Research Article

The Upper Limit on the Minimum Mass of Relic Neutrinos, Allowing for the Equivalence of Their Gravitational Density with the Density of Dark Energy

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A cloud of relativistic material particles is considered, the gravitational interaction between which can be neglected when determining their motion. In proper frame the isotropic Schwarzschild metric defines the field of each particle. The active gravitational mass of the cloud is obtained by applying Lorentz transformations to this metric and using the superposition principle. As the speed of light approaches, the ratio of the active gravitational mass of the cloud to the total rest mass of its particles increases in proportion to the Lorentz factor to the 3rd power. This result is used to re-estimate the contribution of relic neutrinos to the total cosmological energy density as a source of the gravitational field. It provides the upper limit of a neutrino mass $m_\nu < 7.96 \cdot 10^{-5} eV$ sufficient for gravitational density of neutrinos to be the density of dark energy. This is consistent with minimum in neutrino mass hierarchy $m_1 = 0.21^{+1.70}_{-0.21} \cdot 10^{-4} eV$ determined from Koide's relation.

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I. Introduction

The weak equivalence principle of the general theory of relativity equates passive gravitational mass and inertial mass, and these masses are identified with the active gravitational mass of matter [1]. This is undoubtedly satisfied in the static case and establishes the connection between Newton's theory of gravity and general relativity. The special theory identifies inertial mass with energy. By applying Lorentz transformations to the metric of weakly curved space under this condition this principle can serve as a basis for determining the energy density as a source of gravity for dynamic systems. The active gravitational mass of a sparse cloud of material relativistic particles has been obtained [2] based on the properties of Lorentz transformations and the Schwarzschild spacetime geometry.

The standard Big Bang cosmology, based on the existing density of the Universe, accurately specifies the temperature of relic neutrinos regardless of whether they are relativistic or not [3]. Estimating the mass of different types of neutrinos is done based on the difference of squares of their masses [4]. Their masses determined from the improved Koide's relation are strongly hierarchical [5][6]. Using the obtained value of the energy density of gas as a source of gravitational field, we find the upper limit of the minimum neutrino mass at which the relic neutrinos can create a dark energy effect.

II. Weakly gravitating gas cloud

We study a weakly gravitating gas cloud consisting of identical particles with a rest mass m chaotically moving with the same absolute value of velocity v in a certain frame of reference $K' = (t', x', y', z')$. It is assumed that at time $t' = 0$ the distances δr between particles can be neglected to determine the gravity created by this cloud in the considered area. The rarefaction of the gas is given by the condition

$$\alpha_M/\delta r << v^2/c^2, \tag{1}$$

where $\alpha_M = \frac{2GM}{c^2}$ with cloud gravitational mass M and gravitational constant G.

Statistically, the cloud can be represented as a set of systems consisting of two particles A and B, which move in opposite directions. The weak gravitational field of one particle is described approximately [7] in associated coordinates $K = (t, x, y, z)$ by linearised isotropic Schwarzschild metric

$$ds^2 = c^2\left(1 - \frac{\alpha}{r}\right)dt^2 - \left(1 + \frac{\alpha}{r}\right)(dx^2 + dy^2 + dz^2) \tag{2}$$

with $r = \sqrt{x^2 + y^2 + z^2}$ and $\alpha = \frac{2Gm}{c^2}$.

III. Applying Lorentz transformations to Schwarzschild metric

Condition (1) means that the distortions of length and time caused by the presence of the Lorentz factor $\frac{1}{\sqrt{1-\tilde{\beta}^2}}$ with $\tilde{\beta} = \frac{\tilde{v}}{c}$ will be an order of magnitude greater than curvature of space-time by gravity. Therefore, the influence of gravity on the Lorentz transformations

$$t = \frac{t' + \frac{\tilde{\beta}}{c}x'}{\sqrt{1 - \tilde{\beta}^2}}, \quad x = \frac{x' + \tilde{v}t'}{\sqrt{1 - \tilde{\beta}^2}}, \quad y = y', \quad z = z' \tag{3}$$

at

$$\tilde{v} = \begin{cases} v \\ -v \end{cases} \tag{4}$$

is insignificant and they can be applied to the metric (2). Transformation of coordinates with

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$$r' = \sqrt{\left(\frac{x' + \tilde{v}t'}{\sqrt{1-\tilde{\beta}^2}}\right)^2 + y'^2 + z'^2} \tag{5}$$

yields

$$ds^2 = c^2\left(1 - \frac{1+\tilde{\beta}^2}{1-\tilde{\beta}^2}\frac{\alpha}{r'}\right)dt'^2 - \frac{4\tilde{v}}{1-\tilde{\beta}^2}\frac{\alpha}{r'}dt'dx' - \left(1 + \frac{1+\tilde{\beta}^2}{1-\tilde{\beta}^2}\frac{\alpha}{r'}\right)dx'^2 - \left(1 + \frac{\alpha}{r'}\right)(dy'^2 + dz'^2). \tag{6}$$

The applicability of Lorentz transformations to the Schwarzschild metric is confirmed by the analysis of the annual oscillations of the signal from Pioneer 10 [8].

IV. Two-body system

In associated with bodies reference frames K_A, K_B the gravity of each of them separately is described by the metric (2). Let us pass from these coordinate systems to K', using the Lorentz transformations for velocities (4).

If we represent metric coefficients in the form

$$g_{ij} = \eta_{ij} + h_{ij}, \tag{7}$$

where η_{ij} correspond to the Minkovsky metric, then with weak gravity [9] the ratio

$$h_{ij} \approx \sum_n h_{ij}^n \tag{8}$$

is performed for the total field created by n subsystems with metric coefficients

$$g_{ij}^n = \eta_{ij} + h_{ij}^n. \tag{9}$$

Summing the fields obtained after substitutions of velocities (4) into the metric (6), we find approximate path interval in the vicinity of $t' = 0$ in a two-body system

$$ds^2 = c^2\left(1 - \frac{1+\beta^2}{1-\beta^2}\frac{\alpha_1}{r'}\right)dt'^2 - \left(1 + \frac{1+\beta^2}{1-\beta^2}\frac{\alpha_1}{r'}\right)dx'^2 - \left(1 + \frac{\alpha_1}{r'}\right)(dy'^2 + dz'^2) \tag{10}$$

at $\alpha_1 = 2\alpha$ and $\beta = \frac{v}{c}$.

The equations of geodesics

$$\frac{du^i}{ds} + \Gamma^i_{kl}u^k u^l = 0, \tag{11}$$

with Christoffel symbols

$$\Gamma^l_{ij} = \frac{1}{2}g^{lk}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) \tag{12}$$

are used to search for the acceleration of a test material particle in described by metric (10) gravitational field. For spatial coordinates of particle at rest they turn out to be

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$$\frac{du^k}{ds} = \frac{1}{2} g^{kk} \frac{\partial g_{11}}{\partial x^k} \left(u^1\right)^2 \tag{13}$$

with indices $k = 2, 3, 4$. These equations yield coordinate accelerations

$$\ddot{x}' = -\frac{1}{2} \frac{c^2 x'}{\sqrt{1-\beta^2}} \frac{1+\beta^2}{1-\beta^2} \frac{\alpha_1}{(r')^3}, \tag{14}$$

$$\ddot{y}' = -\frac{1}{2} c^2 y' \frac{1+\beta^2}{1-\beta^2} \frac{\alpha_1}{(r')^3}, \tag{15}$$

$$\ddot{z}' = -\frac{1}{2} c^2 z' \frac{1+\beta^2}{1-\beta^2} \frac{\alpha_1}{(r')^3} \tag{16}$$

disregarding small quantities of a larger order.

V. Active gravity mass of the gas cloud

The absolute value of acceleration

$$a' = \sqrt{\left(\ddot{x}'\right)^2 + \left(\ddot{y}'\right)^2 + \left(\ddot{z}'\right)^2} \tag{17}$$

imparted to the test particle by the two-body system will be

$$a' = \frac{1+\beta^2}{2(1-\beta^2)} \frac{c^2 \alpha_1}{(r')^3} \sqrt{\frac{\left(x'\right)^2}{1-\beta^2} + \left(y'\right)^2 + \left(z'\right)^2}, \tag{18}$$

provided that the size of the system is insignificant compared to the distance to the test particle. In spherical coordinate frame $(t', r', \varphi, \theta)$ defined by transformations

$$x' = r' \cos\varphi, y' = r' \sin\varphi \cos\theta, z' = r' \sin\varphi \sin\theta \tag{19}$$

we obtain

$$a' = \frac{1+\beta^2}{2\left(1-\beta^2\right)^{3/2}} \frac{c^2 \alpha_1}{\left(r'\right)^2} \sqrt{1-\beta^2 \sin^2\varphi}. \tag{20}$$

Acceleration a' is caused by the gravitational mass

$$m_2 = 2m \frac{1+\beta^2}{\left(1-\beta^2\right)^{3/2}} \sqrt{1-\beta^2 \sin^2\varphi}. \tag{21}$$

For each pair of particles, the direction of the axes of the coordinate system is chosen so that the axis X' is parallel to the line of their motion. Assuming a uniform distribution of the directions of their motion over the corners, the average gravitational mass of a pair of particles in the gas cloud will be

$$\overline{m}_2 = \frac{2}{\pi} \int_0^{\pi/2} m_2 d\varphi. \tag{22}$$

It determines the gravitational mass of a cloud consisting of n particles

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$$M = nm\frac{2}{\pi}\frac{1+\beta^2}{\left(1-\beta^2\right)^{3/2}}E(\beta), \tag{23}$$

where $E(\beta)$ is complete elliptic integral of the 2nd kind. With $\beta \to 1$ the average gravitational mass of a particle in the cloud will tend to

$$\tilde{m} = \frac{4m}{\pi\left(1-\beta^2\right)^{3/2}}. \tag{24}$$

VI. The relationship between energy densities

In special relativity, with Lorentz factor $\gamma = \left(1-\beta^2\right)^{-1/2}$ the relativistic mass has the form $m_r = m\gamma$. When determining the mass energy density using the principle of equivalence of relativistic and gravitational masses [10], the reduction in the volume of a moving body is taken into account and the density is additionally multiplied by the Lorentz factor:

$$\rho = \rho_0\gamma^2, \tag{25}$$

where ρ_0 is the density in proper frame.

And in the case under consideration, when moving from the mass of a cloud of relativistic material particles to the density of space filled with similar particles, ρ_0 is additionally multiplied by the Lorentz factor. In the relativistic limit, in view of Eq. (24) this yields

$$\tilde{\rho} = \frac{4}{\pi}\rho_0\gamma^4. \tag{26}$$

The ratio of the energy density as a source of the gravitational field to the energy density in the special theory of relativity will be

$$\frac{\tilde{\rho}}{\rho} = \frac{\tilde{m}}{m_r} = \frac{4}{\pi}\gamma^2, \tag{27}$$

coinciding with the ratio between the corresponding masses.

VII. Neutrino mass and density parameter

The upper limit on the sum of the masses of the neutrino's eigenstates in the Big Bang model in energy units is estimated to be 0.58 eV using Wilkinson Microwave Anisotropy Probe (WMAP) data [11] and 0.12 eV from Planck telescope data [12]. The attempt to directly determine the absolute mass of the electron neutrino in the Karlsruhe Tritium Neutrino (KATRIN) laboratory experiment using nuclear beta decay provided an estimate of $m_\nu < 0.8$ eV [13]. The combination of the gravitational influence analysis on galaxy clusters at redshifts z=0.5-7 using Legacy Survey of Space and Time (LSST) data and cosmic microwave background (CMB) lensing should be able to achieve constraints on the neutrino mass sum of 25 meV without optical depth information [14]. To prevent the suppression of matter clustering, it is necessary that neutrino speed remain close to the speed of light throughout cosmic history [15]. According to the standard Big Bang model, temperature of relic neutrinos is T=1.95K [3]. Combination of WMAP data with the distance measurements from Baryon Acoustic Oscillations (BAO) in the distribution of galaxies and Hubble constant measurements [11] yields the relic neutrino density parameter constraint of $\Omega_\nu < 0.0124$. This estimate is made on the assumption that the inertial and gravitational masses are equal.

Measurements by the Planck telescope [11] have given the dark energy density parameter $\Omega_\Lambda = 0.6847 \pm 0.0073$. In order for dark energy to consist of relativistic relic neutrinos with an energy density as a source of gravitational field $\widetilde{\rho}_\nu$, the relation

$$\frac{\widetilde{\rho}_\nu}{\rho_\nu} = \frac{\Omega_\Lambda + \Omega_\nu}{\Omega_\nu} \tag{28}$$

must hold with the left side given by equation (27). From the restriction on Ω_ν, it follows that $\widetilde{\rho}_\nu/\rho_\nu > 56.22$ and from this equation we obtain equivalent inequality $\beta > 0.9888$ for the neutrino velocity.

The average energy of a particle in an ultrarelativistic fermion gas [3] is given by

$$\langle E \rangle = 3.15kT, \tag{29}$$

where k is the Boltzmann constant. For relic neutrinos, it will be $\langle E_\nu \rangle = 5.29 \cdot 10^{-4}$ eV. Following from (27) expression

$$\gamma = \sqrt{\frac{\pi\widetilde{\rho}_\nu}{4\rho_\nu}} \tag{30}$$

gives the upper limit on their minimal mass in energy units $m_\nu = \langle E_\nu \rangle/\gamma$ of $7.96 \cdot 10^{-5}$ eV. The experimentally determined value of the difference of the squares of the rest energies of the electron and muon neutrinos is $|\Delta Q^2_{21}| = 7.4 \cdot 10^{-5}(eV)^2$, and for the muon and tau states is $|\Delta Q^2_{32}| = 2.51 \cdot 10^{-3}(eV)^2$ [4]. The minimal mass in neutrino mass hierarchy determined from Koide's relation has order of magnitude 10^{-5} eV [5]. The absolute neutrino masses are refined for the normal mass hierarchy and the minimum of them is $m_1 = 0.21^{+1.70}_{-0.21} \cdot 10^{-4}$ eV [6]. From these values, it follows that one type of neutrino could correspond to the obtained limit for minimal mass and relativistic neutrinos constitute a significant part of dark energy, more than what is suggested by the Lambda cold dark matter (Λ CDM) model. This is ensured by their contribution to the overall density of matter, as a source of the gravitational field.

Results from the measurement of BAO in galaxy, quasar and Lyman-α forest tracers from the first year of observations from the Dark Energy Spectroscopic Instrument (DESI) and CMB data analysis give a statistical preference for the dark energy model with a time-varying equation of state (EoS) compared to the Λ CDM model [16]. Limits on the neutrino contribution to dark energy are made by assuming that at low redshifts of relevance to DESI, massive neutrinos are non-relativistic and therefore contribute to the total non-relativistic matter density. However, if neutrinos remain relativistic, they will not accumulate in galaxy cluster regions and affect the background geometry and therefore these restrictions will not be valid.

VIII. Conclusions

The active gravitational mass of a rarefied cloud of material relativistic particles was derived from the properties of Lorentz transformations and the geometry of Schwarzschild spacetime. This mass increases faster with their velocity than the total relativistic mass of the cloud particles.

Based on the standard model of the Big Bang, an upper limit of the minimal neutrino mass has been found, sufficient for the density of dark energy to be equal to their gravitational density. It is less than the upper limit of electron neutrino mass directly measured by the KATRIN experiment. It is also consistent with the estimate made on the basis of the BAO and CMB data analysis and is within the limits found for the minimum neutrino mass from Koide's relation. This leaves open the possibility that the energy density of neutrinos as a source of the gravitational field is greater than the standard cosmological model suggests and that it makes up some fraction of dark energy.

However, this study is based on an estimate of the contribution of relic neutrinos to the total density, based on the postulate of equality of inertial and active gravitational masses. Therefore, it can be considered an example calculation, and the result obtained requires further refinement.

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