ON THE FOUNDATION OF QUANTUM DECISION THEORY

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Abstract:

Quantum decision theory is introduced here, and new basis for this theory is proposed. It is first based upon the author's general arguments for the Hilbert space formalism in quantum theory, next on arguments for the Born rule, that is the basis for calculating quantum probabilities. A basic notion behind the quantum theory foundation is that of theoretical variables, that are divided into accessible and inaccessible ones. This is here specialized to decision variables. It is assumed that each accessible variable can be seen as a function of a specific inaccessible variable. Another assumption is that there exist two maximal accessible decision processes in the given situation. Two basic assumptions behind the Born rule are 1) the likelihood principle, 2) the actor in question has motivations that can be modeled by a hypothetical perfectly rational higher being. The theory is illustrated by a medical example.

Key words: Accessible decision variabel; action; actor; Born rule; decision; Hilbert space formalism; illustrating example; quantum probabilities.

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Introduction

The notion of 'decision' is fundamental to all sciences. The notion is analysed in psychological articles and books, but it also lies in the foundation of much economic theory and statistical theory, where optimal decisions often are coupled to expected utility. Of course, in addition the notion has its daily life implications.

Decisions can be made on the basis of knowledge, on the basis of beliefs, or both. They are always made in a concrete context. Single persons can make decisions, and joint decisions can be made by a group of communicating persons.

Political decisions are important, and may have large consequences. In an autocratic country political decisions are made by sovereign leaders, having gained their positions through power exchanges. In democratic countries, at least in principle people may control their leaders' decisions.

In this article we will concentrate on decisions made by single persons. From a psychological point of view, such decisions have been thoroughly reviewed by Newell et al. (2015). The book emphasizes the relationship between learning and decisions, arguing that the best way to understand how and why decisions are made is in the context of learning and knowledge acquisition which precedes them, and the feedback which follows. Some of the discussions in op. cit. are also relevant for the developments in the present article.

Knowledge acquisition is fundamental to the process of making judgements, which precedes any mature decision. According to Newell et al. (2015), knowledge acquisition can be divided into 3 steps: 1) Discovering information; 2) Receiving and searching through information; 3) Combining information.The first step is particularly important, and is modelled through Brunswik's lens model: Variables in the real world go through 'lenses of cues' before they are perceived by our minds. In a simplified way we can say that we discover information by asking questions to nature, and receive answers. The different questions may be coupled.

A person always has his history, and through his history he learns how to handle decisions. Hence learning is important. In standard economic theory and statistical theory, the problem of optimizing behaviour is reduced to optimizing expected reward or utility. We will claim that this does not quite correspond to the way people act during a decision process. This statement can be supported by a number of empirical investigations; see for instance the review articles mentioned below in connection to quantum decision theory.

Admittedly, the decision foundation based upon optimizing expected utility has had and has still a strong theoretical support. There are several axiom systems that can motivate such a basis, and the mathematics behind such axioms can be traced back to Bernoulli and Pascal. Newell et al. (2015) discuss one such axiom, the sure thing principle: If a person prefers option A to option B under the condition X, and he also prefers A to B under the opposite condition not-X, then he always prefers A to B.

One difficulty with the sure thing principle, is that one can find examples where rational people do not seem to follow this principle. Two such examples are given by Allais (1953) and Ellsberg (1961).

A rather primitive form of learning is reinforcement learning, learning governed by rewards or punishments. In Friston et al. (2009) the notion of active inference was introduced as an alternative to reinforcement learning in connection to decision making. This is important, since the way people learn in connection to decisions obviously is more active than what can be described as reinforcement learning. In op cit. a theory of active inference is formalized in terms of a free-energy formulation of perception, quantizised in terms of entropy or average surprise associated with a probability distribution on an agent's state space and his environment.

In Kahneman et al. (2021) decisions are discussed in terms of their noise and skewness. Both are problematic sides of human decisions. Skewness denotes systematic errors made in relation to some defined goal. Noise denotes variability: Imagine two medical doctors in the same city giving different diagnoses to identical patients. Imagine two judges assigning different punishments for the same crimes. Or imagine the same medical doctor or the same judge getting at different results depending upon whether it is before or after lunch, or Monday instead of Friday. According to op. cit. our decisions, also decisions made by socalled experts, have larger random variation than we usually think of. Means for reducing noise are discussed in the book.

Theories of decisions can take many points of departure. In this article we will describe a new foundation of a comparatively recent theory: Quantum decision theory. This theory is completely based upon quantum mechanics, whose foundation is discussed in the book Helland (2021) and in some newer articles Helland (2022a,b, 2023a,b).

Pionering articles on quantum decision theory are Aerts (2009) and Yukalov and Sornette (2014); see also the book by Busemeyer and Bruza (2012). More recent review articles are Ashtiani and Azgomi (2015) and Pothos an Busemeyer (2022). Our approach will be independent of this literature; a completely new entrance to the theory will be aimed at.

The plan of this article is as follows: I first review briefly a new approach to the foundation of quantum theory, and then I use this to derive and motivate a new foundation of quantum decision theory, taking as a basis a simple model of a person's decisions. First, the Hilbert space formalism of decision variables is derived. Then the Born formula, the basis for computing quantum probabilities, is argued for under weak assumptions, and this is specialized to a decision setting. An example of applications are then given, and used to illustrate some of the peculiarities of quantum probabilities. Then the interpretation is briefly discussed before I give some concluding remarks.

A new foundation of quantum decision theory

On the foundation of quantum theory in general

A completely new approach towards quantum foundations is proposed in Helland (2021, 2022a,b, 2023a,b,c,d). The basis of this approach is an observer who is in some physical situation. In this situation there are theoretical variables, and certain of these variables, say $\theta, \lambda, \eta, ...$ are related to the observer C. Some of the variables are *accessible* to him, meaning roughly that it is, in some future, in principle possible to obtain as accurate values as he wishes on the relevant variable. Others are *inaccessible.* Examples of the latter are the vector (position, momentum) of a particle at some time, or the full spin vector of a spin particle, an imagined vector whose discretized projection in the direction a is the spin component in that direction. The terms 'accessible' and 'inaccessible' are primitive notions of the theory.

The main assumption of my theory is then as follows: *Related to an observer C there is an inaccessible variable ϕ such that all accessible variables can be seen as functions of ϕ.* In the two examples above one can take $\phi =$ (position, momentum) and $\phi =$ full spin vector. In the last example, in the spin 1/2 case, one can model the discrete spin component θ^a in direction a as $\text{sign}(\cos(a, \phi))$. Giving ϕ a reasonable distribution here, results in a correct distribution of each θ^a. Here these variables are theoretical variables coupled to some physical situation. Following Zwirn (2016, 2020), every description of reality must be seen from the point of view of some observer. Hence we can assume that the variables also exist relative to C.

But observers may communicate. The mathematical model developed in the articles above is equally valid relative to a group of people that can communicate about the physics and about the various theoretical variables. This gives a new version of the theory, a version where all theoretical variables are defined jointly for such a group. The only difference here is that, for the variables to function during the communication, they must always be possible to define them in words.

In the above two examples there are also maximal accessible variables: In the first example either position or momentum, in the second example the spin component θ^a in some direction a. In general, an accessible variable θ is maximal if there is no other accessible variable λ such that $\theta = f(\lambda)$ for some non-invertible function f.

Two different acccessible variables θ and η are said to be related if there is a transformation k in ϕ-space and a function f such that $\theta = f(\phi)$ and $\eta = f(k\phi)$. In Helland (2023b) it is shown that if both θ and η are maximal and take r values, then ϕ can be constructed in a natural way, there is an accessible variable λ that is in one-to-one correspondence with θ, all this such that λ and η are related according to this definition.

In this sense, one can always think of some relation between θ and η when they are different maximal accessible variables. Such variables are called - following Niels Bohr - for complementary.

In many cases, in particular in the two examples above, λ and θ are identical, so that θ and η are directly related. Two spin components θ^a and θ^b are related, and position and momentum are related theoretical variables. In the first case, ϕ-space can be taken as the plane spanned by the directions a and b, and k can be taken as a 180^o rotation around the midline between a and b. In the last case, k is constructed by a Fourier transform.

In the present paper, we will concentrate on accessible variables that take a finite number of values, say r. Then the following first theorem is proved in the above papers:

Given a situation with two different maximal accessible variables θ and η, there corresponds to every accessible variable a self-adjoint operator in some complex Hilbert space $\mathcal{H}$. With 'different' I here mean really different, not two maximal variables that are one-to-one functions of each other. And a 'self-adjoint operator' is just a complex $r \times r$ matrix with real eigenvalues. In particular then, there is such an operator A^{θ} corresponding to θ and an operator A^{η} corresponding to η.

This theorem is the first step in a new proposed foundation of quantum theory.

The second step is to show the following: If k is the transformation connecting two related variables θ and η, then there is a unitary matrix $W(k)$ such that $A^{\eta} = W(k)^{-1}A^{\theta}W(k)$. (A matrix W is unitary if $W^{-1} = W^{\dagger}$, where the last matrix is obtained from W by transposing it and taking the complex conjugate of all its elements.)

Given these results, a rich theory follows. The set of eigenvalues of the operator A^{θ} is identical to the set of possible values of θ. The variable θ is maximal if and only if all eigenvalues are simple. In general, the eigenspaces of A^{θ} are in one-to-one correspondence with questions 'What is θ'/ 'What will θ be if we measure it?' together with sharp answers $\theta = u$.

If θ is maximal as an accessible variable, the eigenvectors of A^{θ} have a similar interpretation. In my opinion, such eigenvectors, where θ is some meaningful variable, should be taken as the only possible state vectors. These have straightforward interpretations, and from this version of the theory also a number of so-called 'quantum paradoxes' can be illuminated, see Helland (2022b,d).

This not only points at a new foundation of quantum theory, and it also suggests a general epistemic interpretation of the theory: Quantum theory is not directly a theory about the world, but a theory about an actor's knowledge of the world. Versions of such an interpretation already exist, and it is one of the very many suggested interpretations of quantum mechanics.

A simple model of a person's decisions

Consider a person C in some decision situation. Say that he has the choice between a finite set of actions $a_1, ..., a_r$. Relative to this situation we can define a finite-valued decision variable θ, taking the different values $1, ..., r$, such that $\theta = i$ corresponds to the action a_i $(i = 1, ..., r)$. If C really is able to make a decision here and carry out the axtions, we say that θ is an accessible variable. In analogy with the situation in the previous subsection, the variable θ is in correspondence with a person C, in fact, here θ belongs to the mind of C.

We all go through life making decision after decision. Some of these are simple, but some can be really demanding. For the different decisions that C is about to make at some time t, there correspond decision variables $\theta, \eta, \lambda, ...$. Some of these may be accessible to C, but some may be inaccessible: In the given situation, C is simply not able to make up his mind. Consider a fixed time t and let C be in some concrete situation at time t. Assume that he at time t is faced with two different maximal decision problems, both corresponding to r different actions.

This must be made precise. A decision problem is said to be maximal if C is just able to make his mind with respect to this decision; if the problem is made slightly more complicated, he is not able to take a decision. Let two different maximal decision variables be θ and η, where $\theta = i$ corresponds to the action a_i $(i = 1, ..., r)$, and $\eta = j$ corresponds to the action b_j $(j = 1, ..., r)$. Then, by the theory of the previous subsection, we can model the situation by using quantum theory.

Note that this simple model do not cover all situations. Sometimes we have a choice between an infinite number of possibilities, and sometimes the outer context changes during the decision process. Nevertheless, the simple model is a good starting point.

The Hilbert space formalism for decisions

It is well known that our minds may be limited, for instance when faced with difficult decisions. We will first mention a side result in this direction from the present development.

In Helland (2022c), Theorem 2 says essentially: Imagine a person C which in some context has two related maximal accessible variables θ and η in his mind. Impose a specific symmetry assumption. Then C cannot simultaneous have in mind any other maximal accessible varable which is related to θ, but not related to η. It was claimed in Helland (2022c) that the violation of a famous inequality by practical Bell experiments, can be understood on the basis of this theorem. See also Helland (2023c).

Note that this result has the qualification 'at the same time', and indicate a specific restriction to two maximal variables. But the human mind is very flexible. Taking time into account, we can think of very many variables, even ones that are essentially different.

For the present article, however, the direct results from Helland (2022b, 2023b) are more important. Consider again a decision situation, and assume the simple model of the previous Subsection. In particular let C at the same time be confronted with at least two different maximal related decision processes. Then the following hold:

- Each decision variable θ is associated with a self-adjoint operator A^{θ}, whose eigenvalues are that possible values of θ.

- The decision process is maximal if and only if each eigenvalue of the corresponding operator is single.

- In the maximal case, the eigenvectors of the operator can be given interpretation: They are coupled to one particular decision process and a specific choice in this decision process: In concrete terms, the eigenvectors v are in one-to-one correspondence with 1) some maximal accessible decision variable θ, and 2) a specific value u of θ. In other words, the possible eigenvectors are in one-to-one correspondence with 1) the question 'Which decision process?' and 2) 'Which action did this decision process lead to?'.

- In the general case, the eigenspaces of the operator have a similar interpretation.

This can be taken as a starting point of quantum decision theory, but to develop this theory further, we need to be able to calculate probabilities of the various decisions.

The Born rule; general arguments

Born's formula is the basis for all probability calculations in quantum mechanics. In textbooks it is usually stated as a separate axiom, but it has also been argued for by using various sets of assumptions; see Campanella et al. (2020) for some references. In fact, the first argument for the Born formula, assuming that there is an affine mapping from set of density functions to the corresponding probability functions, is due to von Neumann (1927). Here we will try to use assumptions which are as weak as possible, and assumptions that can be related to notions both from statistical theory and quantum theory. We start with assuming the likelihood principle: The experimental evidence from any experiment must be based upon the likelihood l, which is probability density or the point probability of observations, seen as a function of the full parameter.

The likelihood concept is then generalized to a quantum setting; After an experiment is done, and given some context τ, all evidence on the maximal parameter θ^b is contained in the likelihood $p(z^b|\tau, \theta^b)$, where z^b is the data relevant for inference on θ^b, also assumed discrete. This is summarized in the *likelihood effect*:

$$F^b(\mathbf{u}^b; z^b, \tau) = \sum_j p(z^b|\tau, \theta^b = u_j^b)|b; j\rangle\langle b; j|, \tag{1}$$

where the pure state $|b;j\rangle$ corresponds to the event $\theta^b = u_j^b$, and where $\mathbf{u}^b = (u_1^b, ..., u_r^b)$ is a vector of actual values of the parameter θ^b.

The interpretation of the likelihood effect $F^b(z^b, \tau)$ can be formulated as follows: (1) We have posed some inference question on the accessible conceptual variable θ^b. (2) We have specified the relevant likelihood for the data. The question itself and the likelihood for all possible answers of the question, formulated in terms of state vectors, can be recovered from the likelihood effect.

The likelihood effect is closely connected to the concept of an operator-valued measure; see a discussion in Helland (2021). Since the focused question assumes discrete data, each likelihood is in the range $0 \leq p \leq 1$. In the quantum mechanical literature, an effect is any operator with eigenvalues in the range $[0, 1]$.

We will base the discussion upon the following assumption and theorem from Helland (2021), where a further discussion is given.

Assumption 1 *Consider in the context τ an epistemic setting where the the likelihood principle from statistics is satisfied, and the whole situation is observed by an experimentalist C whose decisions can be modelled to be influenced by a superior being D. Assume that D's probabilities for the situation are q, and that D can be seen to be rational in agreement with the Dutch Book Principle.*

The Dutch Book Principle says as follows: No choice of payoffs in a series of bets shall lead to a sure loss for the bettor.

A situation where Assumption 1 holds will be called a *rational epistemic setting.* It will be assumed to be implied by essential situations of quantum mechanics. Below we will discuss whether or not it also can be coupled to certain macroscopic situations, in particular decision situations.

In Helland (2021), a generalized likelihood principle is proved from the ordinary likelihood principle: Given some experiement, or more generally, some context τ connected to an experiment, any experimental evidence will under weak assumptions be a function of the likelihood effect F. In particular, the probability q is a function of F: $q(F|\tau)$.

Theorem 1 *Assume a rational epistemic setting, and assume a fixed context τ. Let F_1 and F_2 be two likelihood effects in this setting, and assume that $F_1 + F_2$ also is an effect. Then the experimental evidences, taken as the epistemic probabilities related to the data of the performed experiments, satisfy*

$$q(F_1 + F_2|\tau) = q(F_1|\tau) + q(F_2|\tau).$$

Corollary 1 *Assume a rational epistemic setting in the context τ. Let F_1, F_2, ... be likelihood effects in this setting, and assume that $F_1 + F_2 + \ldots$ also is an effect. Then*

$$q(F_1 + F_2 + \ldots|\tau) = q(F_1|\tau) + q(F_2|\tau) + \ldots.$$

The further derivations rely on a very elegant recent theorem by Busch (2003): Let in general $\mathcal{H}$ be any separable Hilbert space. Recall that an effect F is any operator on the Hilbert space with eigenvalues in the range $[0, 1]$. A generalized probability measure μ is a function on the effects with the properties

(1) $0 \leq \mu(F) \leq 1$ for all F,

(2) $\mu(I) = 1$,

(3) $\mu(F_1 + F_2 + \ldots) = \mu(F_1) + \mu(F_2) + \ldots$ whenever $F_1 + F_2 + \ldots \leq I$.

Theorem 2 (Busch, 2003) *Any generalized probability measure μ is of the form $\mu(F) = \text{trace}(\rho F)$ for some density operator ρ.*

It is now easy to see that $q(F|\tau)$ on the likelihood effects of the previous Section is a generalized probability measure if Assumption 1 holds: (1) follows since q is a probability; (2) since $F = I$ implies that the likelihood is 1 for all values of the e-variable; finally (3) is a consequence of the corollary of Theorem 1. Hence there is a density operator $\rho = \rho(\tau)$ such that $p(z|\tau) = \text{trace}(\rho(\tau)F)$ for all ideal likelihood effects $F = F(z)$. This is a result which is valid for all experiments, and can be seen as a first general version of Born's formula.

The problem of defining a generalized probability on the set of effects is also discussed in Busch et al. (2016).

Define now a *perfect experiment* as one where the measurement uncertainty can be disregarded. The quantum mechanical literature operates very much with perfect experiments which result in well-defined states $|j\rangle$. From the point of view of statistics, if, say the 99% confidence or credibility region of θ^b is the single point u_j^b, we can infer approximately that a perfect experiment has given the result $\theta^b = u_j^b$.

In our epistemic setting then: We have asked the question: 'What is the value of the accessible e-variable θ^b?', and are interested in finding the probability of the answer $\theta^b = u_j^b$ though a perfect experiment. If u_j^b is a non-degenerate eigenvalue of the operator corresponding to θ^b, this is the probability of a well-defined state $|b;j\rangle$. Assume now that this probability is sought in a setting defined as follows: We have previous knowledge of the answer $\theta^a = u_k^a$ of another maximal question: 'What is the value of θ^a?' That is, we know the state $|a;k\rangle$. (u_k^a is non-degenerate.)

These two experiments, the one leading to $|a;k\rangle$ and the one leading to $|b;j\rangle$, are assumed to be performed in equivalent contexts τ.

Theorem 3 [Born's formula, simple version] *Assume a rational epistemic setting.*

In the above situation we have:

$$P(\theta^b = u_j^b | \theta^a = u_k^a) = |\langle a; k | b; j \rangle|^2. \tag{2}$$

An advantage of using the version of Gleason's theorem due to Busch (Theorem 2) in the derivation of Born's formula, is that this version in valid also in dimension 2. Other derivations using the same point of departure, are Caves et al. (2004) and Wright and Weigert (2019). In Wright and Weigert (2021) the class of general probabilistic theories which also admit Gleason-type theorems is identified. For instance, Auffeves and Granger (2019) derive the Born formula from other postulates.

A survey of quantum probabilities

Here are three easy consequences of Born's formula:

1) If the context of the system is given by the state $|a; k\rangle$, and A^b is the operator corresponding to the e-variable θ^b, then the expected value of a perfect measurement of θ^b is $\langle a; k | A^b | a; k \rangle$.
2) If the context is given by a density operator ρ, and A is the operator corresponding to the e-variable θ, then the expected value of a perfect measurement of θ is $\text{trace}(\rho A)$.
3) In the same situation the expected value of a perfect measurement of $f(\theta)$ is $\text{trace}(\rho f(A))$.

These results give an extended interpretation of the operator $A = A^\theta$: There is a simple formula for all expectations in terms of the operator. On the other hand, the set of such expectations determine the state of the system. Also on the other hand: If A is specialized to the operator of an indicator function, we get back Born's formula, so the consequences are equivalent to this formula.

A consequence of 3) above is that $\theta = \theta^b$ does not need to be maximal in order that a Born formula should be valid. The version 2) of Born's formula is also valid when the basic variable θ^a behind the densitity operator ρ is not maximal, under a certain assumption: Assume that $\theta^a = f(\lambda^a)$, where λ^a is maximal and the conditional probability distribution of λ^a, given θ^a is uniform.

Measurements of conceptual variables is discussed in Helland (2021), but here we will look at the simpler case of a perfect measurement. Assume that we know the state $|\psi\rangle$ of a system, and that we want to measure a new e-variable θ^b. This can be discussed by means of the projection operators $\Pi_j^b = |b;j\rangle\langle b;j|$. First observe that by a simple calculation from Born's formula

$$P(\theta^b = u_j^b|\psi) = \|\Pi_j^b|\psi\rangle\|^2. \tag{3}$$

This formula is also valid when Π_j^b is a multidimensional projection.

It is interesting that Shrapnel et al. (2017) recently simultaneously derived *both* the Born rule and the well-known collapse rule from a knowledge-based perspective. The collapse rule is further discussed in Helland (2021), but in this article we will just assume this derivation as given. Then, after a perfect measurement $\theta^b = u_j^b$ has been obtained, the state changes to

$$|b;j\rangle = \frac{\Pi_j^b|\psi\rangle}{\|\Pi_j^b|\psi\rangle\|}.$$

Successive measurements are often of interest. We find

$$P(\theta^b = u_j^b \text{ and then } \theta^c = u_i^c|\psi) = P(\theta^c = u_i^c|\theta^b = u_j^b)P(\theta^b = u_j^b|\psi)$$

$$= \|\Pi_i^c \frac{\Pi_j^b|\psi\rangle}{\|\Pi_j^b|\psi\rangle\|}\|^2 \|\Pi_j^b|\psi\rangle\|^2 = \|\Pi_i^c\Pi_j^b|\psi\rangle\|^2.$$

In the case with multiple eigenvalues, the formulae above are still valid, but the projectors above must be replaced by projectors upon eigenspaces.

Born's rule in a decision setting

Go back to the person C who is facing at least two related maximal decisions, with corresponding decision variables θ and η. Each of these decisions may be composed of several partial decisions; then the decision variables may be thought of as vectors. However, we will make the simplifying assumption that each partial decision has a finite number of outcomes; then the total decision also has a finite number of outcomes, and without loss of generality, we can order these outcomes, say η, as $1, 2, ..., r$.

Acording to Kahnemam (2011), some of our decisions are slow, and sometimes, in fact in very many cases, we have to make fast decisions, relying on earlier experience, perhaps in a subconscious way, relying on some abstract ideals that we have in our minds. This is consistent with the model assumptions that we have made above in connection to Born's rule. And the fact that such fast decisions may be modelled by quantum probabilities, is consistent with a lot of empirical findings, see Pothos and Busemeyer (2022).

Focus again on the person C and his fast decisions. Depending upon the situation that he is in, he may just have done several irrelevant decisions, collected in the variable η, but at the same moment, he is trying to focus on the problem that he really is interested in, and the corresponding (maximal) variable is called θ. There is not much lack of generality in assuming that these two decision processes have the same number of outcomes r; we can only add some abstract irrelevant outcomes to one of the decision variables.

According to our interpretation of quantum theory and of decision making, each of the two decision processes has an operator connected to it, and the eigenspaces of such an operator correspond to a concrete outcome of the decision process. The probabilities are calculated as in (3). The application of these theoretical considerations is best studied by an example.

Quantum decision theory in practice; an example

Consider a situation with one doctor and one patient. The doctor has the choice between two mutually excluding medicines A and B.

The doctor starts by asking a number of question, and he obtains answers. Result: A state $|\psi\rangle$ for the patient. This is the starting point of our model of what is going on in the mind of the doctor during the decision process. The state is modelled by some vector in an underlying Hilbert space, connected to the mind of the doctor.

The doctor is interested in several epistemic probabilities:

$P(A \text{ helps})$, $P(B \text{ helps})$,

$P(A \text{ helps and } B \text{ helps given that he knows that } A \text{ helps})$.

The simplest case is where these two medicines work completely independently:

$$P(A \text{ helps and also } B \text{ helps}) = P(B \text{ helps and also } A \text{ helps})$$

$$= P(A \text{ helps})P(B \text{ helps}).$$

This can also in principle be modelled in quantum language by using the tensor product $\mathcal{H}^A \otimes \mathcal{H}^B$. But such a model will in this situation be unnecessarily complicated.

However, from our point of view, the most interesting case is where there is some coupling between the two medicines. The doctor must make fast decisions. In his mind he may have a lot of other decisions that he has made or is going to make. As a model, these may be collected in a decision variable η. But now he focuses on the actual patient. The decisions that he is going to make on this patient, are modelled, and collected in the decision variable θ. In the doctor's mind at the moment in question, is also all the information that he posesses about the medicines A and B. As a doctor, he has ideals learned through many years of education and experience; these ideals can be modelled in terms of some abstract perfectly rational being D.

We will propose a quantum model with Hilbert space $\mathcal{H}$. The projector upon the subspace indicating that A helps, is denoted by Π_A.

From Born's rule we get:

$$P(A \text{ helps}) = \|\Pi_A|\psi\rangle\|^2 = \langle\psi|\Pi_A|\psi\rangle.$$

The new state after the doctor knows that A helps is now:

$$|\psi_A\rangle = \frac{\Pi_A|\psi\rangle}{\|\Pi_A|\psi\rangle\|}.$$

To continue the model, consider two orthogonal spaces in $\mathcal{H}$: V_B: Indicating that B helps. The projector upon this space is Π_B. And V_B^{perp}: Indicating that B does not help. The projector upon this space is $\Gamma_B = I - \Pi_B$.

This gives

$$P(A \text{ helps and also } B \text{ helps}) =$$

$$P(A \text{ helps})P(B \text{ helps}|A \text{ helps}) =$$

$$\|\Pi_A\psi\|^2\|\Pi_B\psi_A\|^2 = \|\Pi_B\Pi_A|\psi\rangle\|^2.$$

Some consequences

The marginal probability that B helps is $P(B \text{ helps}) = \|\Pi_B|\psi\rangle\|^2$. Then by a geometric argument it can be seen that it exists situation where

$$P(A \text{ helps and also } B \text{ helps}) > P(B \text{ helps}), \text{ i.e.,}$$

$$\|\Pi_B \Pi_A |\psi\rangle\| > \|\Pi_B |\psi\rangle\|.$$

Also, the law of total probability does not hold. There is additivity in the probability amplitudes:

$$\Pi_A |\psi\rangle = \Pi_B \Pi_A |\psi\rangle + \Gamma_B \Pi_A |\psi\rangle = a + b,$$

so that

$$P(A \text{ helps}) = \|\Pi_A |\psi\rangle\|^2 = |a|^2 + |b|^2 + a^* b + a b^*.$$

But

$$P(A \text{ helps and also } B \text{ helps}) + P(A \text{ helps but } B \text{ does not help})$$

$$= |a|^2 + |b|^2.$$

As a special case of the violation of the law of total probability: The sure thing principle does not hold in quantum decision theory.

Finally, note that one my have

$$P(A \text{ helps and also } B \text{ helps}) \neq P(B \text{ helps and also } A \text{ helps}).$$

In general, events do not commute in relation to calculation of quantum probabilities.

Interpretations

All these probabilities may be thought of as describing what is going on in the mind of the doctor before he makes a decision. Similar qualitative conducts of probabilities in the minds of people have been observed empirically in many investigations; see references in Pothos and Busemeyer (2013). It is interesting in itself that quantum probabilities may be connected to our minds.

In epistemic processes (processes to try to achieve new knowledge) quantum theory is also connected to the variables in the mind of an observer, or variables shared by a group of observers. This is related to our interpretation of the state vectors. But of course this interpretation - as any interpretation - may be controversial.

Concluding remarks

Quantum decision theory seems to be applicable in many cases where decisions are to be made, primarily but not only in the cases which Kahneman (2011) calls decisions of type 1, fast decisions. When a prior is available, an alternative is a Bayesian decision theory.

In this article I have concentrated on decisions made by a single person. But the theory can be extended to joint decisions made by a group of communicating persons. Then these persons must have common ideals, as modelled by a perfectly rational higher being D. The ideals must be common in the setting determined by the relevant decision variables.

It is important that the rules of quantum probabilities are different from the usual Kolmogorov probability rules. The law of total probability does not hold, and the probabilities related to two events may depend on the order in time of the events.

As a basis for quantum decision theory as discussed here, lies the idea of two complementary decision variables at the same time. I repeat that two different variables are complementary if they are maximal as accessible variables. Examples may be found within the mind of any person, but also between persons and between groups of persons. It may in certain situations be related to many pairs of decision variables, leading to partly complementary world views in the last two cases. Then it might imply deep conflicts between those persons or groups if the necessary conditions are there. We can see many examples of this phenomenon in the world today.

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