Commentary

# From the Crisis of Foundations and the Loss of Certainties to a New Open Horizon

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**In the last decades of the nineteenth century, the aim of giving mathematics, starting from arithmetic, certain foundations inspired a formidable research effort. A hard-fought competition developed between the principal schools of foundamentalism: set theory, logicism, formalism and intuitionism. The great effort proved unsuccessful. The failure of attempts to provide a rigorous foundation for mathematical propositions caused the well-known crisis of foundations, which expanded to include philosophy, physics and in general the trust in the capacity of knowledge to achieve definitive certainties. Mathematicians were not too upset by the failure of the various schools. They rightly continued to dedicate themselves to mathematical practice. Once the storm had passed, the crisis of the foundations and the consequent loss of certainties turned into an extraordinary open horizon, first of all in the borderlands between mathematics, logic, physics and philosophy. We are still working in this new frontier.**

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## Introduction

In its history, mathematics has come across very early the non-exact and the non-rational, starting from the incommensurability side-diagonal of the square (Hippasus), and so on with the paradoxes of infinity (Zeno and the Skeptics), the irrational numbers, the case of $\pi$, the enigma of curves, measuring which calls into question of infinity, the understanding of motion and the discovery of analysis. The revelation of the non-rational doesn't only come from outside, but arises from within itself. Not 'everything is number', unlike the former Pythagorean doctrine. It's not necessary to outline the strong development of the discipline, especially in the 17th century (infinitesimal calculus and analytical geometry) and then the golden age of the 19th century (Jacobi, Bolzano, Gauss, Dirichlet, Cauchy, Abel, Riemann, Boole, Hermite, the Analytical Society, Cayley and, for geometry, Brianchon, Poncelet, Steiner, Plücker, Möbius, Grassmann, non-Euclidean geometry with Lobačevskij and Bólyai, Klein with the Erlangen Program). A formidable effort was devoted to the arithmetization of analysis (Weierstrass, Heine, Cantor, Dedekind), assuming "... the real numbers...as conceptual structures rather than as intuitive quantities inherited from Euclidean geometry"[1], similarly to the liberation of geometry from intuitive assumptions about space. The need for rigorization and axiomatization arises from the impressive development of mathematics. Thus, the fundamentalist demand comes to the fore, i.e., the search for secure foundations of the entire building. This context includes the various aspects of the concept of infinity, which have emerged since the beginning of mathematical thought[2]. The same analysis is, in fact, the study of infinite processes, with all the questions, as continuity, limit and convergence, concerning calculus. It was Cantor who paved the way by giving life to set theory in its intuitive formulation[3][4][5]. In the same climate, different lines of research flourished, among which a sort of war of the schools broke out. A passionate mathematical and philosophical debate, between set theory (Cantor, Zermelo), logicism (Frege, Russell), formalism (Hilbert) and, with a different approach, intuitionism (Brouwer, Weil and even before Poincaré himself). These great schools, each with its own framework, face the problem of the foundations, which intersects with some unresolved issues of thought: multiplicity and unity, finite and infinite, discrete and continuum, determinate and indeterminate. Each offers excellent arguments, but none convinces the others, and none prevails. Then, in 1931, the publication of the famous article by Kurt Gödel[6], with incompleteness theorems, unexpectedly comes out. He demonstrates that mathematics cannot prove everything even in mathematics itself, thus exposing the unattainability of Frege's program. The storm rages for a long time. We will examine set theory carefully here (for a detailed analysis of theory, both Cantorian and post-Cantorian, we refer to our previous paper)[7], reasoning concisely about other positions. Meanwhile, mathematicians continued their work, without becoming more than necessary concerned.

## Discussion

A. **SET THEORY.** Cantor breaks down a wall, trying to demonstrate that there is an infinite number of infinite sets, which can have different size, thus giving rise to a new mathematical domain: an arithmetic of transfinite numbers – that is, the *actual infinities* replacing the *potential infinities* -

with a sequence of infinite sets of different sizes. He also introduces transfinite ordinal numbers. All this is what Hilbert calls *Cantor's paradise*. Transfinite numbers were greeted with skepticism and even hostility, particularly by L.Kronecker and also by H. Poincaré, for whom this theory is a disease from which mathematics would be cured, while subsequently they were generally accepted, starting from R.Dedekind, K.Weierstrass, G.Mittag-Leffler, G.Peano, D.Hilbert, G.Frege, B.Russel and A.Whitehead. How does Cantor achieve his results? He uses two essential tools and starts from two fundamental theorems. The tools are the one-to-one correspondence and the diagonal method, through which one can compare the sizes of sets, and hence their cardinalities. Given sets *A* and *B*, the requirements are that for every element *a* in *A* there exists one, and only one, element *b* in $B$, and vice versa. When 1-to-1 correspondence is found, the sets are called equipotent. In the language of Zermelo-Fraenkel axiomatic theory, which is partly different, cardinality is equal when function *f:* $A \rightarrow B$ is *bijective*, that is, at same time both *surjective* (every *b* of the codomain *B* is a range, at least, of an element of the domain *A*), and *injective* (every distinct element in *A* reaches a distinct element in *B*). When function is not bijective, we have $|A| \geq |B|$ or vice versa. Does the domain or the codomain have more elements? It doesn't depends on cardinality, but on formalism of functions. For finite sets, domain is greater than or equal to codomain, if function is surjective and codomain is greater than or equal to domain, if function is injective. Let's now look at the two fundamental theorems. The first establishes that the set of trascendent numbers has greater cardinality than the set of algebraic numbers. The second, usually said *Cantor's theorem*, meaning by *P(A)* the so-called *power set,* i.e. the set of parts or subsets of any set *A*, is written in formal language $|A| < |P(A)|$. The theorem applies to all sets, finite and infinite. Once the applicability to infinite sets is established, the possibility that they have different cardinalities is introduced. From the second theorem a third one follows: there is no set of cardinality greater than any other. It contrasts with the concept of 'the set of all sets' and produces one of the most famous paradoxes. These theorems constitute the heart, the entire building rests on these pillars. It remains to add that a key point is the equalization between finite and infinite sets. Equalization, different from equipotence, means that finite and infinite sets can be treated in the same way. Cantor rejects the notion of *potential infinity* and assumes infinities as *actual existing totalities.* So in fact he, but also axiomatic theory, equalize finite and infinite sets. Cantor does not present a demonstration and merely makes an assumption. It is enough to read the definition of set, that opens *Beiträge* [5] and his letter to Hilbert of 1897[8]. However, the conformity/dissimilarity, between finite and infinite objects, has historically been a point of inexhaustible discussion. Galilei wrote about "... difficoltà che derivano dal discorrer ... intorno a gl'infiniti, dandogli quelli attributi che noi diamo alle cose finite e terminate: il che penso che sia inconveniente"[9]. The point is that every infinite set is thought of not as *potential*, but as *actual infinity*. For Cantor, and also for the axiomatic theory, it is an existing, definite and complete, entity. In general, mathematicians had not admitted actual infinity. It's a discussion that comes from Aristotle's distinction between potential and real infinite. Thus, e.g., Scholastics: *infinitum actu non datur,* unlike various medieval theologians. Besides Galilei, we can remember Descartes, Spinoza, Gauss, who wrote: "infinite is only a *façon de parler*"[10] and so on. Instead, Leibniz had a changing position. In ZFC (Zermelo-Fraenkel- Choice) actual infinity becomes possible through the introduction of the *axiom of infinity* and Russell-Whitehead[11], also to get round the 'Russell's paradox', created a *theory of types* and, in addition, they introduced the axioms of *reducibility*, *infinity* and *choice*. To summarize briefly: 1) there is no set, finite or infinite, that is equipotent to the set of its subsets; 2) sets can be countable or uncountable and those equipotent to *N* are defined as countable; 3) cardinality of *N* set, i.e., cardinality of infinite countable is $|N|:= \aleph_0$; 4) by iterating the formalism, an infinite series, with increasing cardinality, is created ($\aleph_1$, $\aleph_2$, $\aleph_3$ ...$\aleph_n$), from $\aleph_3$ onwards not only uncountable, but also unrepresentable. In this framework many results are derived or included by set theorists: difference between countable-uncountable, Cantor's paradox, transfinite ordinals (already criticized by Burali-Forti[12]), 1-to-1 correspondence, that is an invertible application, ($|A| \leq |B| \wedge |B| \ \leq |A| \implies \ |A| = \ |B|$, between points of a straight line and points of a square or of a n-dimensional space, furthermore Continuum Hypothesis (*CH*) and then, with the exponentiation, the path from arithmetic of transfinite to infinity of absolute, with a deistic meaning for Cantor. Theological topics are not the subject of present work. In this context, continuum cardinality is defined: $C = 2^{\aleph_0}$. For a rigorous analysis of the process, by which Cantor, having stated infinite cardinality of *N* set, affirms that it is the smallest of infinite sets and shows the expansion of infinities, we refer, for brevity, to our paper cited above. So Cantor, and also ZFC, address *CH* (*continuum hypothesis*): $|N|$ is $< |R|$, which is precisely continuum cardinality, furthemore, no cardinality is included among them and the different types of continuum, 1-d., 2-d.,..., n-d. are equipotent. Formally: $\notin A : \aleph_0 < |A| < \aleph_1$ and $|R| = C = 2^{\aleph_0} = \aleph_1$. The rule is that exponentiation changes cardinality. Cantor's point of arrival, with years of exhausting concentration, in a swing of exaltation and depression, is fixed in these formal expressions. Today we can see that Cantor was looking for an impossible demonstration. His tormented search carried forward, surrounded by skepticism. After Cantor, many thinkers have grappled with the enigma of continuum. Among the greatest, in the fields of mathematics and logic, Zermelo, Russell and Whitehead, Banach-Tarski (with the surprising and splendid theorem, according to which, to avoid an inadmissible paradox, one must admit that there are three-dimensional objects that have no volume), Gödel and Cohen. "What Cantor did not know, and he couldn't have known, he was working on an impossible problem. Today we know that *CH* has no solution in our mathematics. Both continuum hypothesis and its negation are true, and both continuum hypothesis and its negation are false: it is undecidable in the domain of our mathematics"[13]. This is the conclusion in short, still accepted by majority of mathematicians, of Gödel (for which *CH* cannot be proven false, starting from the *ZF* axioms, not even by adding the axiom of choice *AC*)[14] and of Cohen (*CH* cannot even be proven true from the same axioms)[15][16]. That means that *CH* is independent of system of *ZF* axioms and of *AC*. Let us add that from mathematical point of view, nothing prevents us from working on a completion of set theory, which might even find out new axioms, that make it possible to prove or disprove CH. Everything can change, depending on the definitions and axioms adopted. Undecidability could be evaluated in a higher system. We can now make an initial assessment. The first difficulty consists in attribution of cardinality to infinite sets. Cardinality is the number of elements in a set. Does an infinite set have a definite size or quantity? Better yet, is infinity a number like any other? One can say that infinity is a mathematical object, but can one handle it as a mathematical object, endowed with a definite quantity? Cantor must introduce the notion of *infinite cardinality*. But cardinality is the magnitude of a quantity. Where do Cantor and the axiomatic theorists demonstrate that the properties of infinite sets are the same as those possessed by finite sets? This constitutes a break with the vision of potential infinite, initiated by Aristotle, but above all established in the mathematical analysis of infinity[17]. Now, we will analyze the diagonal method, which, combined with 1-to-1 correspondence, is the tool that Cantor uses to achieve his crucial results. We will show the critical state of both. Let's take an interval *I,* from *0* to*1*, and assume that all real numbers contained in *I* are enumerated. By representing the elements of *I* in decimal form and arranging the sequence as a matrix, we define a real number $\alpha$, and we get that it is always possible to construct an other real number $\beta$, with all the digits different from diagonal number $\alpha$. Having initially supposed, with a typical *reductio ad absurdum*, that our list contained all real numbers in *I*, it would have been demonstrated that hypothesis is false and therefore interval *I* [0,1] is uncountable. With the words of Courant-Robbins: "We start from the hypothesis that all the real numbers have actually been numbered in a sequence and then we build a number not included in numbering. This leads to a contradiction.... hypothesis is false ...therefore... set of real numbers is not countable”[18]. It was proven that starting list did not contain all real numbers. It is not refuted, at all, that the new real $\beta$ can find a place, in sequence, in infinite set $\aleph_0$ or, to use the famous metaphor, in Hilbert’s[19] Grand Hotel, that there are always rooms available for one unexpected customer or for infinite customers and also for arriving customers on an infinite number of buses, each with an infinite number of passengers. Beyond the observations on the method, there is a fundamental objection. Dedekind’s definition of the infinite set reads like this: “a set in 1-to-1 correspondence with a proper part”[20][21], from which the Cantor-Dedekind theorem follows: an infinite subset of an infinite set has the same cardinality as the set of which it is part. Is the statement $\aleph_1 > \aleph_0$ compatible with this? The highest obstacle is constituted by a theorem of formidable power: the set of points of a square, or of a cube or even of a multi-dimensional space, is not greater than the set of points of a side or an interval *I*, however small. Dedekind was a friendly reference for Cantor, but having a very high logical rigor, he demonstrated that infinite sets are similar to each other, *isomorphic* in current terminology. Essential property of an infinite set and its subsets, with infinite cardinality, is that they never become exhausted, even if infinite new numbers arrive, which weren’t there before, to be put into 1-to-1 correspondence. It can plausibly be said that the diagonal method fails in its task of proving the hierarchy of uncountable sets. We cannot show in detail the passage from the naïve theory to *ZF* (formalization of Zermelo-Fraenkel) or *ZFC* (*C* is for *Axiom of Choice*), with important contributions of Schönflies, Borel, König, Bernstein, Burali-Forti, Lebesgue, to arrive at Skolem’s formulation and other axiomatic theories, such as *NBG* (Von Neumann-Bernays-Gödel), *alternative set theories* (Frege), *MK* (Morse-Kelley, *KP* (Kripke-Platek) and *minimal set theory*. *Model* and *category theories* are, as is well known, of a different type. Diagonal method is Cantor’s typical tool, others use the ‘functional’ formalism, based on injective, surjective and bijective functions. Can this formalism be transferred to the comparison between uncountable sets ($\aleph_1$)? It is demonstrable that relation between domain and codomain cannot be assumed, for $\aleph_1$, as a fixed relation and cannot be determined *a priori*. A new element can always occur, making comparison unstable. To be more precise, with a new element, which is potentially already present in the domain or codomain, no less existing than an actual infinity, a surjection can at any time appear[22]. Diagonal and functional method present similar problems. If you take two real numbers, there is always another real number between them: $\forall x \in R$, $\forall y \in R$, (*x*


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$< y \Rightarrow \ \ni z \in R,\ x < z < y$). The analysis of diagonalization, and functional method in axiomatic theory, leads to the result that hierarchy of infinities and difference in cardinality are not demonstrated for uncountable sets. A clear distinction, however, can be detected, between a sequence developing to infinity in an unilinear way and a path that branches out into a multiplicity, or infinity, of directions. If we remain in the unilinear *sequence paradigm*, the problem remains unsolvable, but it can be tested the *density paradigm*. *Sequence*: after one number there is always another; *density*: between two numbers there is always another. *Sequence*: usually countable; *density*: often uncountable. A similar analysis can be applied to 1-to-1 method. When applied to uncountable sets, it shows a logical fault. It is not possible, between uncountable sets, an application 1-to-1 from the domain to the codomain and vice versa. To count means to match an object and a numeric symbol. Now, *R*, i.e. $\aleph_1$, is uncountable. In set theory counting uncountable is also used to demonstrate the uncountability. Can you use the count, of what you defined as uncountable, to demonstrate the uncountability? It's not *a reductio ad absurdum,* it's a contradiction. Let's summarize the main critical points: **a)** the attribution of cardinality to infinite sets, also uncountable; **b)** the detection of contradictions between some fundamental theorems; **c)** the logical flaws of diagonalization and 1-to-1 correspondence; **d)** the inconsistency of *actual infinity*, as defined and complete object; **e)** the Cantor's paradox. These points constitute a bold challenge to logical thinking. Cantor and axiomatic set theory have started a profound change in mathematics, but they also produced a *factory of antinomies.* For *ZFC* and for *NBG* the set of all sets cannot be a set, then they introduce the distinction between *set* and *proper class*. The latter cannot be an element, but can contain elements. Paradoxes and antinomies can be 'overcome', with various subtleties, operatig on axioms and definitions, but only if you remain on a formal level. The Russell's antinomy is dribbled with the theory of types. Very well, but are these just formal problems or do they go straight to the heart of the relation with the world of experience? Mathematics and logic may legitimately neglect this, but philosophy and physics can't do it.

B. **LOGICISM, FORMALISM, INTUITIONISM.** To integrate the historical picture, let's recall some essential features of the discussion. The beginning of the storm can be taken as the date, June 16 1902, of Russell's famous letter to Frege[23][24][25], then made public by the latter, in which the problem of "the predicate that cannot be predicated of itself" (Russell's antinomy) is posed. The great German logician and mathematician pursued, like other great scholars, as Dedekind, Peano[26][27][28], Russell, "...the goal of making it (mathematics) free from any doubt" [29], that is, to give to arithmetic, and through it perhaps to the whole of mathematics, a character of absoluteness and rigorous validity. In two words, the mathematical certainty. Theories in the field were, in addition to set theory: a) logicism, whose leading thinker was Frege, for whom mathematics can be derived from logic (thesis that can be traced back to Leibniz[30]); in this school we can include, albeit with their own specificities, Russell and Whitehead; b) formalism, of which Hilbert was the inspiration and point of reference: the central property are the coherence of the axioms and deductions and, therefore, the non-contradiction and completeness of system; c) intuitionism, with Brouwer, but also with important personalities, like Poincaré and H. Weil, for whom the foundation is non logic, but intuition, which provides the evidence of concepts. The heated discussion continued for a long time. To put it in Bell's words: "...the paradoxes and contradictions that vitiate the basis of the theory, constitute perhaps the most important contribution that (Cantor) has made and will make to mathematics"[31]. Poincaré, always severe with logicism, nevertheless affirmed that logicism is not sterile, it generates antinomies. The debate primarily involved logic and mathematics, but the discussion affected physics and philosophy and then saw the involvement, just to name a few, of thinkers like Planck, Mach, Wittgenstein, Einstein, Bohr, Heisenberg, Pauli, as well as generations of scholars. What had been considered *certainties of mathematics and physics* were being called into question by irruption of antinomies and non-Euclidean geometries, but also by relativistic and quantum revolution, to the point of questioning the confidence in capacity of human knowledge to describe and explain our world. In this context, in 1931, Gödel's mentioned essay broke out. Its work, "... a milestone in the history of logic and mathematics"[32], was decisive in investigasting the coherence and completeness of mathematics. He demonstrated the impossibility of proving the coherence of axiomatic systems, including arithmetic, unless one resorts to principles, that are in turn even more uncertain and that cannot confer any certainty on branches of mathematics to be free of intrinsic contradictions. The foundations of the axiomatic program are cut off at the root. As M. Kline effectively recaps: "... it is impossible to prove the consistency of arithmetic using any method, or any set of logical principles, that can be translated into the arithmetic system"[33]. And R. Hersch: "The search for secure foundations never recovered from this catastrophe"[34].

C. **FROM THE CRISIS OF FOUNDATIONS TO A NEW OPEN HORIZON.** The declared aim of the foundationalist schools was not achieved. Logicism and set theories, which are very close, often generate antinomies and inconsistencies; formalism, despite the reduction to formal symbolism, is rather ineffective in being able to exclude the possibility of doubts and contradictions; intuitionism has a different approach, but is exposed to well-argued criticism, being based on the empirical level, implying the subject-object relationship. For these reasons, the prevailing interpretation is that of a story of failure. This assessment does not seem to be shared, even if the goal, the undoubted certainty of mathematics, was missed. What has emerged is something substantially new: certainty is neither attainable nor necessary. The foundationalist program is historically understandable, but illusory. Have we lost our certainties? We have lost something we thought, wrongly, we possessed or could conquer. Moreover, “Mathematicians soon decided to ignore the problem of foundations”[35] and Boyer notes: “Mathematicians and physicists were not affected much by this blow”[36]. With the end of the Second World War there was an enormous development in mathematics. We limit ourselves here to a few examples, such as researches on computability, non-standard analysis (hyperreal numbers)[37] and multiform production in mathematical logic. After ZF theory, further refined by Skolem and others, an impressive level production has taken place, with point- set topology, metamathematical theory, alternative axiomatizations, Quine’ works, Cohen’s forcing, Boolean models, large cardinals, Tarski’s elaboration on weakly compact cardinals, cardinals of Ramsey and Erdös, theories of models, categories, and other branches of research. What is most significant to underline is that, after Quantum Mechanics, Special and General Relativity and Gödel’s essays, a new and unknown vision of both the limits and potential of knowledge has gradually taken hold. What is called the crisis of foundations, in reality, was not a negative period, on the contrary it turned into an extraordinary season of scientific and philosophical thought. This concerns research in general, but very much mathematics and borderlands between physics, logic, philosophy and mathematics.

D. **HINTS FOR RESEARCH.** In this new horizon, we put forward, by way of example, some hypotheses.

**1) Density.** When we study the topological spaces, a distinction is made between *dense sets*, as the rational numbers, and *non-dense sets*, such as the natural numbers. A paradigm shift can be tested, from *sequence* to *density*. There is a big difference. Perhaps, it is “…fundamentally question of density that determines the power of set”[38]. The sequence function describes an ordinate series of real numbers, while density refers to topological spaces. It is not identical to the probability density function. Even though density is not measurable in uncountable sets, it offers more possibilities to order sets, in relation to dimensionality, such as 1-space, 2-space, *n*-space varieties. It is interesting to work to obtain a plurality of set theories, and nothing prevents us from testing different ways to deal with infinity. Clearly, we’re not dealing with the ontological infinity, about which we can say very little. **2) From a powerfull theorem.** A counterintuitive result of set theory, generally accepted, is that cardinality of the points of a segment 1-*d*, of a square 2-*d*, of a cube 3-*d*, and of an object n-*d*, is the same. Continuum is equipotent. Now, you can take an interval *I* as small as you want. By applying the appropriate formalism and going to the limit, an unextended point will asymptotically map an infinite set, with other words, a *singularity* would asymptotically 'contain' the largest n-dimensional infinite set. Without needing to enter the mathematical exam of singularities, it's easy to understand that we have a surprising behavior. The infinitely large is *included* in the infinitely small and the opposites coincide. This makes us think of the *coincidentia oppositorum* of Nicholas of Cusa[39] and the *paradox of Albert of Saxony.* **3) On infinite divisibility.** If two points are assumed: that mathematical objects own the property of existence and that the actual infinity is a definite and complete mathematical object, the conclusion would be that Cantor's system represents a real world. Kant had already revealed the sleight of hand of making an object correspond to the logical possibility of a concept[40]. The challenge of counting infinity always ends in an invincible maze. Infinity is an inexhaustible mine of paradoxes and inconsistencies. Antinomies and contradictions of infinity, which we tried to overcome with formal solutions, die and rise again, like the Phoenix or the feathered bird Quetzalcoatl. If I am allowed, I put forward an explanatory conjecture, to be verified. Maybe, *infinite divisibility, like infinite multiplicity, might just be in our mind*. Mathematical research, on infinities and infinitesimals, offers large spaces, with analysis (study of functions, calculation of limits, asymptotic notation, Landau symbols...), and algebra of infinities and infinitesimals, but the refutation of the proposed conjecture would require a demonstration that infinite divisibility and infinite multiplicity, i.e. actual infinity, have a counterpart in nature or phenomenal reality. A demonstration of this -kind, at present, doesn't appear to be done or feasible.

E. **ON MATHEMATICS.** Mathematics is not physics. The discussion often focuses on two questions: is mathematics a discovery or an invention? Do mathematical objects have real existence? It probably could be a single question. Platonism and various forms of neo-Platonism[41] claim that ideas, and also the numbers, are not only existing, but are the true essence of the world. We are among those who think, with respectful attention to opposing positions, that the arguments, in favor of mathematics as discovery of what pre-exists us and of actual existence of mathematical objects, do not seem sufficiently argued. We are more inclined to think that mathematics is a human creation, a historical product of biological brains. In fact, we can develop systems and structures, with the only condition of correctly fixing axioms and definitions, respecting the rules we freely set and ensuring the consistency of the theory. Mathematics is independent from the physical systems, althought it often arises from physical problems, and is limited only by its internal constraints. Mathematics constitutes an indispensable tool for organizing and describing the world of experience, and it has a huge breadth of applicability, but it is not in itself bound to this purpose.

## Conclusions

**1)** We have shown that Cantor's most important theorems (infinite cardinality, equalization between finite and infinite sets, different cardinality of uncountable sets...) are flawed by inconsistencies and contradictions; they are so not proven. Also post-Cantorian set theories have not been recognized, by the majority of mathematicians and philosophers, as capable to address the problems of foundations. Not even the other foundationalist schools have achieved the aim of the mathematical certainty. **2)** The failure of attempts to address the crisis has not slowed down the progress of mathematics, which has not suffered from the lack of solutions. The crisis has turned into an impressive push towards scientific innovation and technological creativity. **3)** The power of mathematics lies maybe precisely in not being strictly a science of nature, although it is indispensable for any natural science. It is pure thought, not a subject trying to grasp the object, but subject trying to explain its own representations and their effectiveness. **4)** The confusion of the fields, mathematics and theology, as in Cantor and elsewhere, doesn't prove useful. It's another thing the openness to all areas of human experience. **5)** A loss of certainties has left us a *new open horizon*, starting in borderlands between mathematics, logic, philosophy and physics. **6)** Each discipline does its job, but the enterprise is common. Natural sciences require predictions, subject to experimental verification. Mathematics works differently. We can maybe think that philosophy is not just a branch, but rather a systematic study of knowledge. We urgently need a strong reintroduction of interdisciplinary dialogue. I like say it, with words of A. Grothendieck, a contemporary mathematician: "I foresee that the renewal sought (if it is yet to come) will come from the soul of a mathematician, well informed on the great problems of physics, rather than from that of a physicist. But above all, it will require a man with a 'philosophical openness' sufficient to grasp the crux of the problem. This is not a technical problem, but a fundamental problem of philosophy of nature"[42].

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