s_{m:1}) = \max\{\mathrm{reg}(J^s_{m:0}) + q_1, \mathrm{reg}(H^s_{m:1}) + q_1 + 1\} = \sum_{i=1}^{m} p_i + q_1 + \frac{s}{2} + 1$.

Suppose that $n > 1$. Then, since

$$\beta_{i,j}(J^s_{m:n}) = \beta_{i,j}(J^s_{m:n-1}) + \beta_{i,j-q_n-1}(H^s_{m:n}) + \beta_{i-1,j-q_n-1}(J^s_{m:n-1}),$$

we have

$$\begin{aligned}\mathrm{reg}(J^s_{m:n}) &= \max\{\mathrm{reg}(J^s_{m:n-1}), \mathrm{reg}(J^s_{m:n-1}) + q_n, \mathrm{reg}(H^s_{m:n}) + q_n + 1\} \\ &= \max\{\sum_{i=1}^{m} p_i + \sum_{i=1}^{n} q_i + \frac{s}{2} + 1, \sum_{i=1}^{m} p_i + \sum_{i=1}^{n} q_i - n + \frac{s}{2} + 2\} \\ &= \sum_{i=1}^{m} p_i + \sum_{i=1}^{n} q_i + \frac{s}{2} + 1,\end{aligned}$$

as desired. □

4.3. **Type three.** In this subsection, we denote the toric ideal of $C_{\underline{p}:\underline{q}:\underline{r}}$ and its initial ideal as $I_{m:n:k}$ and $J_{m:n:k}$ respectively. Similarly, $I_{m:n:k-1}$ and $J_{m:n:k-1}$ represent the toric ideal of $C_{\underline{p}:\underline{q}:\underline{r}'}$ and its initial ideal, respectively. Recall from Subsection 3.3 that $G(J_{m:n:k})$ is the disjoint union

$\left\{e^{''}_i e^{'}_j \mid 1 \le i < j \le m\right\} \cup \left\{f^{''}_i f^{'}_j \mid 1 \le i < j \le n\right\} \cup \left\{g^{''}_i g^{'}_j \mid 1 \le i < j \le k\right\} \cup \left\{e^{'}_i y \mid 1 \le i \le m\right\} \cup \left\{f^{'}_i z \mid 1 \le i \le n\right\} \cup \left\{g^{'}_i x \mid 1 \le i \le k\right\} \cup \left\{e^{'}_i f^{'}_j \mid 1 \le i \le m, 1 \le j \le n\right\} \cup \left\{f^{'}_i g^{'}_j \mid 1 \le i \le\right.$

Proposition 4.11. *Denote by $H_{m:n:k}$ the monomial ideal*

$$(x, e^{'}_1, \ldots, e^{'}_m, f^{'}_1, \ldots, f^{'}_n, g^{''}_1, \ldots, g^{''}_{k-1}).$$

Then $J_{m:n:k} = J_{m:n:k-1} + g^{'}_k H_{m:n:k}$ is an E-K splitting. Furthermore, we have

$$J_{m:n:k-1} \cap g^{'}_k H_{m:n:k} = g^{'}_k J_{m:n:k-1}.$$

Proof: First of all, it is routine to check that $G(J_{m:n:k}) = G(J_{m:n:k-1}) \bigsqcup G(g^{'}_k H_{m:n:k})$ and $J_{m:n:k-1} \cap g^{'}_k H_{m:n:k} = g^{'}_k J_{m:n:k-1}$.

Define a function $\phi: G(g^{'}_k J_{m:n:k-1}) \to G(J_{C_{m:n:k-1}})$ that sends $g^{'}_k u$ to u for all $u \in G(J_{m:n:k-1})$.

Define a function $\psi: G(g^{'}_k J_{m:n:k-1}) \to G(g^{'}_k H_{m:n:k})$ by the following rules:

- $g^{'}_k e^{''}_i e^{'}_j \mapsto g^{'}_k e^{'}_j$ for all $1 \le i < j \le m$ and $g^{'}_k e^{'}_i y \mapsto g^{'}_k e^{'}_i$ for all $1 \le i \le m$;
- $g^{'}_k e^{'}_i f^{'}_j \mapsto g^{'}_k e^{'}_i$ for all $1 \le i \le m$ and $1 \le j \le n$;
- $g^{'}_k f^{''}_i f^{'}_j \mapsto g^{'}_k f^{'}_j$ for all $1 \le i < j \le n$ and $g^{'}_k f^{'}_i z \mapsto g^{'}_k f^{'}_i$ for all $1 \le i \le n$;
- $g^{'}_k f^{'}_i g^{'}_j \mapsto g^{'}_k f^{'}_i$ for all $1 \le i \le k-1$ and $1 \le j \le n$;
- $g^{'}_k g^{'}_i g^{'}_j \mapsto g^{'}_k g^{'}_i$ for all $1 \le i < j \le k-1$ and $g^{'}_k g^{'}_i x \mapsto g^{'}_k x$ for all $1 \le i < j \le k-1$;
- $g^{'}_k g^{'}_i e^{'}_j \mapsto g^{'}_k e^{'}_j$ for all $1 \le i \le k-1$ and $1 \le j \le m$.

It is routine to check conditions (1) and (2) of Definition 1.3 are satisfied. □

Proposition 4.12. *Let $m, n, k \ge 1$. Then $\beta_i(J_{m:n:k}) = (i+1)\binom{m+n+k+1}{i+2}$ for all $i \ge 0$. In particular, $\mathrm{pdim}(J_{m:n:k}) = m + n + k - 1$.*

Proof: We also employ the induction on k. The case that $i = 0$ is straightforward. If $i \ge 1$, then, we have

$$\begin{aligned}\beta_i(J_{m:n:k}) &= \beta_i(J_{m:n:k-1}) + \beta_i(H_{m:n:k}) + \beta_{i-1}(J_{m:n:k-1}) \\ &= (i+1)\binom{m+n+k}{i+2} + \binom{m+n+k}{i+1} + i\binom{m+n+k}{i+1} \\ &= (i+1)\binom{m+n+k+1}{i+2},\end{aligned}$$

as desired. □

Proposition 4.13. *Let $m, n, k \ge 1$. Then $\mathrm{reg}(J_{m:n:k}) = \mathrm{mat}(C_{\underline{p}:\underline{r}}) + 1$.*

Proof: Note that $\mathrm{mat}(C_{\underline{p}:\underline{q}:\underline{r}}) = \sum_{i=1}^{m} p_i + \sum_{i=1}^{n} q_i + \sum_{i=1}^{k} r_i + 1$. We proceed with the induction on k. If $k = 1$, then

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$$\beta_{i,j}(J_{m:n:1}) = \beta_{i,j}(J_{m:n:0}) + \beta_{i,j-r_1-1}(H_{m:n:1}) + \beta_{i-1,j-r_1-1}(J_{m:n:0}).$$

Since $\operatorname{reg}(J_{m:n:0}) = \operatorname{reg}(J^2_{m:n})$, we obtain $\operatorname{reg}(J_{m:n:1}) = \sum_{i=1}^{m} p_i + \sum_{i=1}^{n} q_i + r_1 + 2.$

Suppose that $k > 1$. Then, by Proposition 4.11, we have

$$\beta_{i,j}(J_{m:n:k}) = \beta_{i,j}(J_{m:n:k-1}) + \beta_{i,j-r_k-1}(H_{m:n:k}) + \beta_{i-1,j-r_k-1}(J_{m:n:k-1}).$$

It follows that $\operatorname{reg}(J_{m:n:k})$

$$= \max\{j - i \mid \beta_{i,j}(J_{m:n:k-1}) + \beta_{i-1,j-r_k-1}(J_{m:n:k-1}) + \beta_{i,j-r_k-1}(H_{m:n:k}) \neq 0\}$$

$$= \max\{\operatorname{reg}(J_{m:n:k-1}), \operatorname{reg}(H_{m:n:k}) + r_k + 1, \operatorname{reg}(J_{m:n:k-1}) + r_k\}.$$

Note that $H_{m:n:k}$ is generated by a regular sequence of degrees $p_1 + 1, \ldots, p_m + 1, q_1 + 1, \ldots, q_n + 1, r_1, \ldots, r_{k-1}, 1$, we obtain $\operatorname{reg}(H_{m:n:k}) = \sum_{i=1}^{m} p_i + \sum_{i=1}^{n} q_i + \sum_{i=1}^{k-1} r_i - k + 2.$ Hence,

$$\operatorname{reg}(J_{m:n:k}) = \max\{\sum_{i=1}^{m} p_i + \sum_{i=1}^{n} q_i + \sum_{i=1}^{k} r_i + 2, \sum_{i=1}^{m} p_i + \sum_{i=1}^{n} q_i + \sum_{i=1}^{k} r_i - k + 3\}$$

$$= \sum_{i=1}^{m} p_i + \sum_{i=1}^{n} q_i + \sum_{i=1}^{k} r_i + 2,$$

as desired. □

To complete the proof of Theorem 4.1, we require some additional notation and facts. Recall a connected graph is *planar* if it can be drawn on a 2D plane such that none of the edges intersect. If a planar graph G is drawn in this way, it divides the plane into regions called *faces*. The number of faces is denoted by $f(G)$. The famous Euler formula states that for any planar graph G, we have

$$|E(G)| - |V(G)| = f(G) - 2.$$

If we assume that every edge of G belongs to at most one induced cycle, then there is a one-to-one correspondence between induced cycles and bounded faces of G. Since there is exactly one unbounded face of G, it follows that $f(G) = \iota(G) + 1$.

However, it is worth noting that the formula $f(G) = \iota(G) + 1$ does not hold in general. For example, if G is the complete graph with 4 vertices, then G is planar, but $f(G) = \iota(G) = 4$.

We are now ready to present the proof of Theorem 4.1.

Proof: (1) This is a combination of Propositions 4.5, 4.9 and Proposition 4.12.

(2) It follows immediately from (1).

(3) Since G is a compact graph, G is a planar graph and every edge of G belongs to at most one induced cycle. Hence, because of the discussion above, we have

$$|E(G)| - |V(G)| = \iota(G) - 1.$$

This implies

$$\operatorname{depth}(\mathbb{K}[E(G)]/\operatorname{in}_<(I_G)) = |E(G)| - \operatorname{pdim}(\mathbb{K}[E(G)]/\operatorname{in}_<(I_G))$$

$$= |V(G)| = \dim(\mathbb{K}[G])$$

$$= \dim(\mathbb{K}[E(G)]/\operatorname{in}_<(I_G)).$$

Here, the second last equality follows from [15, Corollary 10.1.21]. Hence, by definition, $\mathbb{K}[E(G)]/\operatorname{in}_<(I_G)$ is Cohen-Macaulay.

(4) This is a combination of Propositions 4.4, 4.10 and Proposition 4.13. □

5. Cohen-Macaulay types and top graded Betti numbers

Assume that G is a compact graph. In this section we will compute the top graded Betti numbers of $\mathbb{K}[G]$. Since $\mathbb{K}[G]$ is Cohen-Macaulay, the regularity of $\mathbb{K}[G]$ is determined by its top graded Betti numbers. Therefore, the regularity formula given in Section 4 could also be deduced from the results of this section. To present the top graded Betti numbers of $\mathbb{K}[G]$, we need to consider three cases. The most complex case is when G is a compact graph of type 3, and we will provide detailed proof specifically for this case. The proofs for the cases when G is a compact graph of type one or type two are similar, with only minor differences, so we will only provide an outline of the proofs for those cases.

The top graded Betti numbers of the edge rings of three types of compact graphs are presented in Propositions 5.2, 5.3 and Proposition 5.4, respectively. By combining the aforementioned results and their proofs, the following conclusion regarding the top total Betti numbers can be immediately derived.

Theorem 5.1. *Let G be a compact graph, and let I_G be the toric ideal of $\mathbb{K}[G]$. Denote by J_G the initial ideal of I_G with respect to the order given in Section 3. Then I_G and J_G share the same top graded Betti numbers. In particular, we have* $\mathrm{type}(\mathbb{K}[G]) = \iota(G) - 1$.

5.1. **type three.** Let C denote the compact graph $C_{p:q:r}$, whose vertex set $V(C)$ and edge set $E(C)$ are given explicitly in Subsection 3.3. In this subsection, we compute the minimal generators of the canonical module $\omega_{\mathbb{K}[C]}$ and then determine the top graded Betti numbers of the toric ring $\mathbb{K}[C]$.

It is easy to see that $|V(C)| = 2\sum_{i=1}^{m} p_i + 2\sum_{i=1}^{n} q_i + 2\sum_{i=1}^{k} r_i + 3$. We use the following notions for all the entries of $\mathbb{R}^{|V(C)|}$:

$$\mathbb{R}^{|V(C)|} = \{\sum_{i=1}^{m}\sum_{j=1}^{2p_i} a_{i,j}\mathbf{u}_{i,j} + a\mathbf{u} + \sum_{i=1}^{n}\sum_{j=1}^{2q_i} b_{i,j}\mathbf{v}_{i,j} + b\mathbf{v} + \sum_{i=1}^{k}\sum_{j=1}^{2r_i} c_{i,j}\mathbf{w}_{i,j} + c\mathbf{w} \mid a_{i,j}, a, b_{i,j}, b, c_{i,j}, c \in \mathbb{R} \text{ for all } i, j\},$$

where $\mathbf{u}, \mathbf{u}_{i,j}, \mathbf{v}, \mathbf{v}_{i,j}, \mathbf{w}, \mathbf{w}_{i,j}$ are the unit vectors of $\mathbb{R}^{|V(C)|}$, each $\mathbf{u}_{i,j}$ (resp. $\mathbf{v}_{i,j}, \mathbf{w}_{i,j}$) corresponds to $u_{i,j}$ (resp. $v_{i,j}, w_{i,j}$) (where $1 \le i \le m$ and $1 \le j \le 2p_i$) (resp. $1 \le i \le n$ and $1 \le j \le 2q_i$, $1 \le i \le k$ and $1 \le j \le 2r_i$) and $\mathbf{u}$ (resp. $\mathbf{v}, \mathbf{w}$) corresponds to u (resp. v, w).

In what follows, we will construct $m + n + k$ integral vectors in $\mathbb{R}^{|V(C)|}$ and then show that they are minimal vectors of $\mathrm{relint}(\mathbb{R}_+(C)) \cap \mathbb{Z}^{|V(C)|}$. Here, an integral vector in $\mathrm{relint}(\mathbb{R}_+(C))$ is called *minimal* if it cannot written as the sum of a vector in $\mathrm{relint}(\mathbb{R}_+(C)) \cap \mathbb{Z}^{|V(C)|}$ and a nonzero vector of $\mathbb{R}_+(C) \cap \mathbb{Z}^{|V(C)|}$. The construction is as follows:

For $\ell = 1, \dots, m$, let

$$\alpha_\ell := \sum_{i=1}^{m}\sum_{j=1}^{2p_i}\mathbf{u}_{i,j} + \sum_{i=1}^{n}\sum_{j=1}^{2q_i}\mathbf{v}_{i,j} + \sum_{i=1}^{k}\sum_{j=1}^{2r_i}\mathbf{w}_{i,j} + \mathbf{v} + \mathbf{w} + 2\ell\mathbf{u}.$$

For $\ell = 1, \dots, n$, let

$$\beta_\ell = \sum_{i=1}^{m}\sum_{j=1}^{2p_i}\mathbf{u}_{i,j} + \sum_{i=1}^{n}\sum_{j=1}^{2q_i}\mathbf{v}_{i,j} + \sum_{i=1}^{k}\sum_{j=1}^{2r_i}\mathbf{w}_{i,j} + \mathbf{w} + \mathbf{u} + 2\ell\mathbf{v}.$$

For $\ell = 1, \dots, k$, let

$$\gamma_\ell = \sum_{i=1}^{m}\sum_{j=1}^{2p_i}\mathbf{u}_{i,j} + \sum_{i=1}^{n}\sum_{j=1}^{2q_i}\mathbf{v}_{i,j} + \sum_{i=1}^{k}\sum_{j=1}^{2r_i}\mathbf{w}_{i,j} + \mathbf{u} + \mathbf{v} + 2\ell\mathbf{w}.$$

We now verify that $\alpha_\ell, \beta_\ell, \gamma_\ell \in \mathrm{relint}\mathbb{R}_+(C)$ for all possible ℓ. For this, we put $u_i^{(1)} = \{u_{i,j} \mid j = 1, 3, \dots, 2p_i - 1\}$ for $i = 1, \dots, m$, $u_i^{(2)} = \{u_{i,j} \mid j = 2, 4, \dots, 2p_i\}$ for $i = 1, \dots, m$ and $v_i^{(1)}, v_i^{(2)}, w_i^{(1)}, w_i^{(2)}$ are defined similarly.

We see the following:

- Each of $u_{i,j}$'s, $v_{i,j}$'s and $w_{i,j}$'s is a regular vertex of C, while u, v and w are not.
- An independent subset T of $V(C)$ is fundamental if and only if T is one of the following sets:
  (i) $\bigcup_{i=1}^{m} u_i^{(f_i)}$, where $(f_1, \dots, f_m) \in \{1,2\}^m$;
  (ii) $\bigcup_{i=1}^{n} v_i^{(g_i)}$, where $(g_1, \dots, g_n) \in \{1,2\}^n$;
  (iii) $\bigcup_{i=1}^{k} w_i^{(h_i)}$, where $(h_1, \dots, h_k) \in \{1,2\}^k$;
  (iv) $\{u\} \cup \bigcup_{i=1}^{m} (u_i^{(f_i)} \setminus \{u_{i,1}, u_{i,2p_i}\}) \cup \bigcup_{i=1}^{n} v_i^{(g_i)} \cup \bigcup_{i=1}^{k} w_i^{(h_i)}\}$, where $(f_1, \dots, f_m) \in \{1,2\}^m$, $(g_1, \dots, g_n) \in \{1,2\}^n$ and $(h_1, \dots, h_k) \in \{1,2\}^k$;
  (v) $\{v\} \cup \bigcup_{i=1}^{n} (v_i^{(g_i)} \setminus \{v_{i,1}, v_{i,2p_i}\}) \cup \bigcup_{i=1}^{m} u_i^{(f_i)} \cup \bigcup_{i=1}^{k} w_i^{(h_i)}$, where $(f_1, \dots, f_m) \in \{1,2\}^m$, $(g_1, \dots, g_n) \in \{1,2\}^n$ and $(h_1, \dots, h_k) \in \{1,2\}^k$;
  (vi) $\{w\} \cup \bigcup_{i=1}^{k} (w_i^{(h_i)} \setminus \{w_{i,1}, w_{i,2p_i}\}) \cup \bigcup_{i=1}^{m} u_i^{(f_i)} \cup \bigcup_{i=1}^{n} v_i^{(g_i)}$, where $(f_1, \dots, f_m) \in \{1,2\}^m$, $(g_1, \dots, g_n) \in \{1,2\}^n$ and $(h_1, \dots, h_k) \in \{1,2\}^k$.

It should be noted that there are 2^m fundamental sets in (i) and 2^{m+n+k} fundamental sets in (iv), and so on. Hence, it follows from (Δ) (see this Subsection 1.3) that a vector of $\mathbb{R}^{|V(C)|}$ of the form:

$$\sum_{i=1}^{m}\sum_{j=1}^{2p_i} a_{i,j}\mathbf{u}_{i,j} + a\mathbf{u} + \sum_{i=1}^{n}\sum_{j=1}^{2q_i} b_{i,j}\mathbf{v}_{i,j} + b\mathbf{v} + \sum_{i=1}^{k}\sum_{j=1}^{2r_i} c_{i,j}\mathbf{w}_{i,j} + c\mathbf{w}$$

belongs to $\mathbb{R}_+(C)$ if and only if the following inequalities are satisfied:

1. $a_{i,j} \ge 0$ for any $1 \le i \le m$ and $1 \le j \le 2p_i$;
2. $b_{i,j} \ge 0$ for any $1 \le i \le n$ and $1 \le j \le 2q_i$;
3. $c_{i,j} \ge 0$ for any $1 \le i \le k$ and $1 \le j \le 2k_i$;

4. $\sum_{i=1}^{m}\sum_{j=1}^{2p_i} a_{i,j} - \sum_{u_{i,j}\in T} a_{i,j} + a \geq \sum_{u_{i,j}\in T} a_{i,j}$ for any $T \in (i)$;

5. $\sum_{i=1}^{n}\sum_{j=1}^{2q_i} b_{i,j} - \sum_{v_{i,j}\in T} b_{i,j} + b \geq \sum_{v_{i,j}\in T} b_{i,j}$ for any $T \in (ii)$;

6. $\sum_{i=1}^{k}\sum_{j=1}^{2r_i} c_{i,j} - \sum_{w_{i,j}\in T} c_{i,j} + c \geq \sum_{w_{i,j}\in T} c_{i,j}$ for any $T \in (iii)$;

7. $\sum + b + c \geq a + 2(\sum_{u_{i,j}\in T} a_{i,j} + \sum_{v_{i,j}\in T} b_{i,j} + \sum_{w_{i,j}\in T} c_{i,j})$ for any $T \in (iv)$;

8. $\sum + c + a \geq b + 2(\sum_{u_{i,j}\in T} a_{i,j} + \sum_{v_{i,j}\in T} b_{i,j} + \sum_{w_{i,j}\in T} c_{i,j})$ for any $T \in (v)$;

9. $\sum + a + b \geq c + 2(\sum_{u_{i,j}\in T} a_{i,j} + \sum_{v_{i,j}\in T} b_{i,j} + \sum_{w_{i,j}\in T} c_{i,j})$ for any $T \in (vi)$.

Here, $\sum$ denotes $\sum_{i=1}^{m}\sum_{j=1}^{2p_i} a_{i,j} + \sum_{i=1}^{n}\sum_{j=1}^{2q_i} b_{i,j} + \sum_{i=1}^{k}\sum_{j=1}^{2r_i} c_{i,j}$. It is straightforward to check that α_ℓ satisfies these inequalities, with strict inequalities holding for each α_ℓ. This implies that $\alpha_\ell \in \mathrm{relint}(\mathbb{R}_+(C)) \cap \mathbb{Z}^{|V(C)|}$.

Next, we show that α_ℓ is a minimal vector in $\mathrm{relint}(\mathbb{R}_+(C)) \cap \mathbb{Z}^{|V(C)|}$, i.e., it cannot be written as a sum of an element in $\mathrm{relint}(\mathbb{R}_+(C)) \cap \mathbb{Z}^{|V(C)|}$ and an element in $\mathbb{R}_+(C) \cap \mathbb{Z}^{|V(C)|} \setminus \{\mathbf{0}\}$ for all $\ell = 1, \ldots, m$.

Suppose on the contrary that $\alpha_\ell = \alpha' + \alpha''$ for some $\alpha' \in \mathrm{relint}(\mathbb{R}_+(C)) \cap \mathbb{Z}^{|V(C)|}$ and $\alpha'' \in \mathbb{R}_+(C) \cap \mathbb{Z}^{|V(C)|} \setminus \{\mathbf{0}\}$. Write

$$\alpha' = \sum_{i=1}^{m}\sum_{j=1}^{2p_i} a'_{i,j}\mathbf{u}_{i,j} + a'\mathbf{u} + \sum_{i=1}^{n}\sum_{j=1}^{2q_i} b'_{i,j}\mathbf{v}_{i,j} + b'\mathbf{v} + \sum_{i=1}^{k}\sum_{j=1}^{2r_i} c'_{i,j}\mathbf{w}_{i,j} + c'\mathbf{w},$$

$$\alpha'' = \sum_{i=1}^{m}\sum_{j=1}^{2p_i} a''_{i,j}\mathbf{u}_{i,j} + a''\mathbf{u} + \sum_{i=1}^{n}\sum_{j=1}^{2q_i} b''_{i,j}\mathbf{v}_{i,j} + b''\mathbf{v} + \sum_{i=1}^{k}\sum_{j=1}^{2r_i} c''_{i,j}\mathbf{w}_{i,j} + c''\mathbf{w}.$$

In view of the inequalities (1) - (3), we see that $a'_{i,j}, b'_{i,j}, c'_{i,j} \geq 1$ for all i, j. Because of the inequalities (4) - (6), we also see that $a', b', c' \geq 1$. Hence, $a''_{i,j} = b''_{i,j} = c''_{i,j} = b'' = c'' = 0$ for all possible i, j. From this together with (7) it follows that $a'' \leq 0$. Hence, $\alpha'' = 0$. This is a contradiction, which shows that α_ℓ is a minimal vector in $\mathrm{relint}(\mathbb{R}_+(C)) \cap \mathbb{Z}^{|V(C)|}$ for $\ell = 1, \ldots, m$. Likewise, so are β_ℓ's and γ_ℓ's.

Proposition 5.2. *Let C be defined as before. Assume $m \leq n \leq k$. Then $\mathrm{type}(\mathbb{K}[C]) = m + n + k$, and the top graded Betti numbers of $\mathbb{K}[C]$ are given by*

$$\beta_{m+n+k,j}(\mathbb{K}[C]) = \begin{cases} 1, & j = \mathrm{mat}(C) + n + m + \ell, \quad \ell = 1, \ldots, k-n; \\ 2, & j = \mathrm{mat}(C) + m + k + \ell, \quad \ell = 1, \ldots, n-m; \\ 3, & j = \mathrm{mat}(C) + k + n + \ell, \quad \ell = 1, \ldots, m; \\ 0, & \text{otherwise.} \end{cases}$$

Proof: Every minimal vector in $\mathrm{relint}(\mathbb{R}_+(C)) \cap \mathbb{Z}^{|V(C)|}$ corresponds to a minimal generator of $\omega_{\mathbb{K}[C]}$. It follows from the above discussion that $\mathrm{type}(\mathbb{K}[C]) \geq m + n + k$. Since $\mathrm{type}(\mathbb{K}[C])$ is equal to the top total Betti number of $\mathbb{K}[C]$, we conclude that $\mathrm{type}(\mathbb{K}[C]) \leq m + n + k$ by Theorem 4.1. Thus the first conclusion follows. From this, we see that $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_k$ are all the minimal vectors of $\mathrm{relint}(\mathbb{R}_+(C)) \cap \mathbb{Z}^{|V(C)|}$. Therefore, the set of monomials

$$\{x^{\alpha_\ell},\ \ell = 1, \ldots, m; \quad x^{\beta_\ell},\ \ell = 1, \ldots, n; \quad x^{\gamma_\ell},\ \ell = 1, \ldots, k\}$$

is a minimal generating set of $\omega_{\mathbb{K}[C]}$, which is an ideal of the edge ring $\mathbb{K}[C] \subset \mathbb{K}[V(C)]$. Note that every monomial x^α belonging to $\mathbb{K}[C]$, which is regarded as a graded module over the standard graded ring $\mathbb{K}[E(C)]$, has a degree of $\frac{1}{2}|\alpha|$. Hence,

$$\beta_{0,j}(\omega_{\mathbb{K}[C]}) = \begin{cases} 3, & j = \sum_{i=1}^{m} p_i + \sum_{i=1}^{n} q_i + \sum_{i=1}^{k} r_i + 1 + \ell, \ell = 1, \ldots, m; \\ 2, & j = \sum_{i=1}^{m} p_i + \sum_{i=1}^{n} q_i + \sum_{i=1}^{k} r_i + 1 + \ell, \ell = m+1, \ldots, n; \\ 1, & j = \sum_{i=1}^{m} p_i + \sum_{i=1}^{n} q_i + \sum_{i=1}^{k} r_i + 1 + \ell, \ell = n+1, \ldots, k; \\ 0, & \text{otherwise.} \end{cases}$$

Since $|E(C)| = 2(\sum_{i=1}^{m} p_i + \sum_{i=1}^{n} q_i + \sum_{i=1}^{k} r_i) + m + n + k + 3$ and $\mathrm{mat}(C) = \sum_{i=1}^{m} p_i + \sum_{i=1}^{n} q_i + \sum_{i=1}^{k} r_i + 1$, the second conclusion follows by Lemma 1.1. □

5.2. **type one.** Let A denote the compact graph $A_{\underline{p}}$, whose vertex set $V(A)$ and edge set $E(A)$ are given explicitly in Subsection 3.1. Then $|V(A)| = 2\sum_{i=1}^{m} p_i + 1$. We may write

$$\mathbb{R}^{|V(A)|} = \{\sum_{i=1}^{m}\sum_{j=1}^{2p_i} a_{i,j}\mathbf{u}_{i,j} + a\mathbf{u} \mid \text{all } a_{i,j}, a \in \mathbb{R}\}.$$

Here, $\mathbf{u}_{i,j}$, $\mathbf{u}$ correspond the vertices of A in a natural way. Then, we could show the following vectors

$$\alpha_\ell := \sum_{i=1}^{m}\sum_{j=1}^{2p_i}\mathbf{u}_{i,j} + 2\ell\mathbf{u},\ \ \ell = 1, \ldots, m-1$$

are all the minimal vectors of $\mathrm{relint}(\mathbb{R}_+(A)) \cap \mathbb{Z}^{|V(A)|}$.

Proposition 5.3. *Let A denote the compact graph A_p. Then $\mathrm{type}(\mathbb{K}[A]) = m-1$, and the top graded Betti numbers of $\mathbb{K}[A]$ are given by*

$$\beta_{m-1,j}(\mathbb{K}[A]) = \begin{cases} 1, & j = \mathrm{mat}(A) + \ell,\ \ell = 1, \ldots, m-1; \\ 0, & \text{otherwise.} \end{cases}$$

5.3. **type two.** Let B^0 and B^s denote the compact graph $B^0_{\underline{p},\underline{q}}$ and $B^s_{\underline{p},\underline{q}}$ respectively. Their vertex sets $V(B^0)$ and $V(B^s)$ and edge sets $E(B^0)$ and $E(B^s)$ are given explicitly in Subsection 3.2. Then $|V(B^0)| = 2(\sum_{i=1}^{m} p_i + \sum_{i=1}^{n} q_i) + 2$. We may write

$$\mathbb{R}^{|V(B^0)|} = \{\sum_{i=1}^{m}\sum_{j=1}^{2p_i} a_{i,j}\mathbf{u}_{i,j} + \sum_{i=1}^{m}\sum_{j=1}^{2p_i} b_{i,j}\mathbf{v}_{i,j} + a\mathbf{u} + b\mathbf{v} \ \mid\ \text{all } a_{i,j}, b_{i,j}, a, b \in \mathbb{R}\}.$$

Here, $\mathbf{u}_{i,j}, \mathbf{v}_{i,j}, \mathbf{u}, \mathbf{v}$ correspond the vertices of B^0 in the natural way. Then, we could show the following vectors

$$\alpha_\ell := \sum_{i=1}^{m}\sum_{j=1}^{2p_i}\mathbf{u}_{i,j} + \sum_{i=1}^{n}\sum_{j=1}^{2q_i}\mathbf{v}_{i,j} + \mathbf{v} + (2\ell+1)\mathbf{u}, \quad \ell = 0, \ldots, m-1$$

and

$$\beta_\ell := \sum_{i=1}^{m}\sum_{j=1}^{2p_i}\mathbf{u}_{i,j} + \sum_{i=1}^{n}\sum_{j=1}^{2q_i}\mathbf{v}_{i,j} + \mathbf{u} + (2\ell+1)\mathbf{v}, \quad \ell = 1, \ldots, n-1$$

are all the minimal vectors of $\mathrm{relint}(\mathbb{R}_+(B^0)) \cap \mathbb{Z}^{|V(B^0)|}$.

On the other hand, we have $|V(B^s)| = |V(B^0)| + s - 1$ and we may write

$$\mathbb{R}^{|V(B^s)|} = \{\sum_{i=1}^{m}\sum_{j=1}^{2p_i} a_{i,j}\mathbf{u}_{i,j} + \sum_{i=1}^{n}\sum_{j=1}^{2q_i} b_{i,j}\mathbf{v}_{i,j} + \sum_{i=1}^{s-1} c_i\mathbf{w}_i + a\mathbf{u} + b\mathbf{v} \ \mid\ \text{all } a_{i,j}, b_{i,j}, c_i, a, b \in \mathbb{R}\}.$$

Here, $\mathbf{u}_{i,j}, \mathbf{v}_{i,j}, \mathbf{w}_i, \mathbf{u}, \mathbf{v}$ correspond the vertices of B^s in the natural way. We may also show the following vectors

$$\alpha_\ell := \sum_{i=1}^{m}\sum_{j=1}^{2p_i}\mathbf{u}_{i,j} + \sum_{i=1}^{n}\sum_{j=1}^{2q_i}\mathbf{v}_{i,j} + \sum_{i=1}^{s-1}\mathbf{w}_i + \mathbf{v} + 2\ell\mathbf{u},\ \ell = 1, \ldots, m$$

and

$$\beta_\ell := \sum_{i=1}^{m}\sum_{j=1}^{2p_i}\mathbf{u}_{i,j} + \sum_{i=1}^{n}\sum_{j=1}^{2q_i}\mathbf{v}_{i,j} + \sum_{i=1}^{s-1}\mathbf{w}_i + \mathbf{u} + 2\ell\mathbf{v},\ \ell = 1, \ldots, n$$

are all the minimal vectors of $\mathrm{relint}(\mathbb{R}_+(B^s)) \cap \mathbb{Z}^{|V(B^s)|}$.

Proposition 5.4. *Let B^0 and B^s be defined as before. Assume $m \le n$. Then the following statements hold:*

- $\mathrm{type}(\mathbb{K}[B^0]) = m + n - 1$ *and* $\mathrm{type}(\mathbb{K}[B^s]) = m + n$*;*
- *the top graded Betti numbers of* $\mathbb{K}[B^0]$ *are given by*

$$\beta_{m+n-1,j}(\mathbb{K}[B^0]) = \begin{cases} 1, & j = \mathrm{mat}(B^0) + m - 1 + \ell, \quad \ell = 1, \ldots, n-m; \\ 2, & j = \mathrm{mat}(B^0) + n - 1 + \ell, \quad \ell = 1, \ldots, m-1; \\ 1, & j = \mathrm{mat}(B^0) + m + n - 1. \\ 0, & \text{otherwise.} \end{cases}$$

,

- *the top graded Betti numbers of* $\mathbb{K}[B^s]$ *are given by*

$$\beta_{m+n,j}(\mathbb{K}[B^s]) = \begin{cases} 1, & j = \mathrm{mat}(B^s) + \ell, \quad \ell = m+1, \ldots, n; \\ 2, & j = \mathrm{mat}(B^s) + n + \ell, \quad \ell = 1, \ldots, m; \\ 0, & \text{otherwise.} \end{cases}$$

6. A question

Let G be a compact graph, and let I_G be the toric ideal of $\mathbb{K}[G]$. Denote by J_G the initial ideal of I_G with respect to the order given in Section 3. As we have seen in the previous section, I_G and J_G share the same top graded Betti numbers. This naturally leads to the following question:

Does I_G and J_G always share the same graded Betti numbers?

Unfortunately, we are unable to provide a general answer to this question, except for a very specific case when G is a compact graph of type one.

In what follows, we use A to denote the compact graph $A_{\underline{p}}$, where $\underline{p} = (\overbrace{p, \dots, p}^{m})$ is a vector in $\mathbb{Z}_+^m$. Let $f(t)$ and $g(t)$ denote the polynomial $\sum_{i,j}\beta_{i,j}(I_A)(-1)^i t^j$ and $\sum_{i,j}\beta_{i,j}(J_A)(-1)^i t^j$, respectively. It is known $f(t) = g(t)$ and $\beta_{i,j}(I_A) \le \beta_{i,j}(J_A)$ for all i, j.

Proposition 6.1. *If* $2 \le m \le p+3$, *then*

$$\beta_{i,j}(I_A) = \beta_{i,j}(J_A) = \begin{cases} \binom{m}{i+2}, & j = (i+2)p + \ell, \ell = 1, \dots, i+1; \\ 0, & \text{otherwise.} \end{cases}$$

Proof: Put $A_i = \{j \in \mathbb{Z} \mid \beta_{i,j}(J_A) \neq 0\}$ for all $i \ge 0$. Then, by Proposition 4.6, we have $A_i = \{(i+2)p + \ell \mid \ell = 1, \dots, i+1\}$ for $0 \le i \le m-2$, and is $\emptyset$ otherwise. Given that $j \notin A_i$ it can be inferred that $\beta_{i,j}(J_A) = \beta_{i,j}(I_A) = 0$. Therefore, we will next consider only the case when $j \in A_i$.

(1) If $m \le p+2$ then it follows that $A_{i_1} \cap A_{i_2} = \emptyset$ for any distinct i_1 and i_2. Consequently, for any $j \in A_i$, the coefficient of t^j in $f(t)$ is $(-1)^i\beta_{i,j}(J_A)$, while in $g(t)$ it is $(-1)^i\beta_{i,j}(I_A)$. Therefore we can deduce that $\beta_{i,j}(J_A) = \beta_{i,j}(I_A)$.

(2) If $m = p+3$, then for any pair $i_1 \neq i_2$, $A_{i_1} \cap A_{i_2} \neq \emptyset$ if and only if $\{i_1, i_2\} = \{m-3, m-2\}$ and in that case $A_{m-3} \cap A_{m-2} = \{mp+1\} = \{(m-1)p + m - 2\}$. If $j \neq mp+1$ then it follows that $\beta_{i,j}(I_A) = \beta_{i,j}(J_A)$ for the same reason as in (1). If $j = mp+1$ then, by comparing the coefficients of t^{mp+1} in polynomials $f(t)$ and $g(t)$, we conclude that

$$\beta_{m-3,mp+1}(I_A) - \beta_{m-2,mp+1}(I_A) = \beta_{m-3,mp+1}(J_A) - \beta_{m-2,mp+1}(J_A).$$

On the other hand, we have $\beta_{m-2,mp+1}(I_A) = \beta_{m-2,mp+1}(J_A)$ by Theorem 5.1. From this it follows that $\beta_{m-3,mp+1}(I_A) = \beta_{m-3,mp+1}(J_A)$, as required. □

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