v1: 18 May 2024

Research Article

Some Aspects of Maxwell's Equations, Klein-Gordon Equations, and Heat and Mass Transfer Equations in an n-Dimensional Maximally Symmetric Space-Time from the Classical and Quantum Mechanical Standpoints

Peer-approved: 18 May 2024

Qeios, Vol. 6 (2024)
ISSN: 2632-3834

Harish Parthasarathy[1,2]

1. Netaji Subhas Inst of Technology, India; 2. ECE division, Netaji Subhas University of Technology, Delhi, India

This manuscript examines Maxwell's equations, Klein-Gordon equations, and heat and mass transfer equations in n-dimensional maximally symmetric space-time. It investigates these equations in spherical and hyperbolic spaces embedded in higher-dimensional Euclidean and Minkowski spaces. The study focuses on the implications of these geometries and symmetries on the behaviour of the equations, highlighting how specific transformations and parametrizations impact their solutions. The findings reveal the underlying connections between geometric symmetries and physical laws, providing insights into their possible applications in theoretical physics. We touch upon both classical and quantum mechanical aspects of density and velocity evolutions with time in the universe. Quantum mechanical aspects of single and two-particle state evolution and statistical moments of the matter four-current are derived from the quantum Boltzmann equation and Feynman's path integral method for fields applied to gravity interacting with electrons and positrons.

Correspondence: papers@team.qeios.com — Qeios will forward to the authors

Consider first a three-dimensional maximally symmetric space defined by the surface equation

$$x^2+y^2+z^2+u^2=S^2$$

or equivalently as

$$r^2+u^2=S^2, r^2=x^2+y^2+z^2$$

This surface is a 3-dimensional spherical surface immersed in $\mathbb{R}^4$. This surface is invariant under the linear transformation

$$\mathbf{r}=R\mathbf{r}'+\mathbf{b}u', u=\mathbf{c}^T\mathbf{r}'+d.u', \mathbf{r}=(x,y,z)^T$$

on $\mathbb{R}^4$, or equivalently, under

$$\begin{pmatrix}\mathbf{r}\\ u\end{pmatrix}=\begin{pmatrix}\mathbf{R} & \mathbf{b}\\ \mathbf{c}^T & d\end{pmatrix}\begin{pmatrix}\mathbf{r}'\\ u'\end{pmatrix}$$

where the matrix

$$\mathbf{T}=\begin{pmatrix}\mathbf{R} & \mathbf{b}\\ \mathbf{c}^T & d\end{pmatrix}$$

is an orthogonal matrix, ie,

$$\mathbf{T}^T\mathbf{T}=\mathbf{I}_4$$

or equivalently,

$$\mathbf{R}^T\mathbf{R}+\mathbf{c}\mathbf{c}^T=\mathbf{I}_3, \mathbf{R}^T\mathbf{b}+\mathbf{c}d=0,$$

$$\mathbf{b}^T\mathbf{b}+d^2=1$$

These equations are equivalent to

$$d=(1-\mathbf{b}^T\mathbf{b})^{1/2}, \mathbf{c}=-(1-\mathbf{b}^T\mathbf{b})^{-1/2}\mathbf{R}^T\mathbf{b},$$

$$\mathbf{R}^T\mathbf{R}+(1-\mathbf{b}^T\mathbf{b})^{-1}\mathbf{R}^T\mathbf{b}\mathbf{b}^T\mathbf{R}=\mathbf{I}_3---(a)$$

The dimension of the space of linear transformations that leave this 3-D surface invariant is thus the same as that of the Lie group $SO(4)$ and this dimension is $6=(3(3+1)/2$. Therefore, this surface with the metric induced from the metric

$$ds^2=dx^2+dy^2+dz^2+du^2=d\mathbf{r}^Td\mathbf{r}+(du)^2$$

on $\mathbb{R}^4$ is also invariant under the induced diffeomorphism, ie, under the diffeomorphism

$$\mathbf{r}=\mathbf{R}\mathbf{r}'+\mathbf{b}\sqrt{S^2-r'^2}$$

where $\mathbf{R},\mathbf{b}$ satisfy the constraint (a). Note that this induced metric is given by

$$dl^2=d\mathbf{r}^Td\mathbf{r}+(d\sqrt{S^2-r^2})^2$$

or equivalently, using polar coordinates,

$$\mathbf{r}=r\hat{n}, \hat{n}=[n_1,n_2,n_3]^T, n_1^2+n_2^2+n_3^2=1,$$

so that

$$n_1=cos(\phi)sin(\theta), y=sin(\phi)sin(\theta), z=cos(\theta),$$

we get

$$d\mathbf{r}=rd\hat{n}+dr.\hat{n}$$

and hence, since $\hat{n}^T\hat{n}=1$, so that $\hat{n}^Td\hat{n}=0$,

$$d\mathbf{r}^Td\mathbf{r}=r^2d\hat{n}^Td\hat{n}+dr^2=$$

$$r^2(d\theta^2+sin^2(\theta)d\phi^2)+dr^2$$

since

$$d\hat{n}^Td\hat{n}=d\theta^2+sin^2(\theta)d\phi^2$$

This gives us the metric of our 3-D maximally symmetric space as

$$dl^2=dr^2+r^2(d\theta^2+sin^2(\theta)d\phi^2)+(d\sqrt{S^2-r^2})^2$$

$$=dr^2(1+r^2/(S^2-r^2))+r^2(d\theta^2+sin^2(\theta)d\phi^2)$$

$$= S^2dr^2/(S^2 - r^2) + r^2(d\theta^2 + sin^2(\theta)d\phi^2)$$

or equivalently, changing the radial coordinate to the "comoving" one r_1, where

$$r = Sr_1,$$

we get

$$dl^2 = S^2dr_1^2/(1 - r_1^2) + S^2r_1^2(d\theta^2 + sin^2(\theta)d\phi^2)$$

An alternate parametrization is to choose

$$r_1 = cos(\chi)$$

to get

$$dl^2 = S^2(d\chi^2 + sin^2(\chi)(d\theta^2 + sin^2(\theta)d\phi^2))$$

The space with this metric is called a spherical maximally symmetric $3 - D$ space. The fact that this metric is invariant under a 6-dimensional Lie group of transformations is usually addressed by saying that the metric admits six Killing vectors. Another kind of maximally symmetric space is a hyperbolic maximally symmetric space defined by the equations

$$x^2 + y^2 + z^2 - u^2 = S^2$$

or equivalently,

$$u^2 - r^2 = -S^2$$

This space is again a 3-D surface imbedded in $\mathbb{R}^4$, invariant under the linear transformations of $\mathbb{R}^4$ defined by

$$\begin{pmatrix} \mathbf{r} \\ u \end{pmatrix} = \mathbf{T}\begin{pmatrix} \mathbf{r}' \\ u' \end{pmatrix}$$

where $\mathbf{T} \in SO(3, 1)$, i.e.,

$$\mathbf{T}'\mathbf{J}\mathbf{T} = \mathbf{T} - - - (b)$$

with

$$\mathbf{J} = diag[1, 1, 1, -1]$$

Actually, this surface has two connected components defined by $u = \pm\sqrt{S^2 + r^2}$ unlike the spherical case where $u = \pm\sqrt{S^2 - r^2}$ got connected at $r = S$. Again, the dimension of the Lie group $SO(3, 1)$ that leaves this surface invariant is six, and the induced transformation on this 3-D surface is given as

$$\mathbf{r} = \mathbf{R}\mathbf{r}' + \mathbf{b}$$

where the $\mathbf{R}, \mathbf{b}$ again satisfy a constraint determined by (b). As before, this is a six-parameter family of diffeomorphisms on the 3-D hyperbolic surface that leaves the metric on the surface invariant, where the metric is that induced by the metric

$$ds^2 = dx^2 + dy^2 + dz^2 - du^2$$

on $\mathbb{R}^4$ with $u = \sqrt{r^2 - S^2}$. The induced metric on the surface is therefore, using polar coordinates for x, y, z,

$$dl^2 = dr^2 + r^2(d\theta^2 + r^2sin^2(\theta)d\phi^2) - r^2dr^2/(r^2 - S^2)$$

$$= S^2dr^2/(S^2 - r^2) + r^2(d\theta^2 + sin^2(\theta)d\phi^2)$$

Since on this surface, $r \geq S$, we can change the variables to $r = S.\,cosh(\chi)$ to get the metric in the form

$$dl^2 = S^2(-d\chi^2 + cosh^2(\chi)(d\theta^2 + sin^2(\theta)d\phi^2))$$

On the other hand, if in the surface equation, we replaced S by iS so that the surface equation becomes

$$u^2 - r^2 = S^2$$

then again this surface is invariant under $SO(3, 1)$ and the metric induced by the $SO(3, 1)$ invariant metric

$$ds^2 = dx^2 + dy^2 + dz^2 - du^2$$

on $\mathbb{R}^4$ would now be given by

$$dl^2 = dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2) - (d\sqrt{S^2 + r^2})^2$$

$$= S^2dr^2/(S^2 + r^2) + r^2(d\theta^2 + sin^2(\theta)d\phi^2)$$

Now we observe that on this hyperbolic surface, $u \geq S$ and there is no constraint on r. Thus, this is a more realistic hyperbolic model for our 3-D space. We can change the variable

$$r = S.\,sinh(\chi), \chi \geq 0$$

to get the metric in the form

$$dl^2 = S^2(d\chi^2 + sinh^2(\chi).\,(d\theta^2 + sin^2(\theta)d\phi^2))$$

A remark

More generally, suppose that we have an n dimensional surface S imbedded in an N dimensional space with the metric on the N-dimensional space being given by

$$ds^2 = dy^TG(y)dy, y \in \mathbb{R}^N$$

Suppose that this metric is invariant under the diffeomorphism $T: \mathbb{R}^N \to \mathbb{R}^N$, so that

$$T'(y)^TG(T(y))T'(y) = G(y), y \in \mathbb{R}^N$$

or equivalently,

$$dT(y)^TG(T(y))dT(y) = dy^TG(y)dy$$

Suppose that the surface S is defined by the equation

$$z = F(x), x \in \mathbb{R}^p, z \in \mathbb{R}^{N-p}$$

Write

$$y = (x, F(x)), x \in \mathbb{R}^p$$

for the equation of the surface S as viewed in $\mathbb{R}^N$. Then, the metric induced on S from the metric ds^2 on $\mathbb{R}^N$ is given by

$$dl^2 = (dx^T, dF(x)^T)G(x, F(x))\begin{pmatrix} dx \\ dF(x) \end{pmatrix}$$

$$= dx^TH(x)dx$$

where

$$H(x) = (I_p, F'(x)^T)G(x, F(x))\begin{pmatrix} I_p \\ F'(x) \end{pmatrix}, x \in \mathbb{R}^p$$

Note that this relationship between the metric on $\mathbb{R}^N$ and the induced metric on S can be expressed equivalently in the form

$$d\begin{pmatrix} x \\ F(x) \end{pmatrix}^T G(x, F(x))d\begin{pmatrix} x \\ F(x) \end{pmatrix}$$

$$= dx^TH(x)dx$$

Note that x parameterises the point on the surface S (which is assumed to be an open subset of $\mathbb{R}^p$) and its coordinates in $\mathbb{R}^N$ are given by $(x, F(x))$. Now suppose, in addition, that T leaves the surface invariant, in the sense that the points $T(x, F(x))$ are again the coordinates of a point on S for any $(x, F(x))$ in S. Then, we can write

$$T(x, F(x)) = (K(x), F(K(x)))$$

where K is a diffeomorphism on $\mathbb{R}^p$. In other words, we can write

$$T(x, F(x)) = (x', F(x')), x' = K(x)$$

Then, we claim that this induced metric on S is also invariant under T, or equivalently, under K. To see this, we observe that

$$dx^T K'(x)^T H(K(x)) K'(x) dx =$$

$$dx^T K'(x)^T (I_p, F'(K(x))^T) G(K(x), F(K(x))) \begin{pmatrix} I_p \\ F'(K(x)) \end{pmatrix} K'(x) dx$$

$$= [dK(x)^T, dF(K(x))^T] G(K(x), F(K(x))) \begin{pmatrix} dK(x) \\ dF(K(x)) \end{pmatrix}$$

$$= dT(x, F(x))^T G(T(x, F(x))) dT(x, F(x)) =$$

$$= d\begin{pmatrix} x \\ F(x) \end{pmatrix}^T G(x, F(x)) d\begin{pmatrix} x \\ F(x) \end{pmatrix}$$

$$= dx^T H(x) dx$$

where in the second last equation, we have used the assumed invariance of the metric on $\mathbb{R}^N$ under T.

This result enables us to construct metrics on manifolds having various kinds of symmetries by embedding the manifold in a larger manifold having a metric with a set of symmetries in such a way that the embedded manifold is invariant under these symmetries and then inducing the metric from the larger manifold to the embedded one, ensuring thereby, by the above result, that the induced metric on the embedded manifold will have the same symmetries as the metric on the larger manifold has.

Now let C be an $n \times n$ real symmetric non-singular matrix with p positive and $q = n - p$ negative eigenvalues. Then, we can write

$$C = ODO^T$$

where D is a diagonal matrix with p diagonal entries positive and q diagonal entries negative, and O is a real orthogonal matrix, i.e., $O^T O = OO^T = -I_n$. Write

$$D = diag[\lambda_1, \ldots, \lambda_p, -\mu_1, \ldots, -\mu_q]$$

so that $\lambda_j > 0, \mu_i < 0$. For $x \in \mathbb{R}^n$, define $y \in \mathbb{R}^n$ by

$$y = |D|^{1/2} Ox$$

where

$$|D| = diag[\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_q]$$

Then we have

$$x^T Cx = y^T Jy$$

where J is the standard $SO(p, q)$ metric, i.e.,

$$J = diag[I_p, -I_q]$$

Thus, the n dimensional surface S imbedded into $\mathbb{R}^{n+1}$ and parametrized by coordinates $x \in \mathbb{R}^n$ with the imbedding defined by the equation

$$x^T Cx + u^2 = S^2$$

can equivalently be parametrized by y with the imbedding defined by

$$y^T Jy + u^2 = S^2$$

When $q = 0$, this surface becomes an n-sphere, and when $q = 1$, it becomes a hyperbolic surface. The metric on this surface is that induced by the $SO(p+1, q)$ metric on $\mathbb{R}^{n+1}$ given by

$$ds^2 = dy^T J dy + du^2 = \sum_{j=1}^{p} y_j^2 + u^2 - \sum_{j=p+1}^{n} y_j^2$$

This induced metric is

$$dl^2 = dy^T J dy + (d\sqrt{S^2 - y^T Jy})^2 =$$

$$dy^T J dy + (y^T J dy)^2 / (S^2 - y^T Jy)$$

$$= dy^T (J + Jyy^T J / (S^2 - y^T Jy)) dy$$

Note that

$$y^T Jy = \sum_{j=1}^{p} y_j^2 - \sum_{j=p+1}^{n} y_j^2$$

The metric ds^2 on $\mathbb{R}^{n+1}$ is invariant under the group $SO(p+1, q)$ and the induced metric dl^2 on the n-dimensional surface S is invariant under the induced transformations

$$y = Ry' + bu', u = c^T y' + du' = \sqrt{S^2 - y^T Jy}$$

or equivalently, under

$$y = Ry' + b\sqrt{S^2 - y'^T Jy'}$$

where

$$T = \begin{pmatrix} R & b \\ c^T & d \end{pmatrix}$$

satisfies

$$T^T J_0 T = J_0$$

with J_0 the $SO(p+1, q)$ metric defined by

$$J_0 = [I_p, -I_q, 1] = diag[J, 1]$$

The metric on the surface S is thus invariant under a $dimSO(p+1, q) = n(n+1)/2$-parameter family of diffeomorphisms and is therefore a maximally symmetric space. Let us now study Maxwell's equations in such a maximally symmetric space after including a time coordinate. In the special case when $n = 3$, as considered at the beginning, the metric is

$$d\tau^2 = dt^2 - dl^2 = dt^2 - S(t)^2 f(r)^2 - S(t)^2 r^2 (d\theta^2 + sin^2(\theta) d\phi^2)$$

where

$$f(r) = 1/(1 - r^2)$$

The coefficients of the metric tensor are thus

$$g_{00} = 1, g_{11} = -S^2(t) f^2(r), g_{22} = -S^2(t) r^2, g_{33} = -S^2(t) r^2 sin^2(\theta)$$

so that

$$\sqrt{-g} = S^3(t) f(r) r^2 sin(\theta)$$

Here, the scale factor $S(t)$of the universe is determined by solving the Einstein field equations with a homogeneous and isotropic energy-momentum tensor

$$T_{ij} = (\rho(t) + p(t)) v_i v_j - p(t) g_{ij}$$

The four-velocity field v_i will be determined by the fluid dynamical equations owing to the Bianchi identity satisfied by the Einstein tensor or, equivalently, by the geodesic equations which turn out to have the comoving solution

$$v_0 = 1, v_i = 0, i = 1, 2, 3$$

so that

$$T_{00} = \rho(t), T_{11} = T_{22} = T_{33} = -p(t), T_{ij} = 0, i \neq j$$

The field equations give us just two independent ordinary differential equations in t for the three variables $\rho(t), p(t), S(t)$ with the third equation being determined by the equation of state $p(t) = h(\rho(t))$. These equations in the radiation-dominated era give a pressure $p(t)$ which corresponds to the isotropic and homogeneous electromagnetic radiation pressure. To obtain the anisotropic and inhomogeneous components of the radiation energy density and flux, and momentum density and flux, we must set up the Maxwell equations in this metric and derive general solutions. The relevant Maxwell equations are

$$(F^{\mu\nu}\sqrt{-g})_{,\nu} = 0, F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

These become

$$(F^{01}\sqrt{-g})_{,1}+(F^{02}\sqrt{-g})_{,2}+(F^{03}\sqrt{-g})_{,3}=0,$$

$$(F^{10}\sqrt{-g})_{,0}+(F^{12}\sqrt{-g})_{,2}+(F^{13}\sqrt{-g})_{,3}=0,$$

$$(F^{20}\sqrt{-g})_{,0}+(F^{21}\sqrt{-g})_{,1}+(F^{23}\sqrt{-g})_{,3}=0,$$

$$(F^{30}\sqrt{-g})_{,0}+(F^{31}\sqrt{-g})_{,1}+(F^{32}\sqrt{-g})_{,2}=0,$$

or equivalently, defining the electric and magnetic field components as

$$E_r=F_{0r}, r=1,2,3, B_1=-F_{23}, B_2=-F_{31}, B_3=F_{12},$$

and noting that

$$g^{11}\sqrt{-g}=-S^3fr^2sin(\theta)/S^2f^2=-Sr^2sin(\theta)/f,$$

$$g^{22}\sqrt{-g}=-S^3fr^2sin(\theta)/S^2r^2=-Sfsin(\theta),$$

$$g^{33}\sqrt{-g}=-S^3fr^2sin(\theta)/S^2r^2sin^2(\theta)=-Sf/sin(\theta)$$

$$g^{11}g^{22}\sqrt{-g}=sin(\theta)/Sf,$$

$$g^{22}g^{33}\sqrt{-g}=f/Sr^2sin(\theta),$$

$$g^{33}g^{11}\sqrt{-g}=1/Sf.sin(\theta)$$

we can express these equations as

$$(Sr^2sin(\theta)E_1/f)_{,1}+(Sfsin(\theta)E_2)_{,2}+(SfE_3/sin(\theta))_{,3}=0,$$

$$(Sr^2sin(\theta)E_1/f)_{,0}-(sin(\theta)B_3/Sf)_{,2}+(B_2/Sfsin(\theta))_{,3}=0,$$

$$(Sf.sin(\theta)E_2)_{,0}+(sin(\theta)B_3/Sf)_{,1}-(fB_1/Sr^2sin(\theta))_{,3}=0,$$

$$(SfE_3/sin(\theta))_{,0}-(B_2/Sfsin(\theta))_{,1}+(fB_1/Sr^2sin(\theta))_{,2}=0$$

These equations are to be supplemented with the homogeneous Maxwell equations that are equivalently a consequence of $F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}$:

$$F_{\mu\nu,\sigma}+F_{\nu\sigma,\mu}+F_{\sigma\mu,\nu}=0$$

and these are

$$F_{01,2}+F_{12,0}+F_{20,1}=0,$$

$$F_{01,3}+F_{13,0}+F_{30,1}=0,$$

$$F_{02,3}+F_{23,0}+F_{30,2}=0,$$

$$F_{12,3}+F_{23,1}+F_{31,2}=0$$

or equivalently,

$$E_{1,2}-B_{3,0}-E_{2,1}=0,$$

$$E_{1,3}+B_{2,0}-E_{3,1}=0,$$

$$E_{2,3}-B_{1,0}-E_{3,2}=0,$$

$$B_{1,1}+B_{2,2}+B_{3,3}=0$$

Now consider the special case when the fields E, B are independent of $x^3=\phi$. Then, these reduce to

$$(r^2sin(\theta)E_1/f)_{,1}+(fsin(\theta)E_2)_{,2}=0---(1)$$

$$(Sr^2sin(\theta)E_1)_{,0}-(sin(\theta)B_3/S)_{,2}=0---(2)$$

$$(Sf.E_2)_{,0}+(B_3/Sf)_{,1}=0---(3)$$

$$(SfE_3/sin(\theta))_{,0}-(B_2/Sfsin(\theta))_{,1}+(fB_1/Sr^2sin(\theta))_{,2}=0---(4)$$

$$E_{1,2}-B_{3,0}-E_{2,1}=0---(5)$$

$$B_{2,0}-E_{3,1}=0---(6)$$

$$B_{1,0}+E_{3,2}=0---(7)$$

$$B_{1,1}+B_{2,2}=0---(8)$$

These equations have a solution with

$$B_1=0, B_2=0, E_3=0$$

so that the above eight equations reduce to the following equations for (B_3, E_1, E_2):

$$(r^2sin(\theta)E_1/f)_{,1}+(fsin(\theta)E_2)_{,2}=0---(1')$$

$$(Sr^2sin(\theta)E_1)_{,0}-(sin(\theta)B_3/S)_{,2}=0---(2')$$

$$(Sf.E_2)_{,0}+(B_3/Sf)_{,1}=0---(3')$$

$$E_{1,2}-B_{3,0}-E_{2,1}=0---(4')$$

The first one implies

$$r^2sin(\theta)E_1/f=\psi_{1,2}, fsin(\theta)E_2=-\psi_{1,1}$$

for some function $\psi_1(t,r,\theta)$. The second then implies

$$(Sf\psi_{1,2})_{,0}-(sin(\theta)B_3)/S)_{,2}=0$$

or equivalently,

$$(Sf\psi_1)_{,02}-(sin(\theta)B_3/S)_{,2}=0$$

and therefore,

$$(Sf\psi_1)_{,0}-sin(\theta)B_3/S=\psi_2(t,r)$$

ie, ψ_2 is independent of θ. This gives

$$B_3=(S/sin(\theta))((Sf\psi_1)_{,0}-\psi_2)$$

Substituting these expressions for E_1, E_2, B_3 into the third equation gives

$$-(S\psi_1)_{,01}+(S\psi_1)_{,01}-(\psi_2/f)_{,1}=0$$

or equivalently,

$$(\psi_2/f)_{,1}=0$$

so that

$$\psi_2(t,r)=f(r)\psi_3(t)$$

i.e., ψ_3 is independent of r,θ. Finally, substituting into the fourth equation gives us

$$(f\psi_{1,2}/r^2sin(\theta))_{,2}-[(S/sin(\theta))((Sf\psi_1)_{,0}-f\psi_3)]_{,0}$$
$$+(\psi_{1,1}/f.sin(\theta))_{,1}=0$$

which simplifies to

$$sin(\theta)(\psi_{1,2}/sin(\theta))_{,2}-r^2[S((S\psi_1)_{,0}-\psi_3)]_{,0}$$
$$+(r^2/f)(\psi_{1,1}/f)_{,1}=0$$

In particular, taking $\psi_3(t)=0$ gives us a linear wave equation for $\psi_1(t,r,\theta)$:

$$sin(\theta)(\psi_{1,2}/sin(\theta))_{,2}-r^2[S(t)((S(t)\psi_1)_{,0}]_{,0}$$
$$+(r^2/f(r))(\psi_{1,1}/f(r))_{,1}=0$$

Use separation of variables to solve this:

$$\psi_1(t,r,\theta)=T(t)R(r)\chi(\theta)$$

Substitution gives

$$(sin(\theta)/\chi(\theta))(\chi'(\theta)/sin(\theta))'$$
$$=r^2(S(t)(S(t)T(t))')'/T(t)-(r^2/f(r)R(r))(R'(r)/f(r))'$$

The LHS is a function of θ only, while the RHS is a function of (t,r) only. Hence, both sides must equal a constant, say β:

$$(\chi'(\theta)/sin(\theta))'-(\beta/sin(\theta))\chi(\theta)=0,$$

$$(S(t)(S(t)T(t))')'/T(t) = \beta/r^2 + (1/f(r)R(r))(R'(r)/f(r))'$$

Again, the LHS is a function of t only, while the RHS is a function of r only. Hence, both sides must equal a constant, say $-\lambda$:

$$(S(t)(S(t)T(t))')' + \lambda.\ T(t) = 0,$$

$$r^2(R'(r)/f(r))' + f(r)(\lambda r^2 + \beta)R(r) = 0$$

2. The general case when fields depend on all the space-time coordinates

We define the operations div and curl in the system $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi$ as

$$divC = C_{1,1} + C_{2,2} + C_{3,3}, C = (C_1, C_2, C_3)$$

$$curlC = (C_{3,2} - C_{2,1}, C_{1,3} - C_{3,1}, C_{2,1} - C_{1,2})$$

and then observe that the equation

$$(Sr^2sin(\theta)E_1/f)_{,1} + (Sfsin(\theta)E_2)_{,2} + (SfE_3/sin(\theta))_{,3} = 0,$$

can be expressed as

$$divD = 0$$

where

$$D_1 = Sr^2sin(\theta)E_1/f, D_2 = Sfsin(\theta)E_2, D_3 = SfE_3/sin(\theta)$$

Thus, there is a 3-vector C such that

$$D = curlC,$$

or equivalently,

$$E_1 = (f/Sr^2sin(\theta))(C_{3,2} - C_{2,3}),$$

$$E_2 = (1/Sf.\ sin(\theta))(C_{1,3} - C_{3,1}),$$

$$E_3 = (sin(\theta)/Sf)(C_{2,1} - C_{1,2})$$

The equations

$$(Sr^2sin(\theta)E_1/f)_{,0} - (sin(\theta)B_3/Sf)_{,2} + (B_2/Sfsin(\theta))_{,3} = 0,$$

$$(Sf.\ sin(\theta)E_2)_{,0} + (sin(\theta)B_3/Sf)_{,1} - (fB_1/Sr^2sin(\theta))_{,3} = 0,$$

$$(SfE_3/sin(\theta))_{,0} - (B_2/Sfsin(\theta))_{,1} + (fB_1/Sr^2sin(\theta))_{,2} = 0$$

can now be expressed as, after substituting the above expressions for E_1, E_2, E_3,

$$[C_{3,0} - sin(\theta)B_3/Sf]_{,2} = [C_{2,0} - B_2/Sf.\ sin(\theta)]_{,3},$$

$$[C_{1,0} - fB_1/Sr^2sin(\theta)]_{,3} = [C_{3,0} - B_3sin(\theta)/Sf]_{,1},$$

$$[C_{2,0} - B_2/Sfsin(\theta)]_{,1} = [C_{1,0} - B_1f/Sr^2sin(\theta)]_{,2}$$

The first implies the existence of a function $\psi_1(t, r, \theta, \phi)$ such that

$$C_{3,0} - sin(\theta)B_3/Sf = \psi_{1,3},$$

$$C_{2,0} - B_2/Sf.\ sin(\theta) = \psi_{1,2}$$

The second therefore implies

$$[C_{1,0} - fB_1/Sr^2sin(\theta) - \psi_{1,1}]_{,3} = 0$$

and hence there is a function $\psi_2(t, r, \theta)$ independent of ϕ such that

$$C_{1,0} - fB_1/Sr^2sin(\theta) - \psi_{1,1} = \psi_2 - - - (a)$$

Likewise, the third implies

$$[C_{1,0} - B_1f/Sr^2sin(\theta) - \psi_{1,1}]_{,2} = 0$$

and hence the existence of a function $\psi_3(t, r, \phi)$ independent of θ such that

$$C_{1,0} - B_1f/Sr^2sin(\theta) - \psi_{1,1} = \psi_3 - - - (b)$$

It follows, therefore, from (a) and (b) that

$$\psi_2(t, r, \theta) = \psi_3(t, r, \phi) = \psi_2(t, r)$$

is independent of both θ and ϕ. Denoting $\psi_1(t, r, \theta, \phi) + \int_0^r \psi_2(t, r)dr$ by $\psi_1(t, r, \theta, \phi)$, it follows, therefore, from the above equations that the magnetic field components can be expressed as

$$B_1 = (Sr^2sin(\theta)/f)(C_{1,0} - \psi_{1,1}]$$

$$B_2 = Sfsin(\theta)(C_{2,0} - \psi_{1,2}),$$

$$B_3 = (Sf/sin(\theta))(C_{3,0} - \psi_{1,3})$$

Substituting for the electric and magnetic field components into the homogeneous Maxwell equations

$$E_{1,2} - B_{3,0} - E_{2,1} = 0,$$

$$E_{1,3} + B_{2,0} - E_{3,1} = 0,$$

$$E_{2,3} - B_{1,0} - E_{3,2} = 0,$$

$$B_{1,1} + B_{2,2} + B_{3,3} = 0$$

their expressions obtained above in terms of C_1, C_2, C_3, ψ_1 then give us

3. Some general remarks about electromagnetic wave equations in a diagonal metric

Consider a metric of space-time for which $g_{\mu\nu} = 0, \mu \neq \nu$. The Maxwell equations in such a metric are

$$\sum_\nu (g^{\mu\nu}g^{\nu\nu}\sqrt{-g}F_{\mu\nu})_{,\nu} = 0,$$

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

The covariant Lorentz gauge condition $(A^\mu\sqrt{-g})_{,\mu} = 0$ now reads

$$\sum_\nu (g^{\nu\nu}\sqrt{-g}A_\nu)_{,\nu} = 0$$

4. Alternate analysis of the Maxwell equations in any diagonal metric

An alternate way of analysing the propagation of electromagnetic waves in the absence of sources in any diagonal metric is to start with the Maxwell equations

$$(F^{\mu\nu}\sqrt{-g})_{,\nu} = 0, F_{\mu\nu,\sigma} + F_{\nu\sigma,\mu} + F_{\sigma\mu,\nu} = 0$$

and to write

$$E = ((E_r)) = (F_{0r})), B = ((B_r)) = -(F_{23}, F_{31}, F_{12})$$

and to note that we can write

$$((F^{0r}\sqrt{-g})) = G.\ E,$$

where G is a 3×3 diagonal matrix whose components are functions of x^μ with $x^0 = t$ and

$$-\sqrt{-g}(F^{23}, F^{31}, F^{12}))^T = K.\ B$$

where K is another 3×3 diagonal matrix whose entries are functions of x^μ and then note that the Maxwell equations $(F^{\mu\nu}\sqrt{-g})_{,\nu} = 0$ can be expressed as

$$div(G.\ E) = 0, curl(K.\ B) = -\partial_t(G.\ E),$$

and the Maxwell equations $F_{\mu\nu,\sigma} + F_{\nu\sigma,\mu} + F_{\sigma\mu,\nu} = 0$ as

$$div(B) = 0, curlE + \partial_t B = 0$$

Thus, there exists a 3-vector field C and a scalar field V such that

doi.org/10.32388/JXIAL3

$$E = G^{-1}.\, curlC, B = K^{-1}.\,(- \partial_t C + \nabla V)$$

which satisfy

$$div[K^{-1}.\,(- \partial_t C + \nabla V)] = 0,$$

$$curl[G^{-1}.\, curlC] + \partial_t[K^{-1}.\,(- \partial_t C + \nabla V)] = 0$$

We write

$$K^{-1} = K_1(S(t), r, \theta) = diag[k_1, k_2, k_3], G^{-1} = G_1(S(t), r, \theta) = diag[g_1, g_2, g_3]$$

where

$$g_j = g_j(S(t), r, \theta), k_j = k_j(S(t), r, \theta)$$

Then, the above equations are in component form, the same as

$$\sum_j (- (k_j C_{j,0})_{,j} + (k_j V_{,j})_{,j}) = 0,$$

$$\sum_{mkjr} [\epsilon(srj)(g_j \epsilon(jkm) C_{m,k})_{,r}$$

$$+(k_s(- C_{s,0} + V_{,s}))_{,0} = 0$$

Note that we can change C to $C' = C + \nabla\chi$ and V to $V' = V - \chi_{,0}$ for any scalar field χ without affecting the values of E, B. This is analogous to the Lorentz gauge transformation in special relativistic electrodynamics.

Remark: We can, for example, choose χ so that the following generalization form of the Coulomb gauge holds:

$$\sum_j (k_j C_j')_{,j} = 0$$

holds, or equivalently, renaming C' as C and V' as V,

$$\sum_j (k_j C_j)_{,j} = 0$$

Then, the first equation above becomes

$$\sum_j (k_j V_{,j})_{,j}) + \sum_j (k_{j,0} C_j)_{,j} = 0$$

Now, observe that

$$g_j \epsilon(srj)\epsilon(jkm)$$

$$= g_j \,|\, \epsilon(srj) \,|\, (\delta(sk)\delta(rm) - \delta(sm)\delta(rk))$$

So

$$\sum_{mkjr} [\,|\, \epsilon(srj) \,|\, (g_j \epsilon(jkm) C_{m,k})_{,r}$$

$$= \sum_{jr} |\, \epsilon(srj) \,|\, (g_j(C_{r,s} - C_{s,r}))_{,r}$$

and hence the second equation can be expressed as

$$\sum_{jr} |\, \epsilon(srj) \,|\, (g_j(C_{r,s} - C_{s,r}))_{,r} + (k_s(- C_{s,0} + V_{,s}))_{,0} = 0$$

Note that if the $g_j's$ were all one, as would be the case in cartesian coordinates in flat space-time, then we would get the usual formula

$$\sum_{jr} |\, \epsilon(srj) \,|\, (g_j(C_{r,s} - C_{s,r}))_{,r}$$

$$= \sum_{jr} |\, \epsilon(srj) \,|\, ((C_{r,s} - C_{s,r}))_{,r}$$

For $s = 1$, this evaluates to

$$\epsilon(123)(C_{2,1} - C_{1,2})_{,2} + |\, \epsilon(132) \,|\, (C_{3,1} - C_{1,3})_{,3}$$

$$= (C_{2,1} - C_{1,2})_{,2} + (C_{3,1} - C_{1,3})_{,3}$$

$$= (C_{1,1} - C_{1,1})_{,1} + (C_{2,1} - C_{1,2})_{,2} + (C_{3,1} - C_{1,3})_{,3}$$

$$= (divC)_{,1} - \nabla^2 C_1$$

as expected.

We rewrite the basic equations now:

$$\sum_{jr} |\, \epsilon(srj) \,|\, (g_j(C_{r,s} - C_{s,r}))_{,r} + (k_s(- C_{s,0} + V_{,s}))_{,0} = 0$$

$$\sum_j (- (k_j C_{j,0})_{,j} + (k_j V_{,j})_{,j}) = 0$$

Note that $|\, \epsilon(srj) \,|$ is one if all the three indices s, r, j are distinct and is zero otherwise.

5. Analysis of Maxwell's equations in a diagonal metric based on electric scalar and magnetic vector potentials

Defining

$$E = ((F_{0r})), B = - (F_{23}, F_{31}, F_{12}),$$

and writing

$$((F^{0r}\sqrt{-g})) = ((g^{rr}\sqrt{-g}F_{0r})) = G.\, E,$$

$$-(F^{23}, F^{31}, F^{12})^T = K.\, B$$

where G, K are diagonal matrices, we obtain from the curved space-time Maxwell equations

$$divB = 0, curlE + \partial_t B = 0, div(G.\, E) = 0, curl(K.\, B) = - \partial_t(G.\, E)$$

where

$$G = ((g^{rr}\sqrt{-g})), K = (g^{22}g^{33}\sqrt{-g}, g^{33}g^{11}\sqrt{-g}, g^{11}g^{22}\sqrt{-g})$$

so that

$$B = curlA, E = - \nabla V - \partial_t A, A = ((- A_r)), V = A_0$$

and

$$curl(K.\, curlA)) - \partial_t(G.\,(\nabla V + \partial_t A)) = 0, div(G.\,(\nabla V + \partial_t A)) = 0$$

In case the medium carries a charge density rho and current density $J = ((- J_r))$, then the generalization would be

$$curl(K.\, curlA) - \partial_t(G.\,(\nabla V + \partial_t A)) = - G.\, J,$$

$$div(G.\,(\nabla V + \partial_t A)) = \rho.\, \sqrt{-g}$$

We require to supplement these with the general relativistic form of the Lorentz gauge condition:

$$\sum_\mu (g^{\mu\mu}\sqrt{-g}A_\mu)_{,\mu} = 0$$

or equivalently, assuming $g_{00} = 1$, as in the case of the Robertson-Walker metric,

$$\partial_t(\sqrt{-g}V) - div(G.\, A) = 0$$

Then, the above equation for V reduces to

$$div(G.\, \nabla V) + \partial_t^2(\sqrt{-g}V) - div((\partial_t G).\, A) = \rho\sqrt{-g}$$

Note that the charge conservation condition can be expressed as

$$(J^\mu\sqrt{-g})_{,\mu} = 0,$$

or equivalently, since the metric is diagonal, as

$$\partial_t(\rho\sqrt{-g}) - div(G.\, J) = 0$$

Remark: In the previous analysis, we had used $G.\, E = curlC, K.\, B - C = \nabla\Phi$. This method would fail if there are sources of charge and current.

Making this choice of gauge, the differential equation satisfied by A is given by

$$curl(K.\,curlA) - \partial_t(G.\,(\nabla((-g)^{-1/2}\int_0^t G.\,Adt) + \partial_t A)) = -G.\,J$$

Note that the first component of $curl(K.\,curlA)$ is

$$-(K_3(A_{2,1} - A_{1,2}))_{,2} + (K_2(A_{1,3} - A_{3,1}))_{,3} + (G_1 g^{11}(A_{1,1} - A_{1,1}))_{,1}$$

and likewise for the other components. This first component can be expressed as

$$(g^{11}G_2.\,A_{1,2})_{,2} + (g^{11}G_3A_{1,3})_{,3} + (g^{11}G_1A_{1,1})_{,1}$$

$$-(K_3A_{2,1})_{,2} - (K_2A_{3,1})_{,3} - (G_1g^{11}A_{1,1})_{,1}$$

$$= (g^{11}G_2.\,A_{1,2})_{,2} + (g^{11}G_3A_{1,3})_{,3} + (g^{11}G_1A_{1,1})_{,1}$$

$$-(G_2g^{11}A_{2,1})_{,2} - (G_3g^{11}A_{3,1})_{,3} - (G_1g^{11}A_{1,1})_{,1}$$

$$= (g^{11}G_2.\,A_{1,2})_{,2} + (g^{11}G_3A_{1,3})_{,3} + (g^{11}G_1A_{1,1})_{,1}$$

$$-(G_2g^{11}A_2)_{,12} - (G_3g^{11}A_3)_{,13} - (G_1g^{11}A_1)_{,11}$$

$$+((G_2g^{11})_{,1}A_2)_{,2} + ((G_3g^{11})_{,3}A_3)_{,1} + ((G_1g^{11})_{,1}A_1)_{,1}$$

The first component of $\partial_t(G.\,(\nabla V + \partial_t A))$ is given by

$$\partial_0(G_1(V_{,1} + A_{1,0}))$$

$$= (G_1V)_{,01} - (G_{1,1}V)_{,0} + (G_1A_{1,0})_{,0}$$

Now suppose instead that we impose the gauge condition

$$div(G.\,\partial_t A) = 0 - - - (\alpha)$$

This equation is the analogue of the Coulomb gauge condition. In this case, the equation for V simplifies drastically to

$$div(G.\,\nabla V) = \rho\sqrt{-g}$$

Since we are assuming the Robertson-Walker metric, we have

$$G = ((g^{rr}\sqrt{-g}) = -S(t)(r^2 sin(\theta)/f(r), f.\,sin(\theta), f(r)/sin(\theta))$$

$$= -S(t)h(r,\theta)$$

where h = $h(r,\theta)$ is independent of t,ϕ. Also recalling that $\sqrt{-g} = S^3 fr^2 sin(\theta)$, we get

$$\nabla(h(r,\phi)\nabla V(t,r,\theta,\phi)) = -\rho(t,r,\theta,\phi).\,S^2(t)f(r)r^2 sin(\theta)$$

It follows from this equation that $V(t,r,\theta,\phi)$ can be expressed in the form

$$V(t,r,\theta,\phi) = S^2(t)\int L(r,\theta,\phi|r',\theta',\phi')\rho(t,r,\theta,\phi)drd\theta d\phi$$

ie V is a matter field, in the language of quantum field theory. Its value at time t at any spatial location is a function of only the matter density over space at that time t. In particular, if $\rho = 0$, the solution will be $V = 0$. So assuming that there is no charge distribution in space, we can assume that $V = 0$. In other words, the electromagnetic field in space-time in an evolving Robertson-Walker space-time, ie, in an expanding universe, is given by the magnetic vector potential only, which satisfies the wave equation

$$curl(K.\,curlA) - \partial_t(G.\,\partial_t A)) = -G.\,J - - - (c)$$

Note that the charge conservation condition with $\rho = 0$ assumed reads $div(G.\,J) = 0$ and this equation is consistent with (c) and our choice of the gauge. In particular, if in addition, $J = 0$, i.e., there are no charges and currents, then using the above gauge, A satisfies the wave equation

$$curl(K.\,curlA) - \partial_t(G.\,\partial_t A)) = 0 - - (\beta)$$

and the electric and magnetic fields are given by

$$E = ((F_{0r} = -\partial_t A,$$

$$B = curlA = -(F_{23}, F_{31}, F_{12})$$

Once we have solved for A and hence E,B, we can calculate the energy-momentum tensor of the electromagnetic field as

$$S_{\mu\nu} = (-1/4)g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + F_{\mu\alpha}F_\nu^\alpha$$

with

$$F^{\alpha\beta}F_{\alpha\beta} = \sum_{\alpha\beta} g^{\alpha\alpha}g^{\beta\beta}F_{\alpha\beta}^2$$

and

$$F_{\mu\alpha}F_\nu^\alpha = \sum_\alpha g^{\alpha\alpha}F_{\mu\alpha}F_{\nu\alpha}$$

where the metric tensor components are those of the RW metric and $F_{\alpha\beta 1}$ have components given by the electric and magnetic fields thus solved for. We are usually interested in the case when the initial electric and magnetic fields had a certain spatial statistical correlation function, i.e., $< F_{\alpha\beta}(t,r)F_{\mu\nu}(t',r') >$ was given to us at $t = t' = 0$ at all r,r' and then we wish to compute this correlation function for all times t,t' at all r,r'. We now outline the procedure for performing this calculation.

6. Maxwell's equations in a maximally symmetric space-time

Now assume that we have an $n+1$-dimensional space-time, with one time dimension and n space dimensions with a maximally symmetric metric

$$d\tau^2 = dt^2 - S^2(t)f^2(r)dr^2 - S^2(t)r^2 d\Omega^2$$

where we write the spatial vector as

$$x = r\hat{n}, \hat{n}^T\hat{n} = 1$$

so that

$$dx = dr\hat{n} + rd\hat{n}, \hat{n}^T d\hat{n} = 0, d\hat{n}^T d\hat{n} = d\Omega^2$$

and the spatial line element becomes

$$dl^2 = dx^T dx + (d\sqrt{S^2 - r^2})^2 = dr^2 + r^2 d\Omega^2 + dr^2/(S^2 - r^2) = S^2 dr^2/(S^2 - r^2) + r^2 d\Omega^2$$

and replacing r by Sr, this becomes

$$dl^2 = S^2 f^2 dr^2 + S^2 r^2 d\Omega^2, f^2 = 1/(1 - r^2),$$

We can parametrize $\hat{n}$ by $n-1$ independent angles $x^2,\ldots,x^n$ so that $d\omega^2$ has a diagonal form

$$d\Omega^2 = \sum_{k=2}^n \chi_k(x^2,\ldots,x^n)(dx^k)^2$$

Then, as before, we define the electromagnetic field tensor components

$$F_{0r} = A_{k,0} - A_{0,k}, k = 1,2,\ldots,n,$$

$$F_{ks} = A_{s,k} - A_{k,s}, k,s = 1,2,\ldots,n$$

Maxwell's equations become

$$\sum_{k\geq 1}(F^{0k}\sqrt{-g})_{,k} = 0,$$

$$(F^{m0}\sqrt{-g})_{,0} + \sum_{k\geq 1}(F^{mk}\sqrt{-g})_{,k} = 0, m = 1,2,\ldots,n$$

or equivalently, since the metric is diagonal with

$$g_{00} = 1, g_{11} = -S^2(t)f^2(r), g_{kk} = -S^2(t)r^2\chi_k(x^2,\ldots,x^n), k = 2,3,\ldots,n$$

so that assuming n is odd,

$$\sqrt{-g} = S^n(t)f(r)r^{n-1}\chi(x^2,\ldots,x^n), \chi = \Pi_{k=2}^n\chi_k$$

We can express the Maxwell equations as

$$(g^{mm}\sqrt{-g}(A_{0,m} - A_{m,0}))_{,0} + \sum_{k\geq 1}(g^{mm}g^{kk}\sqrt{-g}(A_{k,m} - A_{m,k}))_{,k} = 0, m = 1, 2, \ldots, n$$

$$\sum_{k\geq 1}(g^{kk}\sqrt{-g}(A_{k,0} - A_{0,k}))_{,k} = 0$$

We choose our gauge condition as earlier to be the generalization of the Coulomb gauge:

$$\sum_{k\geq 1}(g^{kk}\sqrt{-g}A_{k,0})_{,k} = 0$$

and then the Maxwell equations simplify to

$$\sum_{k\geq 1}(g^{kk}\sqrt{-g})A_{0,k})_{,k} = 0,$$

$$(g^{mm}\sqrt{-g}(A_{0,m} - A_{m,0}))_{,0} + \sum_{k\geq 1}(g^{mm}g^{kk}\sqrt{-g}(A_{k,m} - A_{m,k}))_{,k} = 0,$$

We can write

$$g^{kk}\sqrt{-g} = -S^{n-2}\eta_k(x^1, \ldots, x^n), x^1 = r,$$

and

$$g^{mm}g^{kk}\sqrt{-g} = S^{n-4}v_{mk}(x^1, \ldots, x^n), k, m = 1, 2, \ldots, n,$$

where

$$\eta_1 = r^{n-1}\chi(x^2, \ldots, x^n)/f(r),$$

$$\eta_k = f(r)r^{n-3}\chi(x^2, \ldots, x^n)/\chi_k(x^2, \ldots, x^n), k = 2, 3, \ldots, n$$

$$v_{1k} = \eta_k/f^2, k = 2, 3, \ldots, n,$$

$$v_{mk} = \eta_k/r^2\chi_m = fr^{n-5}\chi/\chi_m\chi_k, 2 \leq m < k \leq n$$

The Maxwell equations

$$\sum_{k\geq 1}(g^{kk}\sqrt{-g})A_{0,k})_{,k} = 0,$$

for the scalar potential, derived above by assuming generalization of the Coulomb gauge, can be expressed as

$$\sum_{k\geq 1}(\eta_k A_{0,k})_{,k} = 0$$

which is a purely spatial equation and hence has a unique solution

$$A_0 = 0$$

Thus, the equations in this gauge satisfied by the vector potential components $A_k, k \geq 1$ are given by

$$-(g^{mm}\sqrt{-g}A_{m,0})_{,0} + \sum_{k\geq 1}(g^{mm}g^{kk}\sqrt{-g}(A_{k,m} - A_{m,k}))_{,k} = 0, m = 1, 2, \ldots, n$$

or equivalently,

$$\eta_m(S^{n-2}(t).A_{m,0})_{,0} + S^{n-4}(t)\sum_{k\geq 1}(v_{mk}(A_{k,m} - A_{m,k}))_{,k} = 0$$

We can solve this using separation of variables: Writing

$$A_m(t, x) = T(t)R_m(x)$$

we get

$$(S^{n-2}(t)T'(t))'/S^{n-4}(t)T(t) = \lambda,$$

$$\lambda.\eta_m(x)R_m(x) + \sum_{k\geq 1}(v_{mk}(x)(R_{k,m}(x) - R_{m,k}(x))_{,k} = 0, m = 1, 2, \ldots, n$$

for some constant λ. Superposition over all possible values of λ then completes the solution. Note that here,

$$x = (r, x^2, \ldots, x^n)$$

with $0 \leq r \leq 1$. This completes our discussion of the solution of the Maxwell equations in a spherically maximally symmetric space-time. We then proceed to a discussion of the same problem in general elliptic-hyperbolic space-times.

7. Magneto-hydro-dynamics in an n+1 dimensional maximally symmetric space-time

The basic Einstein field equations in the presence of a fluid field and an electromagnetic field are

$$R_{\mu\nu} - (1/2)Rg_{\mu\nu} = K[(\rho + p)v_\mu v_\nu - pg_{\mu\nu} + S_{\mu\nu})$$

where $K = -8\pi G/c^2$ and

$$S_{\mu\nu} = (-1/4)F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu} + F_{\mu\alpha}F^\alpha_\nu$$

We assume that the metric of space-time is the RW metric in $n+1$-dimensional space-time and is unaffected by the matter fluid perturbations around the homogeneous and isotropic field and electromagnetic field. This is the zeroth order of approximation. In other words, if this unperturbed metric is denoted by $g^{0\mu\nu}$, then the corresponding Einstein field equations are given by

$$R^0_{\mu\nu} - (1/2)R^0g^0_{\mu\nu} = KT^0_{\mu\nu},$$

where

$$T^0_{\mu\nu} = (\rho_0(t) + p(t))v^0_\mu v^0_\nu - p_0(t)g^0_{\mu\nu}$$

so that $S(t), \rho(t), p(t)$ satisfy the standard unperturbed Einstein field equations for these three functions of time. The perturbations. Note that by the comoving nature of the RW metric, as seen from the associated geodesic equations, we have

$$v^0_i = 0, i = 1, 2, 3, v^0_0 = 1$$

Inhomogeneous and anisotropic perturbations to these quantities involve density and velocity perturbations as well as the presence of an inhomogeneous and anisotropic electromagnetic field. We denote the velocity perturbations by δv_μ, the density perturbations by $\delta\rho$ and the pressure perturbations by δp. These perturbations are all functions of t, x where x denotes the spatial coordinates. The perturbed equations, after taking into account a $-\int J^\mu A_\mu\sqrt{-g}d^{n+1}x$, where

$$J^\mu = \sigma.F^{\mu\nu}v_\nu$$

with σ being the medium conductivity, are

$$(F^{\mu\nu}\sqrt{-g})_{,\nu} + \sigma.F^{\mu\nu}v_{,\nu}\sqrt{-g} = 0,$$

$$((\rho + p)v^\mu v^\nu)_{:\nu} - g^{\mu\nu}p_{,\nu} = F^{\mu\nu}J_\nu$$

$$= \sigma F^{\mu\nu}F^\rho_\nu v_\rho$$

Note that $v_0 = 1 + \delta v_0, v_r = \delta v_r$ because the unperturbed velocity is comoving w.r.t. the RW metric. Also, δv_0 is of the quadratic order of smallness in δv_r because

$$(1 + \delta v^0)^2 + \sum_r g_{rr}(v^r)^2 = 1$$

We shall assume that v_r is small, i.e., much smaller than the electromagnetic field, so that quadratic terms in the v_r can be neglected, but bilinear terms in v_r and the electromagnetic field cannot be neglected. In order to obtain the MHD effects, we shall also not neglect terms that are quadratic in the electromagnetic fields and linear in v_r, i.e., a special sort of trilinear term.

This assumption is based on the hypothesis that we are in the radiation-dominated era. In the transition phase between the radiation-dominated era and the matter-dominated era in the expansion of the universe, we cannot neglect quadratic terms in the v_r. We shall set up the MHD equations in both of these eras.

Then, we get from the above

$$((\rho+p)v^\nu)_{:\nu}v^\mu + (\rho+p)v^\nu v^\mu_{:\nu} - g^{\mu\nu}p_{,\nu}$$

$$= \sigma F^{\mu\nu}F^\rho_\nu v_\rho$$

Multiplying both sides by v_μ gives us

$$((\rho+p)v^\nu)_{:\nu} - p_{,\nu}v^\nu = \sigma F^{\mu\nu}F^\rho_\nu v_\rho v_\mu$$

The term on the rhs can be neglected because it is of second degree in the $v_r^{'}s$ and also of second degree in the electromagnetic field. Thus, we get

$$(\rho+p)v^\nu v^\mu_{:\nu} - g^{\mu\nu}p_{,\nu} + p_{,\nu}v^\nu v^\mu = -\sigma F^{\mu\nu}F^\rho_\nu v_\rho$$

Consider first the case when $\mu = 0$. We have

$$v^0_{:0} = v^0_{,0} + \Gamma^0_{0r}v^r = 0$$

since v^0 is one plus a quadratic term in v^r and hence can be neglected, and $\Gamma^0_{0r} = 0$ for the RW metric. Likewise,

$$v^0_{:r} = v^0_{,r} + \Gamma^0_{rs}v^s = \Gamma^0_{rs}v^s$$

since v^0 is again one plus a term that is quadratic in v^r. Also

$$-g^{0\nu}p_{,\nu} + p_{,\nu}v^\nu v^0 = p_{,r}v^r$$

up to the required order. Thus, we get the fluid energy equation:

$$p_{,r}v^r = \sigma F^{0r}(F^0_r + F^s_r v_s)$$

Note that if we also take into account quadratic terms in the v^r, then the energy equation would become

$$(\rho+p)v^0_{,0} + p_{,r}v^r + p_{,0}(v^0 - 1) = -\sigma F^{0r}(F^0_r + F^s_r v_s)$$

If we neglect the pressure, then this equation approximates the energy equation for the fluid as we learn in non-relativistic fluid dynamics:

$$\rho.\partial_t v^0 = E.J, J = \sigma(E + v\times B)$$

Now, consider the MHD equation for $\mu = r$: Again neglecting quadratic terms in v_r, we get

$$(\rho+p)(v^r_{,0} + 2\Gamma^r_{0r}v^r) - g^{rr}p_{,r} + p_{,0}(v^r - 1) = -\sigma(F^{r0}F^s_0 v_s + F^{rm}F^0_m + F^{rm}F^s_m v_s)$$

Note that

$$\Gamma^r_{0r} = (1/2)g^{rr}g_{rr,0} = S'/S, r = 1,2,3$$

This is the general relativistic analogue of the Lorentz equation in non-relativistic linearized hydrodynamics of a conducting fluid:

$$\rho.\partial_t v = -\nabla p + J\times B, J = \sigma(E + v\times B)$$

If further, we take quadratic terms in v_r into account in the kinetic term as well as in the pressure term, then we obtain

$$(\rho+p)(v^r_{,0} + \Gamma^r_{0r}v^r + v^r_{,s}v^s + (\Gamma^r_{r0}v^r + \Gamma^r_{sk}v^s v^k) - g^{rr}p_{,r} + p_{,0}(v^r - 1) = -\sigma(F^{r0}F^s_0 v_s + F^{rm}F^0_m + F^{rm}F^s_m v_s)$$

Note that

$$\Gamma^r_{sk}v^s v^k = 2\sum_{k\neq r}\Gamma^r_{rk}v^r v^k + \sum_s \Gamma^r_{ss}(v^s)^2$$

for the RW metric.

8. Some general remarks on the Einstein-Maxwell-Klein-Gordon-Dirac equations in a maximally symmetric space

Consider a maximally symmetric space of dimension n defined by the equation

$$\sum_{i=1}^{n+1}(x_i)^2 = S^2$$

The line element on this surface is

$$dl^2 = \sum_{i=1}^{n+1}(dx^i)^2, x^{n+1} = (S^2 - \sum_{i=1^n}(x^i)^2)^{1/2}$$

Thus, writing

$$r = (\sum_{i=1}^n (x^i)^2)^{1/2},$$

and

$$x^i = rn_i, n_i = 1,2,\ldots,n$$

so that

$$\sum_{i=1}^n n_i^2 = 1$$

(and therefore $\sum_{i=1}^n n_i dn_i = 0$) we have

$$dx^i = dr.n_i + rdn_i, i = 1,2,\ldots,n$$

and hence,

$$\sum_{i=1}^n (dx^i)^2 = dr^2 + r^2\sum_{i=1}^n (dn_i)^2$$

This gives

$$dl^2 = dr^2/(S^2 - r^2) + dr^2 + r^2\sum_{i=1}^n (dn_i)^2$$

$$= S^2dr^2/(S^2 - r^2) + r^2\sum_{i=1}^n (dn_i)^2$$

Replacing r by Sr, we get

$$dl^2 = S^2dr^2/(1 - r^2) + S^2r^2\sum_{i=1}^n (dn_i)^2$$

For example, if $n = 3$, we can write

$$n_1 = cos(\theta_1)sin(\theta_2), n_2 = sin(\theta_1)sin(\theta_2), n_3 = cos(\theta_2),$$

giving

$$dn_1^2 + dn_2^2 + dn_3^2 = d\theta_2^2 + sin^2(\theta_1)d\theta_1^2$$

If $n = 4$, then

$$n_1 = sin(\theta_3)cos(\theta_2)sin(\theta_2), n_2 = sin(\theta_3)sin(\theta_1)sin(\theta_2), n_3 = sin(\theta_3)cos(\theta_2), n_4 = cos(\theta_3)$$

giving

$$\sum_{i=1}^4 dn_i^2 = d\theta_3^2 + sin^2(\theta_3)(d\theta_2^2 + sin^2(\theta_1)d\theta_2^2)$$

In general, $n \geq 3$, we can write, after appropriate parametrizations of the $n_i^{'}s$, in terms of angles, just as we do on S^2,

$$\sum_{i=1}^n dn_i^2 =$$

$$d\theta_{n-1}^2 + sin^2(\theta_{n-1}d\theta_{n-2}^2 + sin^2(\theta_{n-1})sin^2(\theta_{n-2})d\theta_{n-3}^2 + \ldots + (\Pi_{m=k+1}^{n-1}sin^2(\theta_m))d\theta_k^2 + ..$$

$$= \sum_{k=1}^{n-1}(\Pi_{m=k+1}^{n-1}sin^2(\theta_m))d\theta_k^2$$

where the coefficient $\Pi_{m=n}^{n-1}sin^2(\theta_m)$ is to be interpreted as 1. Thus, denoting θ_k by x^{k+1}, $k = 1,2,\ldots,n-1$, and r by x^1, our metric can be expressed as

doi.org/10.32388/JXIAL3

$$dl^2 = S^2dr^2/(1-r^2) + S^2r^2\sum_{k=1}^{n-1}\chi_k(x)(dx^{k+1})^2$$

$$= S^2(dx^1)^2/(1-(x^1)^2) + S^2(x^1)^2\sum_{k=1}^{n-1}\chi_k(x)(dx^{k+1})^2$$

$$= S^2(dx^1)^2/(1-(x^1)^2) + S^2(x^1)^2\sum_{k=2}^{n}\chi_{k-1}(x)(dx^k)^2$$

where

$$\chi_k(x) = \chi_k(x^2,\ldots,x^n) = \Pi_{m=k+1}^{n-1}sin^2(x^{m+1})$$

$$= \Pi_{m=k+2}^{n}sin^2(x^m) = \chi_k(x^{k+2},\ldots,x^n), k = 1,2,3,\ldots,n-1$$

Now we compute the Christoffel connection symbols in the space-time metric

$$d\tau^2 = dt^2 - dl^2 = dt^2 - S(t)^2f(r)^2dr^2 - S(t)^2\sum_{k=1}^{n-1}\chi_k(x)(dx^{k+1})^2$$

so that

$$g_{00} = 1, g_{11} = -S^2f^2, g_{kk} = -S^2\chi_k, k = 2,3,\ldots,n$$

where

$$f^2 = f^2(r) = 1/(1-r^2)$$

Remark, defining $r = sin(\theta_0) = sin(x^1)$, we can equivalently express this metric as

$$d\tau^2 = dt^2 - S(t)^2(dx^1)^2 - S(t)^2sin^2(x^1)\sum_{k=2}^{n}\chi_{k-1}(x)(dx^k)^2$$

or equivalently,

$$d\tau^2 = dt^2 - S^2(t)(dx^1)^2 - S(t)^2\sum_{k=2}^{n}sin^2(x^1)(\Pi_{m=k+1}^{n}sin^2(x^m))(dx^k)^2$$

Now define the following permutation of coordinates:

$$y^1 = x^1, y^k = x^{n+2-k}, k = 2,3,\ldots,n$$

Then, we can write

$$d\tau^2 = dt^2 - S^2(t)(dy^1)^2 - S^2(t)\sum_{k=2}^{n}sin^2(y^1)(\Pi_{m=k+1}^{n}sin^2(y^{n+2-m}))(dy^{n+2-k})^2$$

$$= dt^2 - S^2(dy^1)^2 - S^2\sum_{k=2}^{n}sin^2(y^1)\Pi_{m=2}^{n+1-k}sin^2(y^m))(dy^{n+2-k})^2$$

$$= dt^2 - S^2(dy^1)^2 - S^2\sum_{k=2}^{n}(\Pi_{m=1}^{n+1-k}sin^2(y^m))(dy^{n+2-k})^2$$

$$= dt^2 - S^2(dy^1)^2 - S^2\sum_{k=2}^{n}(\Pi_{m=1}^{k-1}sin^2(y^m))(dy^k)^2$$

$$= dt^2 - S^2\sum_{k=1}^{n}(\Pi_{m=1}^{k-1}sin^2(y^m))(dy^k)^2$$

We now rename y^k as x^k, $k = 1,2,\ldots,n$ so that the metric is

$$d\tau^2 = dt^2 - S^2\sum_{k=1}^{n}(\Pi_{m=1}^{k-1}sin^2(x^m))(dx^k)^2$$

$$= dt^2 - S^2\sum_{k=1}^{n}\eta_k(x)(dx^k)^2$$

where

$$\eta_k(x) = \Pi_{m=1}^{k-1}sin^2(x^m) = \eta_k(x^1,\ldots,x^{k-1}), k = 1,2,\ldots,n$$

where $\eta_1(x) = 1$ is understood. Our metric is thus given by

$$g_{00} = 1, g_{kk} = -S^2(t)\eta_k(x), k = 1,2,\ldots,n, g_{\mu\nu} = 0, \mu \neq \nu$$

We write

$$log(sin(x^m)) = f_m(x) = f_m(x^m), m = 1,2,\ldots,n$$

and

$$g_m(x) = cot(x^m) = f_{m,m}(x)$$

Note that

$$log(\eta_k) = 2\sum_{m=1}^{k-1}log(sin(x^m)) = 2\sum_{m=1}^{k-1}f_m(x)$$

and therefore,

$$(logg_{kk})_{,rs} = 0, r \neq s, r,s = 1,2,\ldots,n$$

and the only non-zero Christoffel symbols are

$$\Gamma^k_{0k} = \Gamma^k_{k0} = (logg_{kk})_{,0}/2 = \chi_{k,k}(x)/2\eta_k(x),$$

$$\Gamma^0_{kk} = -g_{kk,0}/2 = SS'\eta_k,$$

For $1 \leq m < k \leq n$,

$$\Gamma^k_{mk} = \Gamma^k_{km} = (\log g_{kk})_{,m}/2 = \eta_{k,m}/2\eta_k = g_m(x), 1 \leq m \leq k \leq n,$$

$$\Gamma^m_{kk} = -g^{mm}g_{kk,m}/2 = -\eta_{k,m}/2\eta_m = -f_m/\eta_m, 1 \leq m \leq k \leq n,$$

We then compute the Ricci tensor: For $k, m \geq 1$,

$$R_{km} = \Gamma^\alpha_{k\alpha,m} - \Gamma^\alpha_{km,\alpha} - \Gamma^\alpha_{km}\Gamma^\beta_{\alpha\beta} + \Gamma^\alpha_{k\beta}\Gamma^\beta_{m\alpha}$$

$$= \Gamma^p_{kp,m} - \Gamma^0_{km,0} - \Gamma^p_{km,p} - \Gamma^0_{km}\Gamma^r_{0r}$$

$$-\Gamma^p_{km}\Gamma^s_{ps} + \Gamma^0_{kp}\Gamma^p_{m0} + \Gamma^p_{k0}\Gamma^0_{mp}$$

$$+\Gamma^p_{ks}\Gamma^s_{mp}$$

It is easily verified that

$$R_{km} = 0, k \neq m, k,m \geq 1$$

because the metric is diagonal, i.e.,

$$d\tau^2 = dt^2 + \sum_{k=1}^{n}g_{kk}(x)(dx^k)^2$$

with

$$(\log g_{kk})_{,km} = 0, k > m \geq 1$$

and

$$(\log g_{mm})_{,k} = 0, k \geq m \geq 1$$

Further, for $k = 1,2,\ldots,n$, we have

$$R_{kk} = \Gamma^\alpha_{k\alpha,k} - \Gamma^\alpha_{kk,\alpha} - \Gamma^\alpha_{kk}\Gamma^\beta_{\alpha\beta} + \Gamma^\alpha_{k\beta}\Gamma^\beta_{k\alpha}$$

$$= \Gamma^p_{kp,k} - \Gamma^0_{kk,0} - \Gamma^p_{kk,p} - \Gamma^0_{kk}\Gamma^s_{0s}$$

$$-\Gamma^p_{kk}\Gamma^s_{ps}$$

$$+2\Gamma^0_{kk}\Gamma^k_{k0} + \sum_p(\Gamma^p_{kp})^2 + 2\sum_{p\neq k}\Gamma^k_{kp}\Gamma^p_{kk}$$

Now,

$$\Gamma^0_{kk,0} = -g_{kk,00}/2,$$

$$\sum_p\Gamma^p_{kp,k} = \sum_{p>k}(\log(g_{pp}))_{,kk}/2 = (n-k)f_{k,k} = (n-k)g_{k,kk} = (n-k)g''_k,$$

$$\sum_p \Gamma^p_{kk,p} = -\sum_{p<k} (g^{pp}g_{kk,p})_{,p}/2$$

$$\sum_s \Gamma^0_{kk}\Gamma^s_{0s} = \sum_s (-1/4)g_{kk,0}g^{ss}g_{ss,0} = \sum_s (-S'/2S)g_{kk,0} = SS'$$

$$= nSS'\eta_k$$

$$\sum_{p,s} \Gamma^p_{kk}\Gamma^s_{ps} = (-1/4)\sum_{p,s}(g^{pp}g^{ss}g_{kk,p}g_{ss,p}$$

$$= ((-1/2)\sum_{s>p} g^{pp}g_{kk,p}f''_p$$

$$\Gamma^0_{kk}\Gamma^k_{k0} = (-1/4)g_{kk,0}(\log g_{kk})_{,0}$$

$$= -g_{kk,0}S'/2S = SS'\eta_k$$

$$\sum_p (\Gamma^p_{kp})^2 = \sum_p ((\log g_{pp})_{,k})^2 = \sum_{p>k}(f'_k)^2 = (n-k)(f'_k)^2$$

$$\sum_{p\neq k} \Gamma^k_{kp}\Gamma^p_{kk}$$

$$= (-1/4)\sum_{p<k}(\log g_{kk})_{,p}g^{pp}g_{kk,p} = (-1/2)\sum_{p<k} f''_p g^{pp}g_{kk,p}$$

Finally,

$$R_{00} = \Gamma^p_{0p,0} + \Gamma^\alpha_{0\beta}\Gamma^\beta_{0\alpha}$$

$$= \sum_p (1/2)(\log g_{pp})_{,00} + \sum_p (\Gamma^p_{0p})^2 =$$

$$(S'/S)' + n(S'/S)^2 = (n-1)(S'/S)^2 - S''/S$$

The Einstein field equations are

$$R_{\mu\nu} = K(T_{\mu\nu} - Tg_{\mu\nu}/2) = KS_{\mu\nu}$$

where

$$T_{\mu\nu} = (\rho + p)v_\mu v_\nu - pg_{\mu\nu}$$

with v_μ being comoving, ie, $v_k = 0, k = 1, 2, \ldots, n$ because it satisfies the geodesic equation

$$dv^k/d\tau + \Gamma^k_{00} = 0, v^0 = 1, \Gamma^k_{00} = 0$$

We compute

$$T = g_{\mu\nu}T^{\mu\nu} = \rho + p - (n+1)p = \rho - np$$

so that

$$S_{00} = T_{00} - Tg_{00}/2 = \rho + p - p - (\rho - np)/2 = (\rho + np)/2$$

The matter conservation equation: The Bianchi identity for the Einstein tensor implies the momentum conservation equation

$$T^{\mu\nu}_{:\nu} = 0$$

which gives

$$((\rho + p)v^\mu v^\nu)_{:\nu} - p^{,\mu} = 0$$

or

$$((\rho + p)v^\nu)_{:\nu} + (\rho + p)v^\nu v^\mu_{:\nu} - p^{,\mu} = 0$$

so that

$$((\rho + p)v^\nu)_{:\nu} - p_{,0} = 0$$

or

$$((\rho + p)\sqrt{-g})_{,0} - p_{,0}\sqrt{-g} = 0$$

or

$$(\rho\sqrt{-g})_{,0} + p(\sqrt{-g})_{,0} = 0$$

Writing

$$g = -S^{2n}\chi^2, \chi = (\Pi^n_{k=1}\chi_k)^{1/2}$$

since the number n of space dimensions is assumed to be odd, we get

$$\sqrt{-g} = S^n\chi$$

and hence the above matter conservation equation reduces to

$$(\rho S^n)' + p(S^n)' = 0$$

Note that we are assuming ρ, p to be functions of only t.

8.1. The KG equation in an n-dimensional maximally symmetric space

The metric is

$$d\tau^2 = dt^2 - S(t)^2\sum_{k=1}^n \chi_k(x)(dx^k)^2,$$

where

$$\chi_k(x) = \Pi^k_{m=1}sin^2(x^m), k = 1, 2, \ldots, n$$

$$(g^{\mu\nu}\phi_{,\mu}\sqrt{-g})_{,\nu} + m^2\sqrt{-g}\phi = 0$$

Writing

$$\Pi^n_{k=1}\chi_k(x) = \chi(x)^2,$$

we have, assuming n odd (i.e., an odd number of space dimensions),

$$\sqrt{-g} = S^n(t)\chi(x),$$

$$g^{kk}\sqrt{-g} = -S^{n-2}\chi(x)/\chi_k(x), k = 1, 2, \ldots, n$$

$$g^{00}\sqrt{-g} = \sqrt{-g} = S^n(t)\chi(x)$$

so the KG equation becomes

$$\chi(S^n\phi_{,0})_{,0} - S^{n-2}\sum_{k=1}^n(\chi\phi_{,k}/\chi_k)_{,k}$$

$$+m^2S^n\chi.\ \phi = 0$$

Or separating variables,

$$\phi(t, x) = T(t)R(x),$$

so

$$(S^n(t)T'(t))'/S^{n-2}(t)T(t) + m^2S^2(t) = \chi(x)^{-1}R(x)^{-1}\sum_{k=1}^n(\chi(x)R_{,k}(x)/\chi_k(x))_{,k}$$

Both sides must equal a constant λ:

$$S^2(t)T''(t) + nS(t)S'(t)T'(t) + (m^2S^2(t) - \lambda)T(t) = 0$$

and

$$\sum_{k=1}^n(\chi(x)R_{,k}(x))/\chi_k(x))_{,k} - \lambda\chi(x)R(x) = 0$$

9. Heat and mass transfer equations in an n+1 dimensional space-time specialized to maximally symmetric spaces

Assume that the metric has the form

$$d\tau^2 = dt^2 - S(t)^2\sum_{k=1}^n \chi_k(x)(dx^k)^2$$

This is a generalization of the spherically symmetric metric

$$d\tau^2 = dt^2 - S(t)^2(\sum_{k=1}^{n}(\Pi_{m=1}^{k-1}sin^2(\theta_m))d\theta_k^2$$

In analogy with this metric, assume that χ_k is a function of only $x^1, \ldots, x^{k-1}$. Further, in analogy with this specialization, that

$$\chi_k(x) = \Pi_{j=1}^{k-1}f_j(x^j)$$

This ensures that

$$(log\chi_k)_{,jm} = 0, j \neq m$$

and hence also ensures that

$$R_{km} = 0, k \neq m$$

As regarding R_{kk}, we have

$$R_{kk} = \Gamma^p_{kp,k} - \Gamma^p_{kk,p} - \Gamma^0_{kk,0} + 2\Gamma^k_{kp}\Gamma^p_{kk} + 2\Gamma^0_{kk}\Gamma^k_{k0} + \sum_{p\neq k}(\Gamma^p_{kp})^2$$

$$= (1/2)\sum_p (logg_{pp})_{,kk} + (1/2)\sum_p (g^{pp}g_{kk,p})_{,p} + (1/2)g_{kk,00} - (1/2)\sum_p g^{pp}g_{kk}((logg_{kk})_{,p})^2$$

$$-(1/2)g_{kk,0}.(log(g_{kk})_{,0} + (1/4)\sum_p((logg_{pp})_{,k})^2$$

$$= ((n-k)/2).log(f_k)'' + (1/2)\sum_p g_{kk}g^{pp}(logf_p)'' - (S^{'2} + SS'')\chi_k + 2(S')^2\chi_k + (1/4)(n-k)((logf_k)')^2$$

The energy-momentum tensor of the matter field: Assume $v^k = 0, k = 1, 2, \ldots, n$. Then $g_{\mu\nu}v^\mu v^\nu = 1$ implies $v^0 =$, or equivalently, $v_0 = 1, v_k = 0, k = 1, 2, \ldots, n$. The energy-momentum tensor of the matter field

$$T^{\mu\nu} = (\rho + p)v^\mu v^\nu - pg^{\mu\nu}$$

has only the following non-vanishing components:

$$T^{00} = \rho + p - p = \rho, T^{kk} = -pg^{kk} = p/S^2\chi_k, k = 1, 2, \ldots, n$$

Now, in the presence of viscous and thermal effects, the energy-momentum tensor acquires a correction $\Delta T^{\mu\nu}$ given by [Steven Weinberg, Gravitation and Cosmology:Principles and Applications of the General Theory of Relativity, Wiley]

$$\Delta T^{\mu\nu} = \chi_1.H^{\mu\alpha}H^{\nu\beta}(v_{\alpha:\beta} + v_{\beta:\alpha}) + (H^{\mu\alpha}v^\nu + H^{\nu\alpha}v^\mu)Q_\alpha$$

where

$$H^{\mu\nu} = g^{\mu\nu} - v^\mu v^\nu,$$

$$Q_\alpha = \chi_2(T_{,\alpha} - Tv_{\alpha:\beta}v^\beta)$$

where χ_1, χ_2 are positive, depending possibly on the temperature T. The energy equation is then

$$(\rho.sv^\mu + \Delta T^{\mu\nu}v_\nu/T)_{:\mu} = \Delta T^{\mu\nu}(v_\nu/T)_{:\mu} - -- (1)$$

where s is the entropy per unit mass. This equation can be derived from the conservation of the total energy-momentum tensor $T^{\mu\nu} + \Delta T^{\mu\nu}$ and the number conservation equation $(nv^\mu)_{:\mu} = 0$, the first law of thermodynamics

$$Td(s/n) = d(\rho/n) + pd(1/n)$$

Note that s/n is the entropy per particle and $\rho = mn$ is the density where n is the number of particles per unit volume and m is the mass per particle. When we impose the condition that the lhs of (1), which represents the rate of entropy increase per unit volume, be non-negative, then we obtain the above form for the correction to the energy-momentum tensor $\Delta T^{\mu\nu}$ due to viscous and thermal effects. If the fluid is adiabatic, we can set $(\rho sv^\mu)_{:\mu}$ to be zero, and the result is the generalization of the temperature diffusion equation after taking into account convective terms, namely the heat transfer equation:

$$\Delta T^{\mu\nu}_{:\nu}v_\mu = 0$$

If the background metric is Robertson-Walker, with comoving velocity, then $v_k = 0, v_0 = 1$ and this equation simplifies to

$$\Delta T^{0\nu}_{:\nu} = 0$$

Now, for the RW metric with comoving velocity,

$$H^{00} = 0, H^{0k} = 0, H^{km} = g^{km}$$

and we find

$$\Delta T^{00} = 0,$$

$$\Delta T^{0k} = g^{kk}Q_k = \chi_2.g^{kk}(T_{,k} - Tv_{k:0}) = \chi_2.g^{kk}(T_{,k} + T\Gamma^k_{k0}) = \chi_2 g^{kk}(T_{,k} + (T/2)(logg_{kk})_{,0})$$

$$= \chi_2 g^{kk}(T_{,k} + TS'/S)$$

and the heat conduction/temperature diffusion equation in the RW metric becomes

$$\Delta T^{0k}_{,k} + \Gamma^0_{kk}\Delta T^{kk} + \Gamma^m_{km}\Delta T^{0k} = 0$$

or

$$\Delta T^{0k}_{,k} + (1/2)(logg_{kk})_{,0}\Delta T^{kk} + (1/2)(logg_{mm})_{,k}\Delta T^{0k} = 0$$

Writing the RW metric as

$$d\tau^2 = dt^2 - S(t)^2((dx^1)^2 + sin^2(x^1))(dx^2)^2 + sin^2(x_2)(dx^3)^2))$$

$$= dt^2 - S(t)^2(\sum_{k=1}^{3}\eta_k(x)(dx^k)^2))$$

where

$$\eta_1(x) = 1, \eta_2(x) = sin^2(x^1), \eta_3(x) = sin^2(x^1)sin^2(x^2),$$

our adiabatic heat conduction equation becomes

$$\Delta T^{0k}_{,k} + (S'/S)\Delta T^{kk} + (1/2)(log(\sqrt{-g}))_{,k}T^{0k} = 0$$

where

$$\sqrt{-g} = S^3(t).sin^2(x^1)sin(x^2)$$

Note that

$$\Delta T^{kj} = -2\chi_1 g^{kk}g^{ij}(\Gamma^0_{kj})$$

which is zero for $k \neq j$ and

$$\Delta T^{kk} = \chi_1(g^{kk})^2 g_{jj,0}$$

for the RW metric with comoving velocities.

10. The linearized Einstein field equations for perturbations in the metric and matter fluid around the RW space-time metric

The energy-momentum tensor of the matter fluid is given by

$$T^{\mu\nu} = T_0^{\mu\nu} + \Delta T^{\mu\nu}$$

where

$$T_0^{\mu\nu} = (\rho + p)v^\mu v^\nu - pg^{\mu\nu},$$

$$\Delta T^{\mu\nu} = H^{\mu\alpha}H^{\nu\beta}\tilde{\Delta}T_{\alpha\beta}$$

$$+(H^{\mu\alpha}v^\nu + H^{\nu\alpha}v^\mu)Q_\alpha$$

where

$$Q_\alpha = \chi_2(T_{,\alpha} - Tv_{\alpha:\beta}v^\beta)$$

and

$$\tilde{\Delta} T_{\alpha\beta} = \chi_0(v_{\alpha:\beta} + v_{\beta:\alpha})$$

$$+\chi_1 v^\rho_{:\rho} g_{\alpha\beta}$$

where $\chi_0, \chi_1, \chi_2\, geq 0$. The unperturbed velocity is comoving, i.e.,

$$v^i = 0, v_i = 0, i = 1, 2, 3, v^0 = v_0 = 1$$

Note that the unperturbed metric is RW:

$$g_{00} = 1.g_{11} = -S^2(t)f(r)^2, g_{22} = -S^2(t)r^2, g_{33} = -S^2(t)r^2 sin^2(\theta)$$

By appropriate choice of coordinates, we can assume that the perturbation to this metric has nonvanishing components

$$\delta g_{rs}, 1 \le r, s \le 3$$

i.e., $\delta g_{0\mu} = 0, \mu = 0, 1, 2, 3$. We denote the velocity perturbations by δv^i and δv_i. Note that

$$\delta v_i = \delta(g_{i\mu} v^\mu) = \delta g_{i\mu}. v^\mu + g_{i\mu}\delta v^\mu$$

$$= g_{ii}\delta v^i$$

since $v^i = 0, \delta g_{i0} = -0$. Also, the equation

$$\delta(g_{\mu\nu} v^\mu v^\nu) = 0$$

implies

$$g_{00}\delta((v^0)^2) + g_{ii}((\delta(v^i)^2) + \delta g_{00} = 0$$

Since $v^i = 0$ implies $\delta * ((v^i)^2) = v^i \delta v^i = 0$ and $\delta g_{00} = 0$, it follows therefore that

$$\delta v^0 = 0$$

($g_{00} = 1, \delta((v^0)^2) = v^0 \delta v^0 = \delta v^0$). The fluid equations are

$$(T^{\mu\nu} + \Delta T^{\mu\nu})_{:\nu} = 0,$$

for ρ, v^i given the equation of state, while the heat transfer equation for the temperature T under adiabatic conditions is

$$(\Delta T^{\mu\nu})_{:\nu} v^\mu = 0$$

Here,

$$H^{\mu\nu} = g^{\mu\nu} - v^\mu v^\nu$$

To study small perturbations around the comoving velocity, density, and metric, we use the linearized field equations:

$$\delta R_{\mu\nu} = K\delta(T_{\mu\nu} - (1/2)Tg_{\mu\nu} + \Delta T_{\mu\nu} - (1/2)\Delta T.\, g_{\mu\nu}),$$

$$\delta[(T^{\mu\nu} + \Delta T^{\mu\nu})_{:\nu}] = 0,$$

$$\delta[(\Delta T^{\mu\nu})_{:\nu} v^\mu] = 0$$

Now let

$$S_0^{\mu\nu} = H^{\mu\alpha} H^{\nu\beta}(v_{\alpha:\beta} + v_{\beta:\alpha}),$$

$$S_1^{\mu\nu} = H^{\mu\alpha} H^{\nu\beta} v^\rho_{:\rho} g_{\alpha\beta}$$

$$S_2^{\mu\nu} = (H^{\mu\alpha} v^\nu + H^{\nu\alpha} v^\mu) Q_\alpha$$

Then,

$$S_0^{\mu\nu} = v^{\mu:\nu} + v^{\nu:\mu} - v^\nu v^\alpha v^\mu_{:\alpha} - v^\mu v^\alpha v^\nu_{:\alpha}$$

since

$$v^\alpha v_{\alpha:\beta} = 0$$

Now

$$\delta v_{\mu:\nu}$$

$$= \delta(v_{\mu,\nu} - \Gamma^\rho_{\mu\nu} v_\rho)$$

In particular,

$$\delta v_{i:j} = \delta v_{i,j} - \Gamma^k_{ij} \delta v_k - \delta\Gamma^0_{ij},$$

$$\delta v_{i:0} = \delta v_{i,0} - \Gamma^k_{i0} \delta v_k$$

$$= \delta v_{i,0} - \Gamma^i_{i0} \delta v_i$$

$$= \delta v_{i,0} - (1/2)(log g_{ii})_{,0} \delta v_i$$

$$= \delta v_{i,0} - (S'/S)\delta v_i$$

Consider next

$$\delta(v^\nu v^\alpha v^\mu_{:\alpha})$$

For $\nu = 0$, this is

$$\delta(v^0 v^\alpha v^\mu_{:\alpha}) =$$

$$\delta(v^\alpha v^\mu_{:\alpha}) =$$

$$= \delta(v^\mu_{:0}) + (\delta v^k) v^\mu_{:k}$$

$$= \delta(v^\mu_{,0} + \Gamma^\mu_{0\nu} v^\nu)$$

$$+\delta v^k (v^\mu_{,k} + \Gamma^\mu_{k0})$$

For $\mu = 0$, this is zero, while for $\mu = r$, this is

$$\delta v^r_{,0} + 2.Gamma^r_{0k} \delta v^k$$

$$= \delta v^r_{,0} + 2.\Gamma^r_{0r} \delta v^r = \delta v^r_{,0} + (S'/S)\delta v^r$$

(No summation over r). For $\nu = k$,

$$\delta(v^\nu v^\alpha v^\mu_{:\alpha}) = \delta(v^k v^\alpha v^\mu_{:\alpha})$$

$$= \delta v^k.\, v^\alpha v^\mu_{:\alpha})$$

$$= \delta v^k.\, v^\mu_{:0} = \delta v^k.\, \Gamma^\mu_{00} = 0$$

Proceeding in this way, we can linearize the differential equations for heat and mass transfer in the expanding universe. Currently, work is going on to generalize these equations to higher dimensional space-time.

11. Quantum noisy Boltzmann equation taking into account quantum noise based on the Hudson-Parthasarathy noisy Schrodinger equation

The HPS (Hudson-Parthasarathy-Schrodinger) equation taking into account a single creation process, a single annihilation process and a single conservation process is given by

$$dU(t) = (-(iH + P)dt + L_1 dA(t) - L_2 dA(t)^* + Sd\Lambda(t))U(t0$$

where for unitarity of $U(t)$, we require that

$$P = L_2^* L_2/2,$$

$$L_1^* - S^* L_2 - L_2 = 0, L_1 - L_2^* - L_2^* S = 0$$

$$S + S^* + S^* S = 0$$

Equivalently, writing $L_2^* = L$, we get for the condition of unitarity,

$$L_1 = L(1 + S), L_2 = L^*, S^* S + S + S^* = 0$$

Writing

$$S = Z - 1,$$

doi.org/10.32388/JXIAL3

this is equivalent to the conditions

$$L_1 = LZ, L_2 = L^*, Z^*Z = 1, P = LL^*/2$$

Thus, the HPS equation becomes for this special case,

$$dU(t) = [-(iH + LL^*/2)dt + LZdA(t) - L^*dA^* + (Z-1)d\Lambda(t)]U(t)$$

Assuming that the system comprises N indistinguishable particles, all connected to the same bath with the same coupling operators, we can write

$$L = \sum_{k=1}^{p} L_k^{\otimes N}, Z = Z_1^{\times N},$$

so that

$$+Z_1^* Z_1 = 1,$$

$$LZ = \sum_k (L_k Z_1)^{\otimes N}, H = \sum_{k=1}^{N} H_k + \sum_{1\le k<j\le N} V_{kj},$$

$$P = LL^*/2 = \sum_{k,j=1}^{p} (L_k L_j^*)^{\otimes N}/2$$

and the HPS equation becomes

$$dU(t) = [(-i(\sum_k H_k + \sum_{k<j} V_{kj}) + \sum_{k,j} (L_k L_j^*)^{\otimes N}/2)dt$$

$$+\sum_k (L_k Z_1)^{\otimes N} dA(t) - \sum_k L_k^{*\otimes N} dA(t)^* + (Z_1^{\otimes N} - 1)d\Lambda(t)]U(t)$$

Owing to the indistinguishability of the particles, the state $\rho(t)$ of the system and bath can be expressed as

$$\rho(t) = \rho_1(t)^{\otimes N} + \sum_{k<j, i\ne k,j} g_{kj}(t) \otimes \rho_i(t)$$

$$+ \sum_{k<j<m, i\ne k,j,m} g_{kjm}(t) \otimes \rho_i(t) + \ldots +$$

$$g_{12\ldots N}(t)$$

where the $\rho_i^{'}s$ are all copies of ρ_1 and likewise, for each $r = 1, 2, \ldots, N$, the $\rho_{i_1\ldots i_r}^{'}s$ are all identical copies of each other for each $1 \le i_1 < \ldots < i_r \le N$. In order to get the correct marginals for the states, we must assume that

$$Tr_2 g_{12} = 0, Tr_3 g_{123} = 0, \ldots, Tr_r g_{123..r} = 0, r = 2, 3, \ldots, N$$

Then, for example,

$$\rho_{123\ldots r} = \rho_1^{\otimes r} + \sum \rho_1^{\otimes r-2} \otimes g_{23}$$

$$+\sum \rho_1^{\otimes r-3} \otimes g_{234} + \ldots + g_{123..r}$$

In particular,

$$\rho_{12} = \rho_1^{\otimes 2} + g_{12},$$

$$\rho_{123} = \rho_1^{\otimes 3} + \rho_1 \otimes g_{23} + \rho_2 \otimes g_{31} + \rho_3 \otimes g_{12},$$

$$+g_{123}$$

Note that ρ_1, ρ_2, ρ_3 are identical copies of each other but act in different Hilbert spaces indexed by the corresponding subscripts.

We now derive the master equation for the system state alone by tracing out over the bath with the bath maintained in a coherent state:

$$\rho(t) = U(t)\rho_s(0) \otimes |\phi(u)><\phi(u)|)U(t)^*$$

so that, using quantum Ito's formulae and properties of the partial trace,

$$\rho_s(t) = Tr_2 \rho(t)$$

Then,

$$d\rho_s(t) = -i[H, \rho_s(t)]dt - P\rho_s(t)dt - \rho_s(t)Pdt$$

$$L_1\rho_s(t)u(t)dt - \bar{u}(t)L_2\rho_s(t)dt + S\rho_s(t)|u(t)|^2 dt$$

$$+\rho_s(t)L_1^*\bar{u}(t)dt - \rho_s(t)L_2^* u(t)dt + \rho_s(t)S^*|u(t)|^2 dt$$

$$-L_2\rho_s(t)S^*\bar{u}(t)dt - S\rho_s(t)L_2^* u(t)dt$$

$$+S\rho_s(t)S^*|u(t)|^2 dt + L_2\rho_s(t)L_2^* dt$$

where $P = L_2^* L_2/2$. Making the above substitutions, we get

$$\rho_s^{'}(t) = -i[\sum_k H_k + \sum_{k<j} V_{kj}, \rho_s(t)]$$

$$+\theta_t(\rho_s(t))$$

where

$$\theta_t(\rho_s) = (-1/2)(L_2^* L_2\rho_s + \rho_s L_2^* L_2 - 2L_2\rho_s L_2^*)$$

$$+u(t)(L_1\rho_s - \rho_s L_2^* - S\rho_s L_2^*) + \bar{u}(t)(\rho_s L_1^* - L_2\rho_s - L_2\rho_s S^*) + |u(t)|^2(S\rho_s + \rho_s S^* + S\rho_s S^*)$$

$$= \theta_1(\rho_s) + u(t)\theta_2(\rho_s) + \bar{u}(t)\theta_3(\rho_s) + |u(t)|^2\theta_4(\rho_s)$$

where

$$\theta_1(\rho_s) = (-1/2)(L_2^* L_2\rho_s + \rho_s L_2^* L_2 - 2L_2\rho_s L_2^*)$$

$$= (-1/2)(LL^*\rho_s + \rho_s LL^* - 2L^*\rho_s L)$$

$$\theta_2(\rho_s) = L_1\rho_s - \rho_s L_2^* - S\rho_s L_2^* = LZ\rho_s - \rho_s L - S\rho_s L$$

$$\theta_3(\rho_s) = \rho_s L_1^* - L_2\rho_s - L_2\rho_s S^* = \rho_s Z^* L^* - L^*\rho - s - L^*\rho_s S^*$$

$$\theta_4(\rho_s) = S\rho_s + \rho_s S^* + S\rho_s S^*$$

In order to derive an approximate second-order Boltzmann equation, we assume $g_{123\ldots r} = 0, r = 3, 4, \ldots, N$ so that only ρ_1, g_{12} and their copies are non-vanishing. Thus, we are assuming that (using the simplified notation ρ for ρ_s)

$$\rho = \rho_1^{\otimes N} + \sum \rho_1^{\otimes N-2} \otimes g_{23}$$

(Note that $\sum \rho_1^{\otimes N} \otimes g_{23}$ is the same as $\sum g_{12} \otimes \rho_3^{\otimes N-2}$). Substitution of this expression, followed by partial tracing, then gives us

$$Tr_{23\ldots N}\theta_1(\rho) =$$

$$= (-1/2)Tr_{23\ldots N}(LL^*\rho + \rho LL^* - 2L^*\rho L)$$

$$= (-1/2)Tr_{23\ldots N}(\sum (L_k L_j^*)^{\otimes N}\rho + \rho\sum (L_k L_j^*)^{\otimes N}$$

$$-2\sum_k L_j^{*\otimes N}\rho L_k^{\otimes N})$$

$$= (-1/2)(Tr(L_k L_j^*\rho_1))^{N-1}(L_k L_j^*\rho_1 + \rho_1 L_k L_j^* - 2L_j^*\rho_1 L_k)$$

$$-((N-1)(N-2)/4)[Tr(L_k L_j^*\rho_1))^{N-3}. [Tr_{23}[(L_k L_j^*)^{\otimes 3}(\rho_1 \otimes g_{23} + \rho_2 \otimes g_{13})]]$$

$$+Tr_{23}[(\rho_1 \otimes g_{23})(L_k L_j^*)^{\otimes 3} + (\rho_2 \otimes g_{13})(L_k L_j^*)^{\otimes 3}]$$

$$-2.Tr_{23}[L_j^{*\otimes 3}(\rho_1 \otimes g_{23} + \rho_2 \otimes g_{13})L_k^{\otimes 3})]]$$

$$-((N-1)(N-2)/4)[Tr(L_k L_j^*\rho_1))^{N-2}. Tr_2[(L_k L_j^*)^{\otimes 2}g_{12} + g_{12}(L_k L_j^*)^{\otimes 2}$$

$$-2L_j^{*\otimes 2}g_{12}L_k^{\otimes 2}]$$

$$= (-1/2)(Tr(L_k L_j^*\rho_1))^{N-1}(L_k L_j^*\rho_1 + \rho_1 L_k L_j^* - 2L_j^*\rho_1 L_k)$$

$$-((N-1)(N-2)/4). Tr((L_k L_j^*)^{\otimes 2}g_{12}))(Tr(L_k L_j^*\rho_1))^{N-3}). [L_k L_j^*\rho_1 + \rho_1 L_k L_j^* - 2L_j^*\rho_1$$

$$-((N-1)(N-2)/2)[Tr(L_k L_j^*\rho_1))^{N-2}. Tr_2[(L_k L_j^*)^{\otimes 2}g_{12} + g_{12}(L_k L_j^*)^{\otimes 2}$$

$$-2L_j^{*\otimes 2}g_{12}L_k^{\otimes 2}]$$

doi.org/10.32388/JXIAL3

Likewise,

$$Tr_{23...N}(\theta_2(\rho)) =$$

$$Tr_{23...N}(LZ\rho - \rho L - (Z-1)\rho L) =$$

$$Tr_{23...N}(\sum_k (L_k Z_1)^{\otimes N}\rho - \rho.\sum_k L_k^{\otimes N} - (Z_1^{\otimes N} - 1)\rho.\sum_k L_k^{\otimes N}))$$

We evaluate the various terms on the rhs using the above second order approximation to ρ:

$$Tr_{23...N}((L_k Z_1)^{\otimes N}\rho)$$

$$= (Tr(L_k Z_1 \rho_1))^{N-1} L_k Z_1 \rho_1$$

$$+((N-1)(N-2)/2).(Tr(L_k Z_1\rho_1))^{N-3}.[Tr_{23}[(L_k Z_1)^{\otimes 3}(\rho_1 \otimes g_{23}) + \rho_2 \otimes g_{13} + \rho_3 \otimes g_{12})]$$

$$= (Tr(L_k Z_1\rho_1))^{N-1} L_k Z_1 \rho_1 + ((N-1)(N-2)/2).(Tr(L_k Z_1 \rho_1))^{N-3}.[Tr_{23}[(L_k Z_1)^{\otimes 3}(\rho_2 \otimes g_{13} + \rho_3 \otimes g_{12})]$$

$$+((N-1)(N-2)/2).(Tr(L_k Z_1)\rho_1))^{N-3}.Tr[(L_k Z_1)^{\otimes 2} g_{23}] L_k Z_1 \rho_1$$

$$= (Tr(L_k Z_1 \rho_1))^{N-1} L_k Z_1 \rho_1 + ((N-1)(N-2)).(Tr(L_k Z_1 \rho_1))^{N-2}.Tr_2[(L_k Z_1)^{\otimes 2} g_{12}]$$

$$+((N-1)(N-2)/2).(Tr(L_k Z_1)\rho_1))^{N-3}.Tr_{23}[(L_k Z_1)^{\otimes 2} g_{23}] L_k Z_1 \rho_1$$

$$Tr_{23..N}[Z_1^{\otimes N}\rho.L_k^{\otimes N}]$$

$$= (Tr(Z_1\rho_1 L_k))^{N-1}.Z_1\rho_1 L_k$$

$$+Tr_{23...N}[Z_1^{\otimes N}(\sum \rho_1^{\otimes N-2} \otimes g_{23}) L_k^{\otimes N}]$$

$$= (Tr(Z_1\rho_1 L_k))^{N-1}.Z_1\rho_1 L_k$$

$$+((N-1)(N-2)/2)(Tr(Z_1\rho_1.L_k))^{N-1} Tr_2[Z_1^{\otimes 2} g_{12} Z_1^{\otimes 2}]$$

Likewise,

$$Tr_{23...N}\theta_3(\rho) = (Tr_{23...N}\theta_2(\rho))^*,$$

and finally,

$$Tr_{23...N}\theta_4(\rho) =$$

$$Tr_{23..N}(S\rho + \rho S^* + S\rho S^*)$$

$$= Tr_{23...N}[(Z_1^{\otimes N} - 1)\rho + \rho(Z_1^{*\otimes N} - 1) + (Z_1^{\otimes N} - 1)\rho.(Z_1^{*\otimes N} - 1)]$$

Now, based on the second-order approximation of the joint state of the particles, consider the term

$$Tr_{23...N}[Z_1^{\otimes N}\rho] = (Tr[Z_1\rho_1])^{N-1} Z_1\rho_1$$

$$+Tr_{23...N}[Z_1^{\otimes N}.\sum(\rho_1^{\otimes N-2} \otimes g_{23})]$$

$$= (Tr[Z_1\rho_1])^{N-1} Z_1\rho_1$$

$$+((N-1)(N-2)/2).(Tr(Z_1\rho_1))^{N-3}.Tr_{23}[Z_1^{\otimes 3}(\rho_1 \otimes g_{23} + \rho_2 \otimes g_{13} + \rho_3 \otimes g_{12})]$$

$$= (Tr[Z_1\rho_1])^{N-1} Z_1\rho_1$$

$$+((N-1)(N-2).(Tr(Z_1\rho_1))^{N-2}.Tr_2(Z_1^{\otimes 2} g_{12})$$

$$+((N-1)(N-2)/2).(Tr(Z_1.\rho_1))^{N-3}.(Tr(Z_1^{\otimes 2} g_{12})).Z_1\rho_1$$

Further,

$$Tr_{23...N} Z_1^{\otimes N}\rho.Z_1^{*\otimes N}$$

can also be evaluated along similar lines. Finally, we substitute these partial trace expressions into the partial traces of the equation

$$\rho'(t) = -i[\sum_k H_k + \sum_{k<j} V_{kj}, \rho(t)]$$

$$+\theta_t(\rho(t))$$

i.e.,

$$\partial_t\rho_1(t) = Tr_{23...N}\rho'(t) = -i[H_1, \rho_1(t)] + (N-1)Tr_2[V_{12}, \rho_1(t) \otimes \rho_1(t)] + Tr_{23...N}\theta_t(\rho(t))$$

and likewise in the equation

$$\partial_t(\rho_1(t) \otimes \rho_1(t) + g_{12}(t)) =$$

$$Tr_{34...N}\rho'(t) + [H_1 + H_2 + V_{12}, \rho_1(t) \otimes \rho_1(t) + g_{12}(t)]$$

$$+(N-2)Tr_3[V_{13} + V_{23}, \rho_1 \otimes \rho_1 \otimes \rho_1 + \rho_1 \otimes g_{23} + \rho_2 \otimes g_{13} + \rho_3 \otimes g_{12}]$$

$$+Tr_{34...N}\theta_t(\rho(t))$$

with $Tr_{34...N}\theta_t(\rho(t))$ being evaluated in a similar way as above. This calculation gives us two nonlinear differential equations for $\rho_1(t), g_{12}(t)$ and can be termed as the second-order quantum Boltzmann equations for an open quantum system comprising N indistinguishable particles.

Some remarks: It should be noted that the form of the Lindblad operators that couple a system of indistinguishable particles to a noisy quantum bath has been selected so that the interaction of the particles with the bath is symmetric with respect to interchange of the particles. For example, such an interaction term involving the annihilation process would have the general form $MdA(t)$ where

$$M = [\sum_{k=1}^{p}\sum_\sigma L_{k,\sigma 1} \otimes \ldots \otimes L_{k,\sigma N}]$$

with $L_{k,j}, k = 1,2,\ldots,p, j = 1,2,\ldots,N$ and σ running over S_N, namely, the group of all $N!$ permutations of $\{1,2,\ldots,N\}$. It is easy to see that M can be expressed in the form

$$M = \sum_k L_k^{\otimes N}$$

where the operators L_k are linear combinations of the $L_{k,j}, j = 1,2,\ldots,N$. For example,

$$L_1 \otimes L_2 + L_2 \otimes L_1 =$$

$$(1/2)[(L_1 + L_2)^{\otimes 2} - L_1^{\otimes 2} - L_2^{\otimes 2}],$$

$$L_1 \otimes L_2 \otimes L_3 + allpermutations$$

$$= (1/6)[(L_1 + L_2 + L_3)^{\otimes 3} - (L_1 + L_2)^{\otimes 3} - (L_2 + L_3)^{\otimes 3} - (L_1 + L_3)^{\otimes 3}$$

$$+L_1^{\otimes 3} + L_2^{\otimes 3} + L_3^{\otimes 3}]$$

12. How to compute the perturbation to the velocity and density/four current density of matter comprising of electrons, positrons, leptons caused and small perturbations to the metric around the RW metric at the quantum level using Feynman's path integrals for fields

1. Let $H_0 + V(t) = H(t)$ be the Hamiltonian of a field with H_0 being time independent and $V(t)$ time dependent. The propagator of the field is an operator Kernel $K(t,t')$ that satisfies the differential equation

$$(i\partial_t - H_0 - V(t))K(t,t') = \delta(t-t').I$$

so we can write its expansion as

$$K = (i\partial_t - H_0)^{-1} + (i\partial_t - H_0)^{-1}.\sum_{n\geq 1}(V.(i\partial_t - H_0)^{-1})^n$$

$$= (i\partial_t - H_0)^{-1}.(1 - (V.(i\partial_t - H_0)^{-1})^{-1}$$

This is well approximated up to linear orders in the time-varying perturbing potential V by the expression

$$K \approx (i\partial_t - H_0)^{-1} + (i\partial_t - H_0)^{-1} V.(i\partial_t - H_0)^{-1}$$

In order to see how this is calculated, we assume that H_0 has a complete orthonormal set of eigenfunctions $|n >= |u_n >= u_n(r)$, with energy

doi.org/10.32388/JXIAL3

eigenvalues E_n. Then, we can write

$$(i\partial_t - H_0)^{-1} = (2\pi)^{-1}\int (E-H_0)^{-1} exp(-iE(t-t^{'}))dE$$

$$= (2\pi)^{-1}\int \sum_n \frac{|n><n|}{E-E_n} exp(-iE(t-t^{'}))dE$$

so, for example, writing

$$G = (i\partial_t - H_0)^{-1}V.(i\partial_t - H_0)^{-1}$$

we find that its kernel is given by

$$G(t,t^{'}) = (2\pi)^{-2}\int dEdE^{'}ds\sum_{n,m}((E-E_n)(E^{'}-E_m))^{-1}.exp(-iE(t-s)).exp(-iE^{'}(s-t^{'}))|n><n|V(s)|m><m|$$

$$= (2\pi)^{-2}\int dEdE^{'}\sum_{n,m}((E-E_n)(E-E_m))^{-1} <n|\hat{V}(E-E^{'})|m> |n><m|$$

where

$$\hat{V}(E) = \int V(s)exp(iEs)ds$$

This formula will play a fundamental role in computing the velocity and density perturbations in our expanding universe determined by the quantum mechanical version of the Einstein field equations linearized around the RW metric based on Feynman's path integral for fields.

Consider now the action functional for metric perturbations around the RW metric obtained from the Einstein-Hilbert action plus the action functional for the metric interacting with the Dirac field of electrons and positrons. The total Lagrangian has the form

$$S_g[V] + \int[\bar{\psi}(x)[\gamma^a V_a^\mu(x)(i\partial_\mu + i\Gamma_\mu(x)) - m]\psi(x)d^4x$$

$$= S_g[V] + S_d[V,\psi,\bar{\psi}]$$

where $S_g[V]$ is the Einstein-Hilbert action as a functional of the tetrad $V_a^\mu(x)$ of the metric field and $\Gamma_\mu(x)$ is the spinor connection of the gravitational field given by

$$\Gamma_\mu(x) = \Gamma_\mu^{ab}(x)\gamma_{ab}/4,$$

$$\Gamma_\mu^{ab}(x) = (1/2)V^{av}V^b_{v:\mu}, \gamma_{ab} = [\gamma_a,\gamma_b]$$

The quantum-averaged four-current density field of matter, comprising electrons and positrons, is then

$$<J^\mu(x)> = Z^{-1}m\int exp(iS_g[V] + iS_d[V,\psi,\bar{\psi}])\bar{\psi}(x)\gamma^\mu\psi(x)DV.D\psi.D\bar{\psi}$$

where

$$Z = \int exp(iS_g[V] + iS_d[V,\psi,\bar{\psi}])DV.D\psi.D\bar{\psi}$$

More generally, the higher-order quantum correlations in the matter current density at the space-time points $x_1,...,.x_r$ are given by

$$<J^{\mu_1}(x_1)...J^{\mu_r}(x_r)> =$$

$$Z^{-1}m^r\int exp(iS_g[V] + iS_d[V,\psi,\bar{\psi}])(\Pi_{k=1}^r\bar{\psi}(x_k)\gamma^{\mu_k}\psi(x_k))DV.D\psi.D\bar{\psi}$$

The integral w.r.t $D\psi.D\bar{\psi}$ is a Fermionic Gaussian integral, and evaluating this part gives us for the average current

$$<J^\mu(x)> =$$

$$C.\int exp(S_g[V]).det(V_a^\mu\gamma^a(i\partial_\mu + i\Gamma_\mu) - m).Tr(\gamma^\mu S(x,x|V))DV$$

where

$$S(x,y|V) =< T(\psi(x)\bar{\psi}(y))>$$

with V being fixed. Note that we have used the identity

$$<\bar{\psi}(x)\gamma^\mu\psi(x)> = Tr(\gamma^\mu <\psi(x)\bar{\psi}(x)>)$$

$$= Tr(\gamma^\mu S(x,x|V))$$

Note that

$$S_{lm}(x,y|V) = \theta(x^0-y^0)<\psi_l(x)\bar{\psi}_m(y)> - \theta(y^0-x^0)<\bar{\psi}_m(y)\psi_l(x)>$$

where $\bar{\psi}(x) = \psi(x)^*\gamma^0$. It follows easily from the canonical equal time anticommutation relations

$$\{\psi_l(x),\psi_m(y)^*\} = \delta^3(x-y), x^0 = y^0$$

that the electron propagator $S(.|V)$ in a gravitational field satisfies

$$DS(x,y) = i\gamma^a V_a^0(x).\delta^4(x-y)$$

where D is the Dirac operator in a gravitational field, i.e.,

$$D = \gamma^a V_a^\mu(i\partial_\mu + i\Gamma_\mu) - m$$

Thus,

$$S(x,y) = (D^{-1}\gamma^a V_a^0)(x,y)$$

$$= [[\gamma^b V_b^\mu(i\partial_\mu + i\Gamma_\mu) - m]^{-1}i\gamma^a V_a^0](x,y)$$

Assuming that the gravitational field is weak, we write

$$V_a^\mu(x) = \delta_a^\mu + \epsilon_a^\mu(x)$$

where $\epsilon_a^\mu(x)$ is of the first order of smallness. Then, up to first order in ϵ, we have

$$\Gamma_\mu^{ab} = (1/2)V^{av}V^b_{v:\mu} = (1/2)V^{av}(V^b_{v,\mu} - \Gamma^\rho_{v\mu}V^b_\rho)$$

$$= (1/2)\eta^{av}(\epsilon^b_{v,\mu} - \Gamma^b_{v\mu})$$

$$= (1/2)(\epsilon^{ba}_{,\mu} - \Gamma^{ba}_{0\mu})$$

where

$$\Gamma^{ba}_{0\mu} = \eta_{av}\Gamma^b_{v\mu} = \eta_{av}\eta_{bc}\Gamma_{cv\mu}$$

Note that for a weak gravitational field, we take

$$g_{\mu v} = \eta_{\mu v} + \delta g_{\mu v} = \eta_{ab}V^a_\mu V^b_v$$

where

$$\delta g_{\mu v} = \epsilon_{\mu v} + \epsilon_{v\mu}$$

so

$$\Gamma_{av\mu} = (1/2)(\epsilon_{av,\mu} + \epsilon_{a\mu,v} - \epsilon_{\mu v,a}$$

$$+ \epsilon_{va,\mu} + \epsilon_{\mu a,v} - \epsilon_{v\mu,a})$$

Thus, we can write up to linear orders in ϵ:

$$D = [i\gamma^a\partial_a - m] + i\epsilon_a^\mu(x)\gamma^a\partial_\mu$$

$$+ i\gamma^a\Gamma_a(x)$$

where

$$\gamma^a\Gamma_a(x) = K(\mu,v,a)\epsilon^\mu_{a,v}(x)$$

summation over all the repeated indices is understood and $K(\mu,v,a)$ consists of constant matrices built out of linear combinations of products of three of the Gamma matrices γ_c. In case the electron is bound by a potential $V(r)$, the Dirac operator for the electron, taking into account gravitational effects, will be given by

$$D = [i\gamma^a\partial_a - m + i\gamma^0 V] + i\epsilon_a^\mu(x)\gamma^a\partial_\mu$$

$$+ iK(\mu,v,a)\epsilon^\mu_{a,v}(x)$$

and then, if $|n>, E_n, n \geq 1$ denote the stationary states corresponding to the energy eigenvalues of the Dirac electron bound by the potential V, we get

doi.org/10.32388/JXIAL3

approximately, for the electron propagator taking into account both the binding potential V and the weak gravitational field ϵ_a^μ,

$$S = (D^{-1}\gamma^a V_a^0) =$$

$$D_0^{-1} - D_0^{-1}.[i\epsilon_a^\mu(x)\gamma^a\partial_\mu$$

$$+iK(\mu,\nu,a)\epsilon_{a,\nu}^\mu(x)].D_0^{-1}$$

where

$$D_0 = i\gamma^a\partial_a - m + i\gamma^0 V = i\gamma^0(\partial_t + V) + (\gamma, i\nabla)$$

where $\gamma = (\gamma^1, \gamma^2, \gamma^3)$. Now, we can write

$$D_0^{-1}(x,y) = \int \gamma^0.[\sum_n < r_x|n >< n|r_y > /(E - E_n)]exp(-iE(t_x - t_y))dE$$

where

$$x = (t_x, r_x), y = (t_y, r_y)$$

Therefore,

$$S(x,y) = D_0^{-1}(x,y) - \int D_0^{-1}(x,z).[i\epsilon_a^\mu(z)\gamma^a\partial_\mu^z$$

$$+iK(\mu,\nu,a)\epsilon_{a,\nu}^\mu(z)].D_0^{-1}(z,y)d^4z$$

In case the binding potential $V = 0$, we have

$$D_0^{-1}(x,y) = \int exp(-ip.(x-y))d^4p/(\gamma.p - m)$$

where

$$p.(x-y) = p_\mu(x^\mu - y^\nu), \gamma.p = \gamma_\mu p^\mu$$

and the above formula reduces to

$$S(x,y) - D_0^{-1}(x,y) =$$

$$-\int exp(-ip.(x-z))[\gamma.p - m]^{-1}.[\epsilon_a^\mu(z)q_\mu\gamma^a + i.K(\mu,\nu,a)\epsilon_{a,\nu}^\mu(z)].[\gamma.q - m]^{-1}exp(-iq.(z-y))d^4pd^4qd^4z$$

This expression can be simplified by defining

$$\int \epsilon_a^\mu(z)exp(ip.z)d^4z = \hat{\epsilon}_a^\mu(p)$$

which implies that

$$\int \epsilon_{a,\nu}^\mu(z)exp(ip.z)d^4z = -ip_\nu\hat{\epsilon}_a^\mu(p)$$

so that

$$S(x,y) - D_0^{-1}(x,y) =$$

$$-\int [\gamma.p - m]^{-1}[\gamma.q - m]^{-1}exp(-ip.x + q.y)[q_\mu\gamma^a + (p_\nu - q_\nu)K(\mu,\nu,a)]\hat{\epsilon}_a^\mu(p-q)d^4pd^4q$$

In particular, for computing the quantum average perturbation to the four current density field, as seen above, we require

$$S(x,x) - D_0^{-1}(x,x)$$

$$-\int [\gamma.p - m]^{-1}[\gamma.q - m]^{-1}exp(-i(p-q).x)[q_\mu\gamma^a + (p_\nu - q_\nu)K(\mu,\nu,a)]\hat{\epsilon}_a^\mu(p-q)d^4pd^4q$$

This completes the formulation of our problem of calculating the approximate average four current field or equivalently, the density and velocity perturbations of the matter field and more generally, the space-time moments of the four current density field caused by metric perturbations around a flat space-time metric. More generally, if we wish to calculate the quantum average/space-time moments of the four current field perturbations caused by small perturbations around a given classical metric like the RW metric, we must first express the metric perturbations in terms of the tetrad perturbations in both the Einstein-Hilbert action and in the Dirac action, and path integrate w.r.t these tetrad perturbations.

13. Conclusions

We analyze various parametrizations of maximally symmetric spaces in higher-dimensional spacetime that provide natural generalizations of the four-dimensional Robertson-Walker metric corresponding to a homogeneous and isotropic expanding universe, and then look at the Maxwell, Klein-Gordon, and fluid dynamical equations in such a space-time. As regards the former two equations, we separate the space and time variables and are able to obtain separate differential equations for the spatial and temporal components. As regards the fluid dynamical and heat transfer equations, we introduce corrections to the energy-momentum tensor of matter caused by viscous and thermal effects and are able to formulate the required differential equations for temperature diffusion and convection. Future work is being directed toward formulating the Einstein field equations for such higher-dimensional maximally symmetric spaces and also analyzing the problem of density, velocity, and metric perturbation evolutions in a homogeneous and isotropic background, which is expected to provide a clue to galactic evolution in higher-dimensional space-times. To this end, in this paper, we explain how to compute the Ricci tensor components in higher-dimensional maximally symmetric space-times and also how to compute the partial differential equations satisfied by the velocity and temperature fluctuations using linearized heat and mass transfer equations in general relativity. It should be mentioned that as regards computing the contribution to the energy-momentum tensor of the matter fluid caused by viscous and thermal effects, we use the existing results in the literature based on the second law of thermodynamics, which yields the general form of the viscous and thermal contribution to the energy-momentum tensor of the matter fluid. We also present some computations on the quantum Boltzmann equation for open quantum systems and touch upon the problem involving computing the quantum average of the matter four-current (i.e., density and velocity perturbations) when the matter consists of only electrons and positrons, using the Feynman path integral formula for the Dirac field interacting with the quantum gravitational field via the spinor connection of gravity. The importance of the quantum Boltzmann equation stems from quantum cosmology, wherein we have a very large number of particles in a volume and we are interested only in the dynamics of a single, or at most a small finite number of particles, from the quantum mechanical angle.

References

- Steven Weinberg, "The quantum theory of fields, vol.1, Cambridge University Press, 2000.
- Steven Weinberg, "Gravitation and Cosmology:Principles and applications of the general theory of relativity", John Wiley and Sons, 2004.
- V.S.Varadarajan, "Supersymmetry for Mathematicians, An Introduction", Courant University lecture notes, American Mathematical Society, 2004.
- Landau and Lifshitz, "The classical theory of fields", Butterworth and Heineman, 2004.
- R.W.Newcomb, "A Clifford Algebra Representation of N-dimensional Maxwell Equations",
- R.W.Newcomb, "Psychic fields: Basics and Main Ideas".
- P.A.M. Dirac, "The principles of quantum mechanics, Oxford, 1930.
- Thomas Thiemann, "Modern canonical quantum general relativity", Cambridge University Press.
- S.Ramanan, "Global Calculus", American Mathematical Society, 2005.
- M.Green, J.Schwartz and E.Witten, "Superstring theory", Cambridge University Press, 2012.
- Riener Lenz, "Group theoretical methods in image processing", Springer, 1990.
- Raman Arora and Harish Parthasarathy, "Optimal estimation and detection in homogeneous spaces", IEEE transactions on signal processing, 58(5), 2623-2635, 2010.
- V.S.Varadarajan, "An introduction to Harmonic analysis on semisimple Lie groups", Cambridge Studies in Advanced Mathematics, 1989.
- Barry Simon, "Representations of finite and compact groups", American Mathematical Society.
- Harish Parthasarathy, "Handbook of problems in engineering mathematics and physics", I.K.International Publishing House, 2013.

- Harish Parthasarathy, "Supersymmetry and Superstring theory with engineering applications", Taylor and Francis, 2023.

Declarations

Funding: No specific funding was received for this work.
Potential competing interests: No potential competing interests to declare.