Research Article

Two Undecidable Decision Problems on an Ordered Pair of Non-Negative Integers

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For $n \in \mathbb{N}$, let $E_n = \{1 = x_k, x_i + x_j = x_k, x_i \cdot x_j = x_k : i, j, k \in \{0, \ldots, n\}\}$. For $n \in \mathbb{N}$, $f(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $\mathcal{S} \subseteq E_n$ has a solution in $\mathbb{N}^{n+1}$, then $\mathcal{S}$ has a solution in $\{0, \ldots, b\}^{n+1}$. The author proved earlier that the function $f : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every computable function $g : \mathbb{N} \to \mathbb{N}$. We present a short program in MuPAD which for $n \in \mathbb{N}$ prints the sequence $\{f_i(n)\}_{i=0}^{\infty}$ of non-negative integers converging to $f(n)$. Since f is not computable, no algorithm takes as input non-negative integers n and m and decides whether or not $\forall (x_0, \ldots, x_n) \in \mathbb{N}^{n+1} \exists (y_0, \ldots, y_n) \in \{0, \ldots, m\}^{n+1} (\forall k \in \{0, \ldots, n\} (1 = x_k \Rightarrow 1 = y_k)) \wedge (\forall i, j, k \in \{0, \ldots, n\} (x_i + x_j = x_k \Rightarrow y_i + y_j = y_k)) \wedge (\forall i, j, k \in \{0, \ldots, n\} (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k))$. Similarly, no algorithm takes as input non-negative integers n and m and decides whether or not $\forall (x_0, \ldots, x_n) \in \mathbb{N}^{n+1} \exists (y_0, \ldots, y_n) \in \{0, \ldots, m\}^{n+1} (\forall j, k \in \{0, \ldots, n\} (x_j + 1 = x_k \Rightarrow y_j + 1 = y_k)) \wedge (\forall i, j, k \in \{0, \ldots, n\} (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k))$. For $n \in \mathbb{N}$, $\beta(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $\mathcal{S} \subseteq E_n$ has a unique solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0, \ldots, b\}^{n+1}$. The author proved earlier that the function $\beta : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation. The computability of β is unknown. We present a short program in MuPAD which for $n \in \mathbb{N}$ prints the sequence $\{\beta_i(n)\}_{i=0}^{\infty}$ of non-negative integers converging to $\beta(n)$.

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 doi.org/10.32388/JCNAAA

1. The Collatz problem leads to a short computer program that computes in the limit a function $\gamma : \mathbb{N} \to \{0,1\}$ of unknown computability

Definition 1. *(cf. [1]). A computation in the limit of a function $f : \mathbb{N} \to \mathbb{N}$ is a semi-algorithm which takes as input a non-negative integer n and for every $m \in \mathbb{N}$ prints a non-negative integer $\xi(n,m)$ such that $\lim\limits_{m \to \infty} \xi(n,m) = f(n)$.*

By Definition 1, a function $f : \mathbb{N} \to \mathbb{N}$ is computable in the limit when there exists an infinite computation which takes as input a non-negative integer n and prints a non-negative integer on each iteration and prints $f(n)$ on each sufficiently high iteration.

It is known that there exists a limit-computable function $f : \mathbb{N} \to \mathbb{N}$ which is not computable, see Theorem 1. Every known proof of this fact does not lead to the existence of a short computer program that computes f in the limit. So far, short computer programs can only compute in the limit functions from $\mathbb{N}$ to $\mathbb{N}$ whose computability is proven or unknown.

Lemma 1. For every $n \in \mathbb{N}$,

$$\frac{\mathrm{sign}(n-1) \cdot (2n + (1 - (-1)^n) \cdot (5n+2))}{4} = \begin{cases} 0, & \text{if } n = 1 \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \\ 3n+1, & \text{if } n \text{ is odd and } n \neq 1 \end{cases}$$

MuPAD is a part of the Symbolic Math Toolbox in MATLAB R2019b. By Lemma 1, the following program in MuPAD computes in the limit a function $\gamma : \mathbb{N} \to \{0,1\}$.

```
input("Input a non-negative integer n",n):

while TRUE do

print(sign(n)):

n:=sign(n-1)*(2*n+(1-(-1)^n)*(5*n+2))/4:

end_while:
```

The computability of γ is unknown, see [2]. The Collatz conjecture implies that $\gamma(n) = 0$ for every $n \in \mathbb{N}$.

doi.org/10.32388/JCNAAA

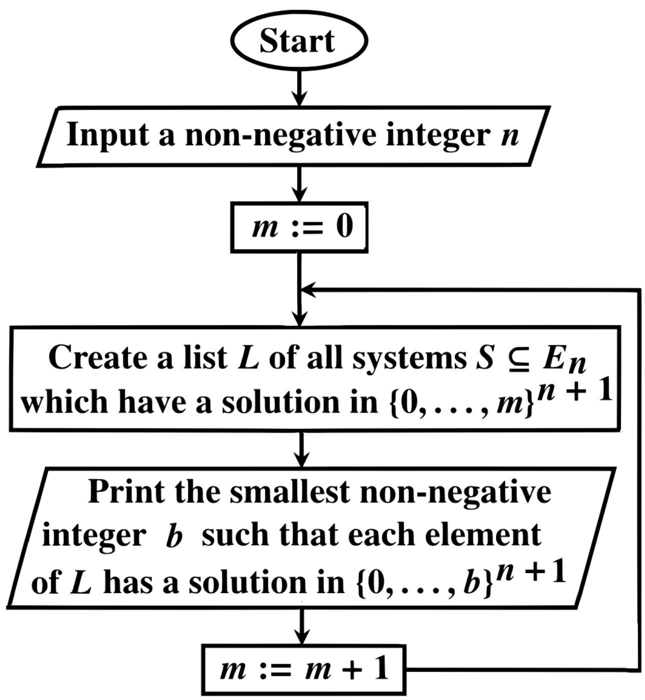
2. A limit-computable function $f : \mathbb{N} \to \mathbb{N}$ which eventually dominates every computable function $g : \mathbb{N} \to \mathbb{N}$

For $n \in \mathbb{N}$, let

$$E_n = \{1 = x_k, x_i + x_j = x_k, x_i \cdot x_j = x_k : i, j, k \in \{0, \ldots, n\}\}$$

Theorem 1. [3]. *There exists a limit-computable function $f : \mathbb{N} \to \mathbb{N}$ which eventually dominates every computable function $g : \mathbb{N} \to \mathbb{N}$.*

We present an alternative proof of Theorem 1. For $n \in \mathbb{N}$, $f(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $\mathcal{S} \subseteq E_n$ has a solution in $\mathbb{N}^{n+1}$, then $\mathcal{S}$ has a solution in $\{0, \ldots, b\}^{n+1}$. The function $f : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every computable function $g : \mathbb{N} \to \mathbb{N}$, see [4]. The term "dominated" in the title of [4] means "eventually dominated". Flowchart 1 shows a semi-algorithm which computes $f(n)$ in the limit, see [4].



Flowchart 1. A semi-algorithm which computes $f(n)$ in the limit

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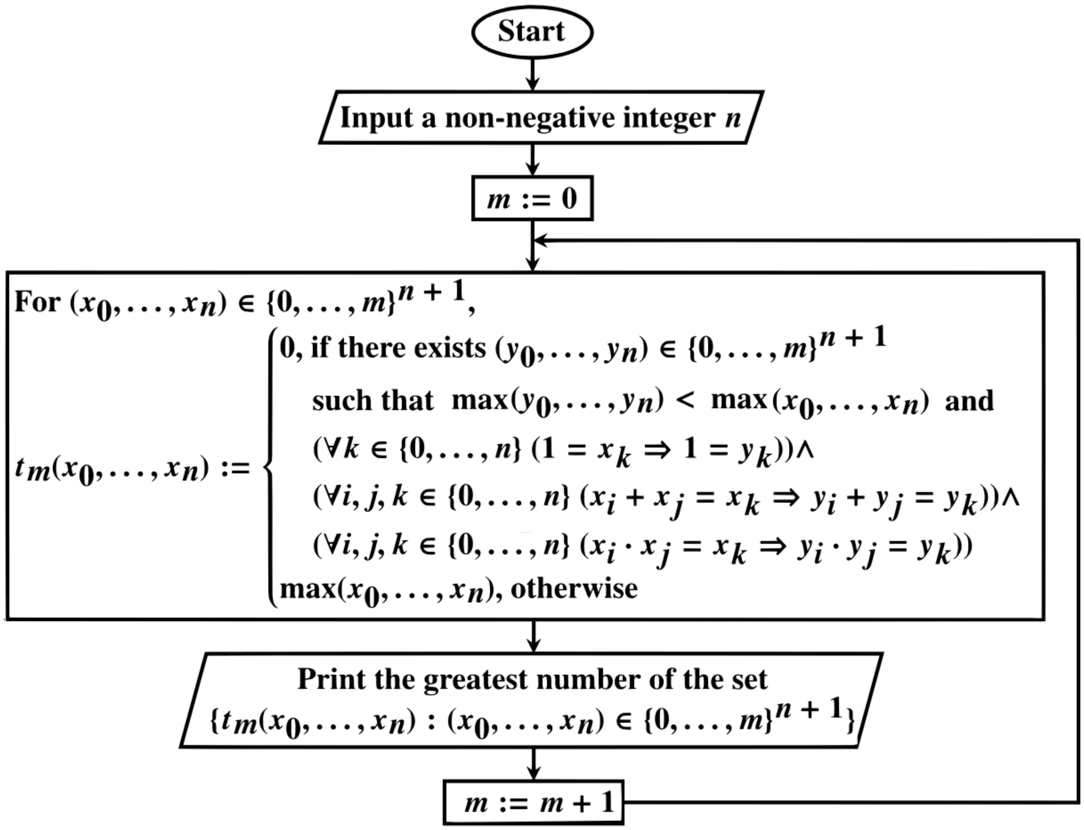
3. The first undecidable decision problem on an ordered pair of non-negative integers

Flowchart 2 shows a simpler semi-algorithm which computes $f(n)$ in the limit.



Flowchart 2. A simpler semi-algorithm which computes $f(n)$ in the limit

Lemma 2. *For every $n, m \in \mathbb{N}$, the number printed by Flowchart 2 does not exceed the number printed by Flowchart 1.*

Proof. For every $(a_0, \ldots, a_n) \in \{0, \ldots, m\}^{n+1}$,

$$E_n \supseteq \{1 = x_k : (k \in \{0, \ldots, n\}) \wedge (1 = a_k)\} \cup$$

$$\{x_i + x_j = x_k : (i, j, k \in \{0, \ldots, n\}) \wedge (a_i + a_j = a_k)\} \cup$$

$$\{x_i \cdot x_j = x_k : (i, j, k \in \{0, \ldots, n\}) \wedge (a_i \cdot a_j = a_k)\}$$

doi.org/10.32388/JCNAAA

□

Lemma 3. *For every $n, m \in \mathbb{N}$, the number printed by Flowchart 1 does not exceed the number printed by Flowchart 2.*

Proof. Let $n, m \in \mathbb{N}$. For every system of equations $\mathcal{S} \subseteq E_n$, if $(a_0, \dots, a_n) \in \{0, \dots, m\}^{n+1}$ and $(a_0, \dots, a_n)$ solves $\mathcal{S}$, then $(a_0, \dots, a_n)$ solves the following system of equations:

$$\{1 = x_k : (k \in \{0, \dots, n\}) \wedge (1 = a_k)\} \cup$$

$$\{x_i + x_j = x_k : (i, j, k \in \{0, \dots, n\}) \wedge (a_i + a_j = a_k)\} \cup$$

$$\{x_i \cdot x_j = x_k : (i, j, k \in \{0, \dots, n\}) \wedge (a_i \cdot a_j = a_k)\}$$

□

Theorem 2. *For every $n, m \in \mathbb{N}$, Flowcharts 1 and 2 print the same number.*

Proof. It follows from Lemmas 2 and 3. □

Definition 2. *An approximation of a tuple $(x_0, \dots, x_n) \in \mathbb{N}^{n+1}$ is a tuple $(y_0, \dots, y_n) \in \mathbb{N}^{n+1}$ such that*

$$(\forall k \in \{0, \dots, n\}(1 = x_k \Rightarrow 1 = y_k)) \wedge$$

$$(\forall i, j, k \in \{0, \dots, n\}(x_i + x_j = x_k \Rightarrow y_i + y_j = y_k)) \wedge$$

$$(\forall i, j, k \in \{0, \dots, n\}(x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k))$$

Observation 1. *For every $n \in \mathbb{N}$, there exists a set $\mathcal{A}(n) \subseteq \mathbb{N}^{n+1}$ such that*

$$card(\mathcal{A}(n)) \leqslant 2^{card(E_n)} = 2^{n+1+2\cdot(n+1)^3}$$

and every tuple $(x_0, \dots, x_n) \in \mathbb{N}^{n+1}$ possesses an approximation in $\mathcal{A}(n)$.

Observation 2. *For every $n \in \mathbb{N}$, $f(n)$ equals the smallest $b \in \mathbb{N}$ such that every tuple $(x_0, \dots, x_n) \in \mathbb{N}^{n+1}$ possesses an approximation in $\{0, \dots, b\}^{n+1}$.*

Observation 3. *For every $n, m \in \mathbb{N}$, Flowcharts 1 and 2 print the smallest $b \in \{0, \dots, m\}$ such that every tuple $(x_0, \dots, x_n) \in \{0, \dots, m\}^{n+1}$ possesses an approximation in $\{0, \dots, b\}^{n+1}$.*

Theorem 3. *No algorithm takes as input non-negative integers n and m and returns the logical value of the following sentence: every tuple $(x_0, \dots, x_n) \in \mathbb{N}^{n+1}$ possesses an approximation in $\{0, \dots, m\}^{n+1}$.*

Proof. Since the function f is not computable, it follows from Observation 2. □

4. A short program in MuPAD that computes f in the limit

The following program in MuPAD implements the semi-algorithm shown in Flowchart 2.

```
input("Input a non-negative integer n",n):
m:=0:
while TRUE do
X:=combinat::cartesianProduct([s $s=0..m] $t=0..n):
Y:=[max(op(X[u])) $u=1..(m+1)^(n+1)]:
for p from 1 to (m+1)^(n+1) do for q from 1 to (m+1)^(n+1) do
v:=1:
for k from 1 to n+1 do
if 1=X[p][k] and 1<>X[q][k] then v:=0 end_if:
for i from 1 to n+1 do for j from i to n+1 do if X[p][i]+X[p][j]=X[p][k] and X[q]
[i]+X[q][j]<>X[q][k] then v:=0 end_if:
if X[p][i]*X[p][j]=X[p][k] and X[q][i]*X[q][j]<>X[q][k] then v:=0 end_if:
end_for:
end_for:
end_for:
if max(op(X[q]))<max(op(X[p])) and v=1 then Y[p]:=0 end_if:
end_for:
end_for:
print(max(op(Y))):
m:=m+1:
end_while:
```

5. The second undecidable decision problem on an ordered pair of non-negative integers

For $n \in \mathbb{N}$, $h(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $\mathcal{S} \subseteq \{x_j + 1 = x_k, x_i \cdot x_j = x_k : i,j,k \in \{0,\ldots,n\}\}$ has a solution in $\mathbb{N}^{n+1}$, then $\mathcal{S}$ has a solution in $\{0,\ldots,b\}^{n+1}$. From [4] and Lemma 3 in [5], it follows that the function $h : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every computable function $g : \mathbb{N} \to \mathbb{N}$. A bit shorter program in MuPAD computes h in the limit.

Theorem 4. *No algorithm takes as input non-negative integers n and m and returns the logical value of the following sentence:*

$$\forall(x_0, \ldots, x_n) \in \mathbb{N}^{n+1} \exists(y_0, \ldots, y_n) \in \{0, \ldots, m\}^{n+1}$$

$$(\forall j, k \in \{0, \ldots, n\}(x_j + 1 = x_k \Rightarrow y_j + 1 = y_k))\wedge$$

$$(\forall i, j, k \in \{0, \ldots, n\}(x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k))$$

Proof. It holds because the function h is not computable. □

6. A limit-computable function $\beta : \mathbb{N} \to \mathbb{N}$ of unknown computability which eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation

The Davis-Putnam-Robinson-Matiyasevich theorem states that every listable set $\mathcal{M} \subseteq \mathbb{N}^n$ $(n \in \mathbb{N} \setminus \{0\})$ has a Diophantine representation, that is

$$(a_1, \ldots, a_n) \in \mathcal{M} \Longleftrightarrow \exists x_1, \ldots, x_m \in \mathbb{N} W(a_1, \ldots, a_n, x_1, \ldots, x_m) = 0 \qquad \text{(R)}$$

for some polynomial W with integer coefficients, see [6]. The representation (R) is said to be single-fold, if for any $a_1, \ldots, a_n \in \mathbb{N}$ the equation $W(a_1, \ldots, a_n, x_1, \ldots, x_m) = 0$ has at most one solution $(x_1, \ldots, x_m) \in \mathbb{N}^m$.

Hypothesis 1. ([7][8][9][10][11][12]). *Every listable set $\mathcal{X} \subseteq \mathbb{N}^k$ $(k \in \mathbb{N} \setminus \{0\})$ has a single-fold Diophantine representation.*

For $n \in \mathbb{N}$, $\beta(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $\mathcal{S} \subseteq E_n$ has a unique solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0, \ldots, b\}^{n+1}$. The computability of β is unknown.

Theorem 5. *The function $\beta : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. This is proved in [4]. Flowchart 3 shows a semi-algorithm which computes $\beta(n)$ in the limit, see [4].

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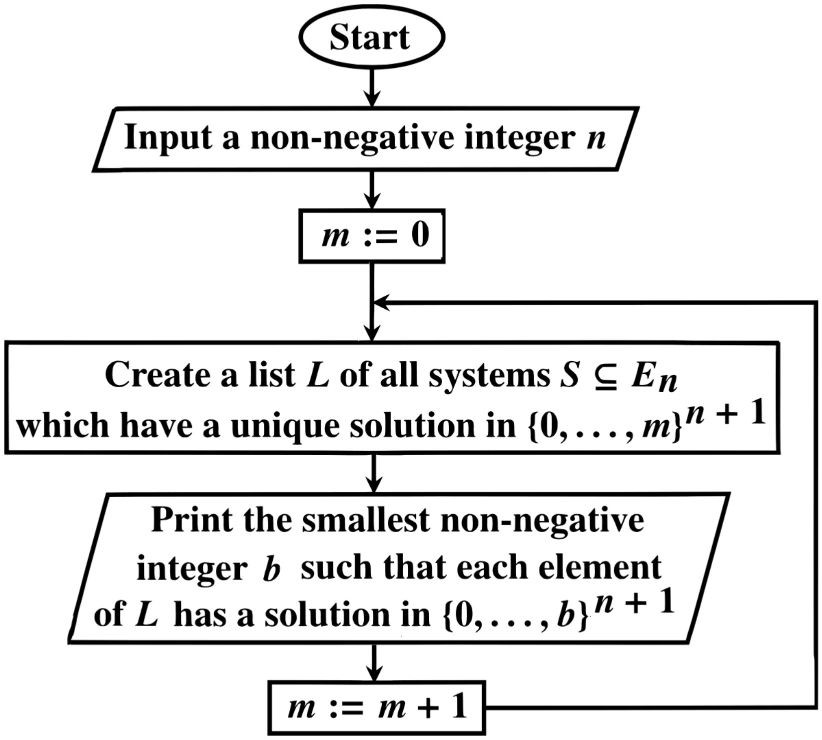
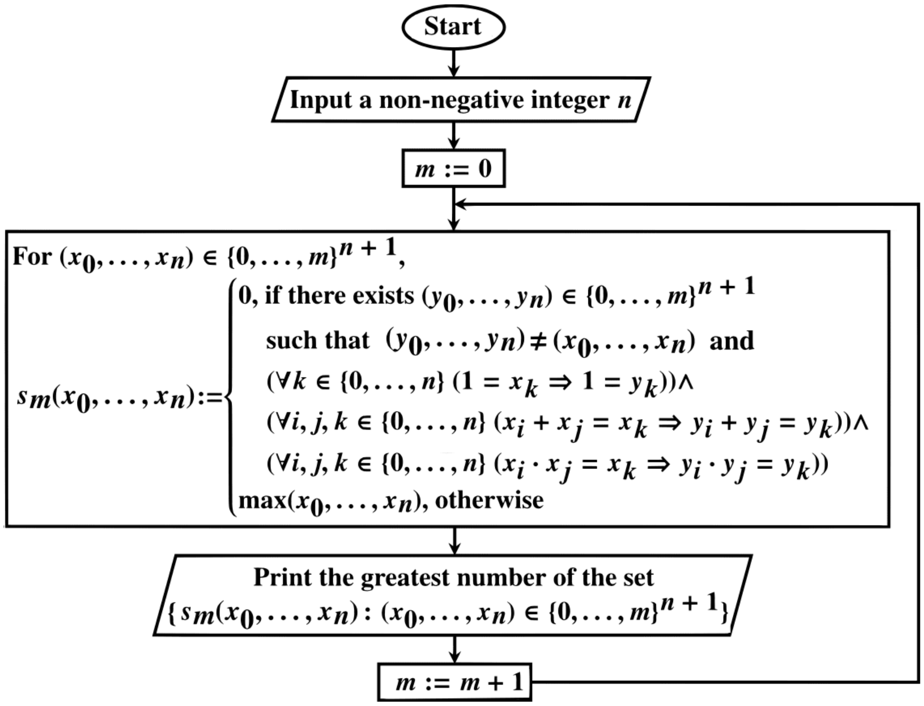


Flowchart 3. A semi-algorithm which computes $\beta(n)$ in the limit

□

7. A short program in MuPAD that computes β in the limit

Flowchart 4 shows a simpler semi-algorithm which computes $\beta(n)$ in the limit.



Flowchart 4. A simpler semi-algorithm which computes $\beta(n)$ in the limit

Lemma 4. *For every $n, m \in \mathbb{N}$, the number printed by Flowchart 4 does not exceed the number printed by Flowchart 3.*

Proof. For every $(a_0, \ldots, a_n) \in \{0, \ldots, m\}^{n+1}$,

$$E_n \supseteq \{1 = x_k : (k \in \{0, \ldots, n\}) \wedge (1 = a_k)\} \cup$$

$$\{x_i + x_j = x_k : (i, j, k \in \{0, \ldots, n\}) \wedge (a_i + a_j = a_k)\} \cup$$

$$\{x_i \cdot x_j = x_k : (i, j, k \in \{0, \ldots, n\}) \wedge (a_i \cdot a_j = a_k)\}$$

□

Lemma 5. *For every $n, m \in \mathbb{N}$, the number printed by Flowchart 3 does not exceed the number printed by Flowchart 4.*

Proof. Let $n, m \in \mathbb{N}$. For every system of equations $\mathcal{S} \subseteq E_n$, if $(a_0, \ldots, a_n) \in \{0, \ldots, m\}^{n+1}$ is a unique solution of $\mathcal{S}$ in $\{0, \ldots, m\}^{n+1}$, then $(a_0, \ldots, a_n)$ solves the system $\hat{\mathcal{S}}$, where

$$\hat{\mathcal{S}} = \{1 = x_k : (k \in \{0, \ldots, n\}) \wedge (1 = a_k)\} \cup$$

doi.org/10.32388/JCNAAA

$$\{x_i + x_j = x_k : (i,j,k \in \{0,\ldots,n\}) \wedge (a_i + a_j = a_k)\} \cup$$

$$\{x_i \cdot x_j = x_k : (i,j,k \in \{0,\ldots,n\}) \wedge (a_i \cdot a_j = a_k)\}$$

By this and the inclusion $\hat{\mathcal{S}} \supseteq \mathcal{S}$, $\hat{\mathcal{S}}$ has exactly one solution in $\{0,\ldots,m\}^{n+1}$, namely $(a_0,\ldots,a_n)$. □

Theorem 6. *For every $n, m \in \mathbb{N}$, Flowcharts 3 and 4 print the same number.*

Proof. It follows from Lemmas 4 and 5. □

The following program in MuPAD implements the semi-algorithm shown in Flowchart 4.

```
input("Input a non-negative integer n",n):

m:=0:

while TRUE do

X:=combinat::cartesianProduct([s $s=0..m] $t=0..n):

Y:=[max(op(X[u])) $u=1..(m+1)^(n+1)]:

for p from 1 to (m+1)^(n+1) do

for q from 1 to (m+1)^(n+1) do

v:=1:

for k from 1 to n+1 do

if 1=X[p][k] and 1<>X[q][k] then v:=0 end_if:

for i from 1 to n+1 do

for j from i to n+1 do

if X[p][i]+X[p][j]=X[p][k] and X[q][i]+X[q][j]<>X[q][k] then v:=0 end_if:

if X[p][i]*X[p][j]=X[p][k] and X[q][i]*X[q][j]<>X[q][k] then v:=0 end_if:

end_for:

end_for:

end_for:

if q<>p and v=1 then Y[p]:=0 end_if:

end_for:

end_for:

print(max(op(Y))):

m:=m+1:

end_while:
```


doi.org/10.32388/JCNAAA

Notes

2020 Mathematics Subject Classification: 03D20, 11U05.

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Declarations

Funding: No specific funding was received for this work.

Potential competing interests: No potential competing interests to declare.