Research Article

The $V_2^{(2r-1)}$ of PG(2r, q) as a Representation of PG(2, q)

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In the André/Bruck and Bose representation of $PG(2,q^2)$ in $PG(4,q)$ (cf. [1], [2]) a non-affine Baer subplane B corresponds to a ruled variety V_2^3 (cf. [3], [4]). In [5] is proved that a non degenerate conic in B is a rational normal curve of V_2^3.

Using that technique, in [6] is studied the representation of the projective plane $PG(2,q^r)$ in $PG(2r,q)$ and of a non-affine subplane $PG(2,q)$ in a variety V_2^{2r-1}.

In this note are studied sections of V_2^{2r-1} by hyperplanes giving rise to caps related to some arcs in $PG(2,q)$. Then is determined a partition of the affine points of V_2^{2r-1} in caps corresponding to a partition in conics of the affine points of $PG(2,q)$.

The V_2^{2r-1} of $PG(2r,q)$ as a representation of $PG(2,q)$: sections and partitions.

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1. Introduction

It is known that a projective translation plane can be represented in a projective space of even order (cf. André [1], Bruck and Bose [2]).

More precisely, if Π is the projective plane $PG(2,q^r)$ with kernel $F = GF(q)$, then it can be represented by a $2r$-dimensional projective space $\Sigma = PG(2r,q)$, fixing a hyperplane $\Sigma' = PG(2r-1,q)$ and a spread $\mathcal{S}$ of Σ' with $(r-1)$-dimensional subspaces, $|\mathcal{S}| = q^r + 1$. The affine points of Π are represented by the points of $\Sigma \setminus \Sigma'$, the points at infinity by the elements of $\mathcal{S}$, the affine lines by the r-subspaces S_r of Σ such that $S_r \cap \Sigma' \in \mathcal{S}$, the line at infinity by $\mathcal{S}$. If Π is Desarguesian, the spread $\mathcal{S}$ is regular (cf. [1], [2], and [3], [4] for $r = 2$, [7] for $r = 3$, [6] for the general case).

A subplane of Π is *affine* or *non-affine* (or, *tangent*) depending on whether it intersects the line at infinity in a *subline* or in one point, respectively.

An *affine* subplane of order q is represented by a *transversal* plane to the spread, that is, a plane of Σ intersecting $q+1$ elements of the spread.

A *non-affine* subplane π of $\Pi = PG(2, q^r)$ of order q is represented by a variety V_2^{2r-1}, that is, a ruled variety of $\Sigma = PG(2r, q)$ with the minimum order directrix a rational curve of order $r-1$ and a maximum order directrix a rational curve of order r, the two curves lying in two complementary spaces of dimension $r-1$ and r, respectively. The variety V_2^{2r-1} can be obtained by joining points of the two directrix curves corresponding via a projectivity (cf. [8], Cap.13, 8., 9. and [6], Section 4).

After the results obtained in [5], Theorem 3.1 and Theorem 3.2 for $r=2$, in this note is studied a generalization for $r>2$. Some properties about hyperplanes of Σ and their intersections with the variety V_2^{2r-1} are proved, a procedure to show how to represent substructures of the plane π in the space Σ (cf. Section 3.1) and viceversa (cf. Section 3.2), namely some kind of arcs and caps, respectively.

In Theorem 3.6 is proved how to construct in V_2^{2r-1} a rational normal curve of order $r+1$ representing a conic in π. The paper concludes with Theorem 3.7 with a partition in caps of the affine points of V_2^{2r-1} corresponding to a partition in conics of π.

2. Preliminary Notes and Results

Denote $F = GF(q)$ a finite field, $q = p^s$, p an odd prime, $\overline{F}$ the algebraic closure of the field F, F^{n+1} the $(n+1)$-dimensional vector space over F, $PG(n,q) = PrF^{n+1}$ the n-dimensional projective space contraction of F^{n+1} over F. The geometry $PG(n,q)$ is considered a sub-geometry of $\overline{PG(n,q)}$, the projective geometry over $\overline{F}$. A subspace of $PG(n,q)$ of dimension h (an h-space) is denoted S_h (cf. [7], Section 2).

Definition 2.1. *A k-arc $\mathcal{K}$ in $PG(n,q)$ is a set of $k \geq n+1$ points no $n+1$ of which lie in a hyperplane.*

A k-cap $\mathcal{K}$ of $PG(n,q)$, $n \geq 3$ is a set of k points no three of which are collinear.

A tangent of $\mathcal{K}$ is a line which has exactly one point in common with $\mathcal{K}$.

See Thas [9].

A curve of order r is denoted $\mathcal{C}^r$.

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Definition 2.2. *A rational normal curve $\mathcal{C}^n$ of $PG(n,q)$ consists of $q+1$ points ($q \geq n$) no $n+1$ of which in a hyperplane S_{n-1} (that is, a hyperplane meets $\mathcal{C}^n$ in at most n points).*

See Hirschfeld [10] p.229, Theorem 21.1.1, (iv).

Consequence - n points lie in no S_{n-2}, $n-1$ points in no S_{n-3},..., no three points in a line (that is, an S_{n-2} meets the curve in at most $n-1$ points, an S_{n-3} in $n-2$ points,..., a line in 2 points).

In $PG(3,q)$, q odd, a $(q+1)$-arc is a twisted cubic, that is, a rational normal curve of degree 3 (cf. [10], p.242–243, Theorem 21.2.3).

Definition 2.3. *A variety V_u^v of dimension u and of order v of $PG(n,q)$ is the set of the rational points of a projective variety $\overline{V}_u^v$ of $\overline{PG(n,q)}$ defined by a finite set of polynomials with coefficients in the field F.*

Definition 2.4. *The ruled variety V_2^{n-1} of $PG(n,q)$, $n \geq 4$ and $n \neq 5$, is generated by the $q+1$ lines joining the corresponding points of two birationally (projectively) equivalent curves of order m and $n-1-m$, respectively, lying in two complementary subspaces of the same dimensions, m and $n-1-m$ respectively. As such directrix curves have no point in common, then the number of points of V_2^{n-1} is $(q+1)^2$ and the order is the sum of the orders of the curves.*
The $q+1$ lines are generatrices (or, generatrix lines).

See Bertini [8], Cap.9, n.1–3, Cap.13, n.1–8, p.290, 7., Vincenti [7], Lemma 2.2, and [6].

From [8], p.287, 3., follows

RESULT 1 - *In $PG(n,q)$ a hyperplane S_{n-1} meets a ruled variety V_2^{n-1} either in*

1. *a rational normal curve of order $n-1$ (with $q \geq n-1$)*

or, in

2. *a curve of order $m < n-1$ met by all the generatrix lines and in $n-1-m$ generatrix lines and does not consist of two or more curves.*

1. *Every irreducible curve $\mathcal{C}^m$, $m \leq n-1$ contained in V_2^{n-1} is a rational normal curve, that is, it exists in a space S_m.*

Note that since throughout the paper we will speak of rational normal curves $\mathcal{C}^h$ of $PG(h,q)$ for some $h \geq 2$, we choose $q \geq h$ (see Definition 2.2).

Let Σ be the projective space $PG(2r,q)$, $r \geq 2$, $\Sigma' = PG(2r-1,q)$ a hyperplane of Σ, $\mathcal{S}$ a *regular spread* of $(r-1)$-spaces of Σ'. It is $|\mathcal{S}| = q^r + 1$. For the definition of spread, regulus and regular spread see [2] and [7], Definition 2.3 and the representation.

The Desarguesian plane $PG(2,q^r)$ is represented by Σ and by the spread $\mathcal{S}$ of Σ' according the André/Bruck and Bose method (cf. [1], [2]).

Let $r = 2$. In such a case the projective plane is $PG(2,q^2)$, $\Sigma = PG(4,q)$, $\Sigma' = PG(3,q)$, $\mathcal{S}$ is a regular spread of q^2+1 lines of Σ'. If $= PG(2,q)$ denote a non-affine Baer subplane of $PG(2,q^2)$, $P_\infty \in$ the unique point on the line at infinity, then B is represented by a variety V_2^3 having as the linear directrix a line $r_\infty \in \mathcal{S}$ and as a conic directrix $\mathcal{C}$ of a plane meeting Σ' in a line $l_\infty \in \mathcal{S}$, $l_\infty \neq r_\infty$ and with $\mathcal{C} \cap l_\infty = \varnothing$ (cf. [3], [5], [4]).

RESULT 2 – *A non degenerate conic $\mathcal{C}$ of a non-affine Baer subplane B through P_∞ is represented on the variety V_2^3 by either a twisted cubic curve (and viceversa), or by a normal rational curve of order 4 depending on whether P_∞ belongs to $\mathcal{C}$ or not.*

See [5], Theorem 3.1 and Theorem 3.2.

Let $r \geq 2$, $\Pi = PG(2,q^r)$. It is $\Sigma = PG(2r,q)$, $\Sigma' = PG(2r-1,q)$, $\mathcal{S}$ is a regular spread of $(r-1)$-subspaces, $|\mathcal{S}| = q^r + 1$.

RESULT 3 – *A non-affine subplane $\pi = PG(2,q)$ of Π having only one point P_∞ at infinity is represented in Σ by a ruled variety V_2^{2r-1}. Such a variety is the locus of the lines connecting corresponding points (via a projectivity) of a curve $\mathcal{C}_\infty^{r-1}$ of a subspace $S_{r-1}^\infty \in \mathcal{S}$ and of a curve $\mathcal{C}_0^r$ of a subspace $S_r^0 \subset \Sigma$ such that $S_r^0 \cap \Sigma' = S_{r-1}^0 \in \mathcal{S} \setminus S_{r-1}^\infty$ and $\mathcal{C}_0^r \cap S_{r-1}^0 = \varnothing$. Such lines are the generatrix lines, the curves $\mathcal{C}^r = S_r \cap V_2^{2r-1}$ with $S_r \cap \Sigma' \in \mathcal{S}$ are directrices of V_2^{2r-1}. The $q+1$ lines of π through P_∞ are represented by the generatrix lines, the other lines by the q^2 directrix curves of V_2^{2r-1}.*

See [6], Theorems 4.7, 4.8.

RESULT 4 – *A subspace S_i of $S_r = PG(r,q)$ with $i \geq r-k$, meets a variety V_k^n in a variety V_{i+k-r}^n.*

See [8], p.191, 3., comma 2.

3. Main results

3.1. From Σ to Π

Represent $\Pi = PG(2, q^r)$ in $\Sigma = PG(2r, q)$ with $r \geq 2, q \geq 2r-1$.

Let $\mathcal{S}$ be a regular spread of $(r-1)$-spaces of a hyperplane $\Sigma' = PG(2r-1, q)$, $|\mathcal{S}| = q^r + 1$. The elements of $\mathcal{S}$ are the points at infinity of Π, the r-spaces S_r such that $S_r \cap \Sigma' \in \mathcal{S}$ are the affine lines of Π, $\mathcal{S}$ is the line at infinity l_∞ of Π. Note that a *transversal r-space*, that is, an S_r with $S_r \cap \Sigma' \notin \mathcal{S}$ might represent a Baer subplane β of Π only if r is even, as β would have order $q^{\frac{r}{2}}$, and if the points of $S_r \cap \Sigma'$ can be equally divided into $q^{\frac{r}{2}} + 1$ elements of $\mathcal{S}$.

Choose and fix an $(r-1)$-space S^∞_{r-1} of $\mathcal{S}$, a curve $\mathcal{C}^{r-1}_\infty \subset S^\infty_{r-1}$ of order $r-1$, an r-space S^0_r such that $S^0_r \cap \Sigma' = S^0_{r-1} \in \mathcal{S} \setminus S^\infty_{r-1}$, a curve $\mathcal{C}^r_0 \subset S^0_r$ with $\mathcal{C}^r_0 \cap S^0_{r-1} = \varnothing$.

Let $\pi = PG(2, q)$ be a non-affine subplane of Π of order q with $P_\infty \in l_\infty$ as its unique point at infinity corresponding to the $(r-1)$-space S^∞_{r-1}. Then π is represented by the ruled variety $\mathcal{V} = V^{2r-1}_2$ obtained by connecting corresponding points of $\mathcal{C}^{r-1}_\infty$ and of $\mathcal{C}^r_0$ (see Result 3). The $q+1$ generatrix lines of $\mathcal{V}$ represent the bundle (P_∞) of the $q+1$ lines of π with center P_∞, the curve $\mathcal{C}^r_0$ is a directrix and represents one of the remaining q^2 lines of π.

NOTE 1 - *The q^2 lines of π not belonging to (P_∞), identify q^2 points on the line l_∞, one by one. Therefore there exists a subset $\overline{\mathcal{S}} \subseteq \mathcal{S} \setminus \{S^\infty_{r-1}\}$, $|\overline{\mathcal{S}}| = q^2$ corresponding to such points. For $r=2$, $\overline{\mathcal{S}} = \mathcal{S} \setminus \{S^\infty_{r-1}\}$.*

About an element $S_{r-1} \in \overline{\mathcal{S}}$ there is only one r-space S_r meeting $\mathcal{V}$ in a directrix curve $\mathcal{C}^r$, which is a line of π, the only line of π with the point at infinity S_{r-1}. Then about S^0_{r-1} there is only S^0_r meeting $\mathcal{V}$ in the directrix curve $\mathcal{C}^r_0 \subset S^0_r$.

As $\mathcal{C}^{r-1}_\infty$ is a rational normal curve, it consists of $q+1$ points no r of which in a hyperplane S_{r-2} and with $q \geq r-1$ as by hypothesis $q \geq 2r-1$ (cf. Definition 2.2).

Choose a subset $\mathcal{P} = \{P_1, \ldots, P_{r-1}\} \subset \mathcal{C}^{r-1}_\infty$ of $r-1$ independent points. Denote $S^{\mathcal{P}}_{r-2} = <\mathcal{P}>$ the $(r-2)$-space of S^∞_{r-1}, generated by the points of $\mathcal{P}$. For $r=2$ the set $\mathcal{P}$ is a singleton.

Let $\mathcal{G}_{\mathcal{P}} = \{g_1, \ldots, g_{r-1}\}$ be the set of the $r-1$ generatrix lines through the points of $\mathcal{P}$ and the corresponding $r-1$ points of $\mathcal{C}^r_0$, $\mathcal{G}' = \{g'_r, \ldots, g'_{q+1}\}$ the set of the remaining $q+2-r$ generatrix lines.

The hyperplane $H = S^0_r + S^{\mathcal{P}}_{r-2}$ meets $\mathcal{V}$ in the union of $\mathcal{C}^r_0 \subset S^0_r$ with the set $\mathcal{G}_{\mathcal{P}}$ (cf. Result 1, 2)).

Consider the subspace $S_{2r-2} = S^0_{r-1} + S^{\mathcal{P}}_{r-2}$ direct sum of the two subspaces.

In the bundle $\mathcal{B}_{2r-2}$ of hyperplanes with axes S_{2r-2} there are $q+1$ hyperplanes, one being Σ', another one being $H = S_r^0 + S_{r-2}^{\mathcal{P}}$.

Each hyperplane $H' \in \mathcal{B}_{2r-2} \setminus \{\Sigma'\}$ meets all the $q+1$ generatrix lines of $\mathcal{G}_{\mathcal{P}} \cup \mathcal{G}'$. If $H' \neq H$ and contains no generatrix line, the $r-1$ lines of $\mathcal{G}_{\mathcal{P}}$ are met by H' in the points of $\mathcal{P} \subset \mathcal{C}_{\infty}^{r-1}$. The remaining $q+2-r$ lines of $\mathcal{G}'$ are met in affine points. Denote $\mathcal{Q} = \{Q_r, \ldots, Q_{q+1}\}$ such a set of points.

Lemma 3.1. *If $r > 2$, in $\mathcal{B}_{2r-2} \setminus \{\Sigma', H\}$ there are both hyperplanes containing one generatrix and hyperplanes without any generatrix line of $\mathcal{G}_{\mathcal{P}}$.*

If $r = 2$, there exists only one hyperplane in $\mathcal{B}_{2r-2} \setminus \{\Sigma'\}$ containing a generatrix line.

Proof. From Result 1 follows that a hyperplane $H' \in \mathcal{B}_{2r-2} \setminus \{\Sigma', H\}$ meets $\mathcal{V}$ either in a rational normal curve $\mathcal{C}^{2r-1}$ of degree $2r-1$, or in a curve of order $m < n-1$ met by all the generatrices and in $n-1-m$ generatrix lines.

Let $r > 2$. Assume $H' \cap \mathcal{V}$ contains at least two generatrix lines, $g_i, g_j \in \mathcal{G}_{\mathcal{P}}$. As all the generatrix lines meet each directrix curve, denote A_i, A_j the points of g_i, g_j, respectively, belonging to the directrix curve $\mathcal{C}_0^r \subset S_r^0$.

As $S_{r-1}^0 \subset H'$, the line A_iA_j meets Σ' in a point of S_{r-1}^0, then H' would contain the whole S_0^r and $H' = H$, a contradiction.

Hence $H' \cap \mathcal{V}$ contains at most one generatrix line.

Assume each of the $q-1$ hyperplanes of $\mathcal{B}_{2r-2} \setminus \{\Sigma', H\}$ contains one generatrix line of $\mathcal{G}_{\mathcal{P}}$. Denote H', H " two different hyperplanes of $\mathcal{B}_{2r-2} \setminus \{\Sigma', H\}$, $g_1 \subset H', g_2 \subset H$ " with $g_1, g_2 \in \mathcal{G}_{\mathcal{P}}$.

Two different hyperplanes of $\mathcal{B}_{2r-2} \setminus \{\Sigma', H\}$ can have in common only the space S_{2r-2}, therefore such generatrix lines must be different. As $q \geq 2r-1 > r$, then $q-1 = |\mathcal{B}_{2r-2} \setminus \{\Sigma', H\}| > r-1 = |\mathcal{P}|$, so that we get a contradiction.

Hence in $\mathcal{B}_{2r-2} \setminus \{\Sigma', H\}$ there are both hyperplanes containing one generatrix and hyperplanes without any generatrix line of $\mathcal{G}_{\mathcal{P}}$.

If $r = 2$, $\mathcal{P} = \{P\}$, then there is only one generatrix line g through it. Hence there exists only one hyperplane in $\mathcal{B}_{2r-2} \setminus \{\Sigma'\}$ containing g, and, consequently, a conic directrix.

Theorem 3.2. *Let H' be a hyperplane of $\mathcal{B}_{2r-2} \setminus \{\Sigma', H\}$.*

$i)$ $r \geq 2$. If $H' \cap \mathcal{V}$ contains no generatrix line, then $H' \cap \mathcal{V}$ is a rational normal curve $\mathcal{C}^{2r-1}$.

$ii_1)$ $r > 2$. If $H' \cap \mathcal{V}$ contains one generatrix line $g \in \mathcal{G}_{\mathcal{P}}$, then $H' \cap \mathcal{V} = g \cup \mathcal{C}^{2r-2}$.

In both cases the curve consists of the $r-1$ points of $\mathcal{P}$ and of $q+2-r$ affine points which distribute, one for *each, on the $q+2-r$ generatrix lines of $\mathcal{G}'$ and no two of them belong to any directrix.*

Let H' be a hyperplane of $\mathcal{B}_{2r-2} \setminus \Sigma'$.

ii_2) $r=2$. If $H' \cap \mathcal{V}$ contains no generatrix, then $H' \cap \mathcal{V}$ is a rational normal cubic curve, if $H' \cap \mathcal{V}$ contains the generatrix line g, then $H' \cap \mathcal{V} = g \cup \mathcal{C}$ where $\mathcal{C}$ is the conic directrix $\mathcal{C}_0^2$ and $H' = H,$.

Proof. i) $r \geq 2$. Assume H' contains no generatrix of $\mathcal{G}_{\mathcal{P}}$.

The hyperplane H' meets all the $q+1$ generatrix lines of $\mathcal{G}_{\mathcal{P}} \cup \mathcal{G}'$. The $r-1$ lines of $\mathcal{G}_{\mathcal{P}}$ are met in the points of $\mathcal{P} \subset \mathcal{C}_\infty^{r-1}$. The remaining $q+2-r$ lines of $\mathcal{G}'$ are met in affine points. Denote $\mathcal{Q} = \{Q_r, \ldots, Q_{q+1}\}$ the set of such points.

From Result 1, 1), follows $H \cap \mathcal{V} = \mathcal{C}^{2r-1}$, where $\mathcal{C}^{2r-1} \supset \mathcal{P} \cup \mathcal{Q}$, is a rational normal curve. The condition $q \geq 2r-1$ can be proved although it has been assumed as a hypothesis (cf. [10], Theorem 21.1.1, (i)).

Let us assume a point $Q \in \mathcal{Q}$ belongs to a generatrix $g \in \mathcal{G}_{\mathcal{P}}$. Then the line g, having two points in H', should belong to H', a contradiction. We get analogous contradiction if Q and a point $P_k \in \mathcal{P}$, or, if two points $Q_j, Q_h \in \mathcal{Q}$, belong to a same generatrix g.

Hence the $q+2-r$ affine points of $\mathcal{Q}$ distribute, one for each, on the $q+1-(r-1) = q+2-r$ generatrix lines $\{g'_r, \ldots, g'_{q+1}\}$.

Assume H' contains two affine points $A, B \in \mathcal{Q}$ of a directrix curve. As $H' \supset S_{2r-2} \supset S_{r-1}^0$ such a directrix necessarily should be $\mathcal{C}_0^r$ as the line AB meets S_{r-1}^0. Therefore H' would contain the whole S_0^r and $H' = H$, a contradiction. Hence no two affine points belong to a directrix curve.

ii_1) $r > 2$. If H' contains one generatrix $g \in \mathcal{G}_{\mathcal{P}}$, then from Result 1, 2), follows $2r-1-m=1$ so that $m = 2r-2$. Therefore $H' \cap \mathcal{V} = g \cup \mathcal{C}^{2r-2}$. From Result 3, $\mathcal{C}^{2r-2}$ being irreducible, is a rational normal curve, so that it lives in a subspace S'_{2r-2} of H'. Note that $\mathcal{C}^{2r-2}$ in addition to $\mathcal{Q}$, contains the whole set $\mathcal{P}$, including the point $g \cap \mathcal{C}_\infty^{r-1} \in \mathcal{P}$, otherwise it would have q points. Moreover, it meets all the generatrix lines. There are no affine point of $\mathcal{C}^{2r-2}$ on g otherwise $\mathcal{C}^{2r-2}$ would have $q+2$ points, so that the line g meets $\mathcal{C}^{2r-2}$ in one point and does not belong to S_{2r-2}. The proof of the first property for $\mathcal{C}^{2r-2}$ is analogous to the proof in i) where contradictions arise from the possibility that H' contains one generatrix more than g. The proof of the second assertion is analogous.

ii_2) $r=2$. It is $\mathcal{P} = \{P\}$, $\mathcal{C}_\infty^{r-1}$ is a line $\mathcal{C}_\infty^1$, H' has dimension 3, g is the unique generatrix line of $\mathcal{G}_{\mathcal{P}}$. If H' does not contain g then $H' \cap \mathcal{V}$ is a rational normal cubic curve. If $g \subset H'$ then H' the residual curve of $H' \cap V_2^3$ is a conic $\mathcal{C}$, that is, $H' \cap V_2^3 = g \cup \mathcal{C}$.

As $\mathcal{C}$ is a directrix curve and $H' \supset S^0_{r-1} = S^0_1 \in \mathcal{S}$, the plane α of $\mathcal{C}$ must meet Σ' in the line S^0_1, $H' = H$, $\alpha = S^0_2$ and $\mathcal{C}$ is the conic $\mathcal{C}^2_0$.

Let H' be a hyperplane of the bundle $\mathcal{B}_{2r-2} \setminus \{\Sigma', H\}$ containing no generatrix line. Denote $\mathcal{Q} = \{H' \cap g'_i = Q_i | i = r, \dots, q+1\}$ the set of the $q+2-r$ affine points met by H'.

Let $\mathcal{Q}' = \{Q'_r, \dots, Q'_{q+1}\}$ be the subset of the points of $\pi \subset \Pi$ represented in Σ by the points of $\mathcal{Q}$ to which we add the point P_∞ represented by S^∞_{r-1}. Denote $\mathcal{K} = \mathcal{Q}' \cup \{P_\infty\}$ such a subset of points of π.

Proposition 3.3. *$\mathcal{K}$ is a $(q+3-r)$-arc of π. It is maximal, that is, a $(q+1)$-arc, when $r = 2$, q is odd, in which case $\mathcal{K}$ is a conic.*

Proof. First note that $\mathcal{K}$ has cardinality $q+1-(r-1)+1 = q+3-r$.

If three affine points of $\mathcal{K}$ were collinear, then the corresponding points of $\mathcal{Q}$ should belong to a directrix curve, a contradiction to Theorem 3.2. If Q'_i, Q'_j, P_∞ were collinear, their line would be a line through P_∞ so that the two affine points Q_i, Q_j of $\mathcal{C}^{2r-1}$ should belong to a generatrix line of $\mathcal{V}$, a contradiction to our assumption and to Theorem 3.2.

The arc $\mathcal{K}$ is maximal when $q+3-r = q+1$, that is, when $r = 2$, q is odd, $\Pi = PG(2, q^2)$, π is a Baer subplane of Π and $\mathcal{K}$ is a conic.

Denote t the tangent line of $\mathcal{K}$ at the point P_∞, g_t the corresponding line of the variety $\mathcal{V}$ in Σ.

Corollary 3.4. *The line t is represented in Σ by g_t, a generatrix line with $g_t \notin H'$.*

Proof. The tangent t of $\mathcal{K}$ at the point P_∞ is a line through it and obviously belongs to π. As all the lines through P_∞ in the subplane π are represented in Σ by the generatrix lines of $\mathcal{V}$, then g_t is a generatrix line. Such a line cannot belong to H' by hypothesis.

3.2. From Π to Σ

Let $\mathcal{C} \subset \pi$ be a non degenerate conic containing the unique infinite point P_∞ of π. Denote $\{P'_1, \dots, P'_q\}$ the q affine points of $\mathcal{C}$, $\{g'_1, \dots, g'_q\}$ the q affine lines of π connecting P_∞ with the points of $\{P'_1, \dots, P'_q\}$, t the tangent line to $\mathcal{C}$ at P_∞.

Referring to the previous notations, let $\mathcal{V} = V^{2r-1}_2$, denote $\mathcal{K}_\mathcal{C}$ the subset of $\mathcal{V}$ corresponding in Σ to the points of $\mathcal{C}$, $\mathcal{K} = \{P_1, \dots, P_q\}$ is the set of the q affine points of $\mathcal{K}_\mathcal{C}$ representing $\{P'_1, \dots, P'_q\}$, $\mathcal{G} = \{g_1, \dots, g_q, g_t\}$ the set of the $q+1$ generatrix lines of $\mathcal{V}$ corresponding to $\{g'_1, \dots, g'_q, t\}$ where g_t is the generatrix representing t, T the point $g_t \cap \mathcal{C}^{r-1}_\infty$. Note that if $r = 2$ the curve $\mathcal{C}^{r-1}_\infty$ is a line and coincides with S^∞_{r-1}.

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Theorem 3.5. $i)$ $\mathcal{K}_\mathcal{C}$ consists of $q+1$ points of $\mathcal{V}$, of which the q affine points of $\mathcal{K}$ belong each on one generatrix of $\{g_1,\ldots,g_q\}$, T, the unique point belonging to the curve $\mathcal{C}_\infty^{r-1}$, is the point at infinity of $\mathcal{K}_\mathcal{C}$.
$ii)$ $\mathcal{K}_\mathcal{C}$ is a $(q+1)$-cap, the line g_t being the tangent to $\mathcal{K}_\mathcal{C}$ at T.

Proof. $i)$ Obviously no point of $\mathcal{K}$ belongs to the generatrix g_t as no affine point of the conic $\mathcal{C}$ belongs to the tangent t.

Assume two points of $\mathcal{K}$ belong to one generatrix $g\in\{g_1,\ldots,g_q\}$. That would mean that the two corresponding points of $\mathcal{C}$ would be collinear with P_∞, a contradiction. Hence the q affine points of $\mathcal{K}$ are distributed each on one of the q generatices $g_1,\ldots,g_q$ different from g_t, that is, $P_i\in g_i$ for $i=1,\ldots,q$.

Assume $\mathcal{K}_\mathcal{C}$ contains a point $T'\in\mathcal{C}_\infty^{r-1}$, $T'\neq T$. Then the generatrix to which T' belongs should be a line $g_i\in\mathcal{G}$ for some $i=1,..,q$. As $P_i\in g_i$, then the whole generatrix $g_i=T'P_i$ would be a line of the configuration. This line would add to $\mathcal{K}$ the $q-1$ affine points of the generatrix g_i, whose corresponding points in π collinear with P_∞, would be added to $\mathcal{C}$, a contradiction. Therefore $T'=T$ and it is the unique point at infinity of $\mathcal{K}_\mathcal{C}$.

$ii)$ First note that no two points of $\mathcal{K}$ are on a generatrix line as no two affine points of $\mathcal{C}$ corresponding to them are collinear with P_∞.

Assume $P_i,P_j,P_h\in\mathcal{K}$ are collinear. Denote l their line and let P_i',P_j',P_h' be the corresponding points of $\mathcal{C}$. The line l is not a generatrix therefore it selects with the point $l\cap\Sigma'$ an element $S'_{r-1}\in\mathcal{S}$ and then a space S'_r with $S'_r\cap\Sigma'=S'_{r-1}$, containing a unique directrix $\mathcal{C}'^r\subset S'_r$. The line of π through the points P_i,P_j is represented in Σ by $\mathcal{C}'^r$, as well the line of π through P_i,P_h and P_j,P_h. That is, the three points $P_i',P_j',P_h'\in\mathcal{C}$ would be collinear, a contradiction.

The line $g_t\in\mathcal{V}$, representing the tangent t to $\mathcal{C}$ at the point P_∞, has only T in common with $\mathcal{K}_\mathcal{C}$, that is, it is the tangent line to $\mathcal{K}_\mathcal{C}$ at T.

Choose a subset $\mathcal{P}=\{P_1,\ldots,P_{r-2},T\}$ of $r-1$ points of $C_\infty^{r-1}\subset S_{r-1}^\infty$.

Denote $S'_{r-2}=<\mathcal{P}>\subset S_{r-1}^\infty$ the subspace generated by $\mathcal{P}$, let $\mathcal{G}=\{g_1,\ldots,g_{r-2},g_t\}$ be the set of the generatrix lines through the points of $\mathcal{P}$ with $T=g_t\cap\mathcal{P}$.

Theorem 3.6. *There exists a hyperplane H' with $g_t\notin H'$ containing a subspace S_{r+1} with a rational normal curve $\mathcal{C}^{r+1}$, which is a cap of $\mathcal{V}$ with $q+1$ points, of which T is at infinity.*
$\mathcal{C}^{r+1}$ represents a conic of the subplane π of Π, through P_∞.

Proof.

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$r > 2$. Choose $S_{r-1} \in \mathcal{S} \setminus \overline{\mathcal{S}} \cup \{S^{\infty}_{r-1}\}$ (cf. Note 1). Consider the set $\mathcal{G} \setminus \{g_t\} = \{g_1, \ldots, g_{r-2}\}$ of the $r-2$ generatrices. Denote S^*_{r-1} the subspace generated by them. Let $H' = S^*_{r-1} + S_{r-1}$ be the hyperplane defined by the sum of such two subspaces. Obviously $H' \neq \Sigma'$ as in Σ' there are no generatrix. As H' contains $2r-1-m = r-2$ generatrix lines, then in $H' \cap \mathcal{V}$ there is also a residual curve $\mathcal{C}^{r+1}$ of order $m = r+1$ meeting all the generatrices (cf. Result 1).

The subspace $H' \cap S^{\infty}_{r-1}$ has dimension $r-2$, therefore H' contains the whole space $S'_{r-2} = <\mathcal{P}>$, the point T included.

The curve $\mathcal{C}^{r+1}$ is a rational normal curve, then it lives in a subspace $S_{r+1} \subset H'$ that meets S'_{r-2} in one point. If $\mathcal{C}^{r+1} \subset S_{r+1}$ met each generatrix in an affine point, it would be a directrix, but the maximum order of a directrix is $r < r+1$, a contradiction. Hence it must meet one generatrix of $\mathcal{G}$ in a point $P = S_{r+1} \cap S'_{r-2}$ of $\mathcal{P}$.

Assume $P \neq T$. Then $\mathcal{C}^{r+1}$ should meet g_t in an affine point, so that $g_t \subset H'$, a contradiction, as it would add one more dimension to S^*_{r-1}. Therefore $P = T$.

The remaining q generatrices are met one each at an affine point so that $\mathcal{C}^{r+1}$ has q affine points and one point at infinity. Since $\mathcal{C}^{r+1}$ is a normal rational curve consisting of $q+1$ points, it is clear that it represents a $(q+1)$-cap.

Denote $\mathcal{C}$ the subset of points of π represented in Σ by the points of $\mathcal{C}^{r+1}$. Let P'_i, P'_j, P'_h be three affine points of $\mathcal{C}$. Denote P_i, P_j, P_h the corresponding affine points of $\mathcal{C}^{r+1}$. Assume P'_i, P'_j, P'_h are collinear. Then P_i, P_j, P_h should belong to a directrix d^r of $\mathcal{V}$.

Each directrix of $\mathcal{V}$ lives in a subspace S of dimension r such that $S \cap \Sigma' \in \mathcal{S} \setminus \{S^{\infty}_{r-1}\}$ and represents a line of π. Then it should be $S \cap \Sigma' \in \overline{\mathcal{S}}$. On the contrary $H' \supset S_{r-1}$ where $S_{r-1} \in \mathcal{S} \setminus \overline{\mathcal{S}} \cup \{S^{\infty}_{r-1}\}$, a contradiction. Therefore no three affine points of $\mathcal{C}^{r+1}$ belong to a directrix curve, that is, the corresponding points of $\mathcal{C}$ in π are not collinear.

Assume three points $P'_i, P'_j, P_{\infty} \in \mathcal{C}$ are collinear. Then the corresponding points $P_i, P_j, T \in \mathcal{C}^{r+1}$ should belong to a generatrix line, a contradiction to what was proved above.

Hence $\mathcal{C}^{r+1}$ represents a conic of π through the point P_{∞}.

$r = 2$. In this case $r-1 = 1$, $2r-1 = r+1 = 3$, $\mathcal{S}$ is a spread of lines. Denote s_{∞} the line $S^{\infty}_{r-1} = \mathcal{C}^{r-1}_{\infty}$. It is $\mathcal{S} \setminus \{s_{\infty}\} = \overline{\mathcal{S}}$ (cf. Note 1). Moreover, $\mathcal{P} = \{T\}$, $\mathcal{G} = \{g_t\}$. Choose a line $s_0 \in \overline{\mathcal{S}}$. Denote α the plane $s_0 + T$, β the plane with $\beta \cap \Sigma' = s_0$, containing the unique conic directrix $\mathcal{C}^2$ with the line s_0 at infinity. The hyperplane H defined by β and T contains also the generatrix g_t so that $H \cap \mathcal{V} = g_t \cup \mathcal{C}^2$.

Let H' be one of the $q-1$ hyperplanes around α different from both Σ' and H. It meets the line s_∞ in the point T. If $g_t \subset H'$, then H' should contain a conic directrix $\mathcal{C}$ living in a plane through s_0. But $\mathcal{C}^2$ is the unique conic directrix with s_0 at infinity, so that $H' = H$, a contradiction. Therefore g_t does not belong to H'.

Hence $H' \cap \mathcal{V}$ is an irreducible curve of order $2r-1=3$, that is, a cubic curve $\mathcal{C}^3$ living in the hyperplane H' that in such a case coincides with a space of dimension $r+1$ of the general case. It contains the unique point T at infinity, g_t is *tangent* $\mathcal{C}^3$ at the point T, but it does not belong to H'.

Denote $\mathcal{C}'$ the set of the $q+1$ points of π corresponding to the points of $\mathcal{C}^3$. If three points of $\mathcal{C}'$ were collinear, then the corresponding three points of $\mathcal{C}^3$ would belong to a directrix curve, that is, to a conic. In such a case the hyperplane H' would contain also g_t and $H' = H$, a contradiction. Therefore $\mathcal{C}'$ is a conic with P_∞ as its unique point at infinity (cf. also [5], Lemma 2.1). Note that the tangent t to $\mathcal{C}'$ at P_∞ corresponds to the generatrix g_t and belongs to π.

Theorem 3.7. *There exists a partition of the affine points of* $\mathcal{V} = V_2^{2r-1}$, $r \geq 2$, *consisting of* q *rational normal curves of order* $r+1$ *and one generatrix line.*

Proof. In the non-affine subplane π, denote $\mathcal{F}$ a bundle of hyperosculating conics at P_∞ and the line $t \ni P_\infty$ as their common tangent. It is easy to check that $|\mathcal{F}| = q$ so that $\mathcal{F} \cup t$ is a covering of π, the affine points of $\mathcal{F} \cup t$ is a partition of the q^2+q affine points of π.

Denote $\mathcal{L}_\mathcal{F}$ the q rational normal curves of order $r+1$ in $\mathcal{V}$ corresponding to the conics of $\mathcal{F}$ and g_t the generatrix line representing t (cf. Theorem 3.5). As two conics of $\mathcal{F}$ meet only in P_∞, the corresponding two curves have no affine point in common. The affine points of $\mathcal{V}$ are q^2+q, the affine points of $\mathcal{V}$ on the curves of $\mathcal{L}_\mathcal{F}$ are $q \cdot q = q^2$ to which, adding the affine points of g_t, we get q^2+q.

Notes

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