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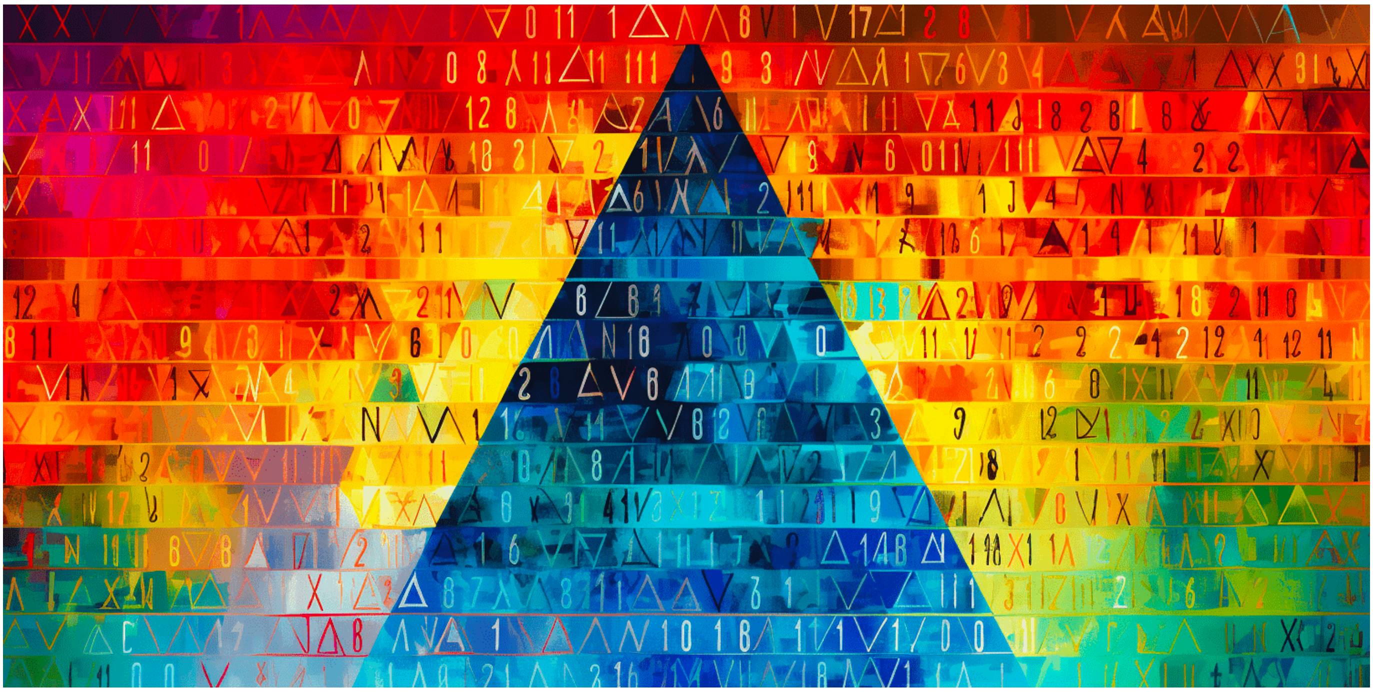

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TERNARY GOLDBACH CONJECTURE IMPLIES ABC CONJECTURE

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Abstract. Several crucial properties of abc conjecture are presented and proven. One of the resultsis that the ternary Goldbach Conjecture implies the existence of three numbers (a, b, c) that satisfy the abc conjecture for an arbitrary value of the sum $c = a + b$.
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The abc conjecture says the following. For every positive real number ϵ, and a triplet (a, b, c) of pairwise coprime positive integers with $a+b = c$, one has $k < K(\epsilon) < \infty$, with $k = c/r^{1+\epsilon}$, where $r = \mathrm{rad}(a\,b\,c)$. The conjecture is regarded as unproven [1].

The ternary Goldbach Conjecture was proven in Ref. [2]. Why? Even if the paper is not published in a journal, the consensus of experts says that the article is accurate. So, any number c (odd or even) can be presented as a sum $a + q + p + y = c$, where a, q, p are primes, and for the number y one has $b = q + p + y \geq 1$, where y can be any integer (prime, non-prime, positive, negative). This means that c can have any value.

Let me arrange the numbers in the way $a \geq q \geq p \geq y$. Then $a + 1 \leq c \leq 4\,a$, and

$$k = \frac{c}{r^{1+\epsilon}} \leq \frac{4\,a}{a^{1+\epsilon}\,(\mathrm{rad}((q+p+y)\,c))^{1+\epsilon}} \leq \frac{4\,a}{(2a)^{1+\epsilon}} < \frac{4}{2^{1+\epsilon}} \tag{1}$$

because $\mathrm{rad}((q + p + y)\,c) \geq 2$. Therefore, if a is any prime, and if $1 \leq b \leq 3\,a$ is any number, the abc conjecture holds.

In conclusion, $a + b = c$ satisfies the abc conjecture for any value of c. Other proofs of abc conjecture are written below.

1. The signature of abc conjecture

The abc conjecture implies that in the limit $c \to \infty$, one has $r = \infty$. Otherwise, for every single $\epsilon > 0$ one has $K(\epsilon) = \infty$. For arbitrary

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$m > 0$ one has

$$c/r^{1+m} = U\,W\,, \tag{2}$$

where

$$U = c/r^{1+\epsilon}\,, \quad W = r^{\epsilon}/r^{m}\,, \tag{3}$$

and $\epsilon > 0$ is arbitrary. For $\epsilon > m$, in the limit $r \to \infty$ the abc conjecture implies $U = 0$, as $W = \infty$; because the abc conjecture implies finiteness of $c/r^{1+m} < \infty$ as well. One concludes that in the limit $r \to \infty$, the abc conjecture implies $k = c/r^{1+\epsilon} = 0$. If, for some triplet, the $U \neq 0$ happens in the limit $r \to \infty$, the abc conjecture is wrong because then $c/r^{1+m} = \infty$. Therefore, the limit exists. Accordingly, in this limit there is an infinite number of triplets (a, b, c) with k arbitrarily close to zero. In other words, the abc conjecture implies that for an arbitrary constant $\delta > 0$ there is an infinite number of triplets (a, b, c) satisfying $c/r^{1+\epsilon} < \delta$, $c < \delta\, r^{1+\epsilon}$.

1.1. Realization of the signature.

Because a, b, c have no common factors, one has $r = \text{rad}(a\,b)\,\text{rad}(c)$.

Accordingly, $c < \delta\,(\text{rad}(ab))^{1+\epsilon}\,(\text{rad}(c))^{1+\epsilon}$. Here and in the following, δ is a fixed parameter. Let us study such numbers c which are prime numbers, namely $c = 2, 3, 5, \ldots, \infty$. Then $c = \text{rad}(c)$. Therefore, $1 < \delta\,(\text{rad}(ab))^{1+\epsilon}\,(\text{rad}(c))^{\epsilon}$. By increasing c, $\text{rad}(c)$ tends to infinity, $(\text{rad}(ab))^{1+\epsilon} \geq 1$, and there is an infinite amount of different primes. Therefore, the infinite amount of triplets satisfies $1 < \delta\,(\text{rad}(ab))^{1+\epsilon}\,(\text{rad}(c))^{\epsilon}$. This holds for any combination of a and b for a given $c = a + b$.

In the following, c is again an arbitrary integer. Because there are several ways to put $c = a + b$, k can take several values for a given c. The maximum value $S(c) = \max k(c)$ saturates at zero. This means the limit $k(c) \leq S(c) = 0$, $c \to \infty$.

2. No transitions between $k = 0$ and $k = \infty$

The first part of the paper has shown that there are infinitely many triplets at $k < 1$. Therefore, if the abc conjecture fails, the k starts endless bouncing (while the increase of c) between near zero and large values ($k \gg 1$). There are an infinite number of forth (in values of k) and back trans-passings; each one leaves behind a trace of the triplets. Hence, an infinite number of triplets would be expected within a gap $k_1 < k < k_2$, where $k_1 \neq 0$. An alternative formulation of the abc conjecture is that for $k \geq 1$, there is a finite number of triplets [3]. Hence, the number of triplets within $1 < k < k_2$ has to be finite.

Otherwise, even if $k < K(\epsilon)$ the conjecture fails because there is an infinite amount of triplets with $k \geq 1$. But if $k < K(\epsilon)$, the conjecture cannot fail. We came to a disagreement. Hence, the number of triplets within $1 < k < k_2$ is finite.

3. The boundary of limit

Let us define

$$Z = \frac{r(c+Y)}{r(c)} \frac{r(c)}{r(c-1)} = \frac{r(c-1+1+Y)}{r(c-1)} \,. \tag{4}$$

Such an integer Y exists within $2 - c \leq Y < \infty$ so that

$$Z > G \tag{5}$$

together with

$$\frac{r(c+Y)}{r(c)} < M \tag{6}$$

because non-vanishing G can be arbitrarily small, and the finite M can be arbitrarily large. The $Y = Y(c)$.

Eqs. (4), (5), (6) imply

$$\frac{r(c)}{r(c-1)} > \frac{G}{M} \,, \tag{7}$$

which implies

$$\frac{r(c+1)}{r(c)} > L \neq 0 \,. \tag{8}$$

The ratio reads

$$\frac{c}{c+1} \left(\frac{r(c+1)}{r(c)} \right)^{1+\epsilon} = \frac{k(c)}{k(c+1)} = \beta \,. \tag{9}$$

Let us assume for a moment that the abc conjecture fails. Because there are infinitely many triplets at $k = 0$ while increasing c, k starts to jump abruptly from nearly zero to unlimitedly large values. Then if abc conjecture fails, β changes repeatedly from zero to infinity and from infinity to zero in the limit $c \to \infty$. Therefore, $r(c+1)/r(c)$ changes repeatedly from zero to infinity and from infinity to zero during the growth of c. But this comes into a disagreement with Eq. (8).

References

[1] D. Castelvecchi, "Mathematical proof that rocked number theory will be published," Nature (3 April 2020).

[2] H. A. Helfgott, The ternary Goldbach conjecture is true, arXiv:1312.7748 [math.NT]
https://doi.org/10.48550/arXiv.1312.7748

[3] D. W. Masser, "Open problems", Proceedings of the Symposium on Analytic Number Theory, W. W. L. Chen., London: Imperial College, 1985, Vol. 25.