Cosmological multi-connected structures and Hammerstein integral equations

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Abstract: The criteria for the formation of stationary quasi-periodic structures in a system of gravitating particles, described by the system of Vlasov–Poisson equations. Conditions of branching of solutions of a nonlinear integral equation for a generalized gravitational potential are investigated. The arisen non-analytic solutions leading to the origin of structures of relative equilibrium in the cosmological systems consisting galactics, clusters and superclusters.

Keywords: Vlasov–Poisson equation, Fredholm equation, Newton potential, Gurzadyan theorem, cosmological term

1. Introduction

At present, the description of the origin of large cosmological structures is based on two main methods, based on the use of the sequential hydrodynamic approach (in which the influence of gravitational fields of particles, of which the structures are composed, is taken to be secondary), and the approach using the methods of nonequilibrium thermodynamics and kinetic equations for systems of massive interacting particles. It is usually assumed that the primary structure formation in the Universe is caused by deviations from the Friedmann flow leading to the appearance of the topological features of the type of caustic surfaces, for example, which are consequences of crossing of fronts of multicenter baryonic acoustic waves. These two-dimensional manifolds form, at certain stages of their evolution, relatively long-lived three-dimensional "web-like" formations. The classical models of Ya.B. Zel'dovich [1]–[3] related to the ordered (in one or two directions) emergence of a set of density fluctuations can be referred to the same technique. of density or velocity fluctuations in an initially homogeneous moving system of particles (the theory of "pancakes" or "walls"). The approach using nonequilibrium statistical mechanics, is considered by most authors as a basis for the formation of secondary cosmological (meso)structures with smaller (compared to the "caustic" approach) dimensions, and actually playing an essential role only for explaining the decay rate or pseudocollapse of the instability developing in megastructures. It is obvious what is the fundamental difference between the two above-described techniques: 1) the hydrodynamic approach assumes little influence of the intrinsic gravity of the particles in the considered system compared to the influence on the geometry of inhomogeneities (2-dimensional caustics) of the modified (by introducing perturbations in the De Sitter epoch) Friedmann flow; 2) the kinetic-field approach is assumed to be suitable for relatively fast non-equilibrium phenomena in an external quasi-ordered medium (in which the wave fronts overturning and intersection of three-dimensional matter flows channels for kinetic processes are formed).

Both approaches, to all appearances, do not have strictly defined limits of applicability, and are suitable only with certain physical assumptions (in particular, the authors can emphasize

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here the following the assumption of coherence of velocity fluctuations in the selected directions for the models of void walls, as well as the universality of the internal structure of the known star superclusters). However, the generally accepted separation on scales of the hydrodynamic and kinetic approach in the light of new approaches in modeling may be inappropriate at all. It is usually assumed that the Friedmann flow rate exceeds the rate of structure ordering in the astrophysical system associated with kinetic temperature processes. This assumption is due to the very definition of temperature in statistical mechanics due to double, triple, ... collisions of particles (the thermal equilibrium in this case in a rarefied set of particles is established very slowly, and the geometry of the system is connected with it). the geometry of arising long-lived subsystems is connected with it). If we assume that the thermalization in the system occurs due to the interaction of the of particles with the self-consistent gravitational field of its particles and the external field from neighboring systems (the temperature takes the status of "kinetic" [4]), then the quasi-stationary states of the of the system (states of relative equilibrium) associated with the emergence of ordered structures in the system can be reached in time intervals smaller than in the case of formation of a web of caustics for the hydrodynamic equations without taking into account gravitational fields and acoustic modes. At that, taking into account the increasing antigravity forces between particles on sufficiently large distances, one should expect a discrepancy between the results of kinetic modeling and the results of hydrodynamic modeling due to the lack of conformality of structures of different cosmological scales: The zones of dominating influence of ascending and descending branches of the modified Newton gravitational potential form a topologically different multi-connected structure of matter in the Universe (in particular, one can note dipole objects "'Shapley Attractor/Dipole Repeller" [5] for the maximum currently observable scales).

In the present paper, the authors continue to develop the method of describing cosmological structures on the basis of the integral form of the gravitational potential equation, which has been previously considered in the papers [6]–[8]. The main idea of the proposed approach is deterministic ordering of large structures, arising in the course of evolution of systems of massive particles interacting by means of a self-consistent gravitational field. The description of quasi-static processes occurring in the system near its state of relative equilibrium is possible using the system of Vlasov-Poisson equations. However, in this case it is reasonable to consider not the local version of the equation for the potential — the differential Poisson equation (in the Liouville–Gelfand form, as the the particle distribution density is represented as a Maxwell exponent) — but its global variant in the form of an integral equation of Hammerstein type. For the latter, the corresponding boundary problem is posed based on the use of Gurzadyan's [9] theorem (this is due to the fact that the structure of the of solutions of the integral equation is more illustrative, and their possible singularities can be expressed using known formalizable representations). At it is necessary to investigate the properties of the spectrum of the nonlinear integral operator and its relation with the spectrum of the Fredholm operator for the linear potential.

2. Integral Hammerstein form of the Poisson–Liouville–Gelfand equation for the system of massive particles

The starting point for obtaining hydrodynamic models with singularities–caustics and kinetic models without taking into account the influence of gravity is the introduction of the "Friedmann flow" characterized by a field of velocities whose values for each pair of points in space will be proportional to the distance between these points $\mathbf{v} = H\Delta\mathbf{r}$. The Friedmann flow in the above-mentioned models was considered as some analog of an equilibrium state, which is perturbed either by inhomogeneity in the initial data set, or random fluctuations of the density leads to the formation of geometrical "small" inhomogeneity (adiabatically stable) of the spatial distribution of particles of the system on the background of this flow (the "smallness" is caused by random deviations, which a random deviations, which are assumed a priori to be essentially limited in norm in comparison with the basis values of mean parameters).

The system of Vlasov–Poisson equations for describing of dynamics in a system of N cosmological objects (with masses $m_{i=1,\dots,N} = m \equiv 1$) may be represented as

$$\frac{\partial F(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathrm{div}_{\mathbf{x}}(\mathbf{v}F) + \widehat{G}(F;F) = 0, \quad \widehat{G}(F;F) \equiv -\big(\nabla_{\mathbf{v}}F\big)\big(\nabla_{\mathbf{x}}(\Phi[F(\mathbf{x})]\big), \tag{1}$$

$$\Delta^{(3)}_{\mathbf{x}}\Phi[F(\mathbf{x})]\big|_{t=t_0} = AS_3G\int F(\mathbf{x},\mathbf{v},t_0)\,d\mathbf{v} - \frac{c^2\Lambda}{2}, \tag{2}$$

$$S_3 \equiv \mathrm{meas}\,\mathcal{S}^2 = 4\pi, \quad \mathcal{S}^d = \{\mathbf{x}\in\mathbb{R}^d, |\mathbf{x} = 1|\},$$

where $F(\mathbf{x},\mathbf{v},t)$ is the distribution function of gravitationally interacting particles, A is a normalization factor for particle density, t_0 is a fixed moment of time, G is a gravitational constant System of objects/particles is considered in the finite domain of configurational space $\Omega \subseteq \mathbb{R}^3$ ($\mathrm{diam}\,\Omega \le \infty$), with a C^2-smooth boundary $\partial\Omega$.

Equation (2) is the nonlinear Poisson equation, with account the cosmological term. The third term on the right side of the kinetic equation (1) may be represented as

$$\widehat{G}(F;F) = \mathbf{G}(F)\frac{\partial F}{\partial \mathbf{v}}, \quad \mathbf{G}(F) = -\nabla_{\mathbf{x}}\Phi[F(\mathbf{x})], \tag{3}$$

$$\Phi[F(\mathbf{x})] = AS_3G\int\int \mathfrak{K}_3(\mathbf{x}-\mathbf{x}')F(\mathbf{x}',\mathbf{v}',t_*)\,d\mathbf{x}'d\mathbf{v}' + \frac{\Lambda c^2}{12}|\mathbf{x}|^2 + \widehat{\mathfrak{B}}_3(\mathbf{x},\mathbf{x}'),$$

where: $\mathfrak{K}_3(\mathbf{x}-\mathbf{x}') = -|\mathbf{x}-\mathbf{x}'|^{-1}$, $\widehat{\mathfrak{B}}_3(\mathbf{x},\mathbf{x}')$ is an operator term that takes into account the influence of boundary conditions. Classical Newtonian potential $\Phi_N(r) = -Gm/r$ increases monotonically on the interval $r \in (0,+\infty)$ ($\Phi_N \in (-\infty,0)$); a generalized (with cosmological term) Newton gravity potential $\Phi_{GN}(r) \equiv -Gm/r - \frac{1}{2}c^2\Lambda r^2$ has a maximum $\Phi^{(max)}_{GN}(r_c) = -\frac{1}{2}G(3mc^{2/3})\Lambda^{1/3}$, $r_c = \big(Gm/(3\Lambda c^2)\big)^{1/3}$ (it increases on the interval $r \in (0;\, r_c]$ and decreases on the interval $r \in (r_c;\, \infty)$).

We will consider the stationary case of dynamics: $F = F(\mathbf{x},\mathbf{v})$. However, it should be pointed out that further analysis will mainly concern the second equation of the system (1)–(2), which is the Poisson equation relative to the potential, and no explicit time dependence is observed in it. That is why, when varying the Hilbert–Einstein–Maxwell action (or Hilbert–Poisson–Poisson) [6]–[8] maybe separate variation over fields (for a fixed particle distribution) and variation over distribution functions (with fixed fields); thus, approach considered in this paper, is applicable for adiabatic processes at a quasi-equilibrium (weakly varying) particle distribution functions. In this case, we can use the energy substitution for unique variable of the distribution function [10]: $F(\mathbf{x},\mathbf{v}) = f(\varepsilon) \in C^1_+(\mathbb{R}^1)$, where $\varepsilon = m\mathbf{v}^2/2 + \Phi(\mathbf{x})$. Thus, the particle density in the right side of the Poisson equation can be expressed in terms of the equilibrium solution of Vlasov equations. This solution is identical in form of Maxwell–Boltzmann distributions $f = f_0(\varepsilon) = AN\exp(-\varepsilon/\theta)$. However the physical meaning of the equilibrium solution of the

Vlasov equation is essentially different from that of the Boltzmann equation. This solution must meet the following requirements: 1) the maximal possible statistical independence, 2) isotropy of velocity distribution, 3) stationarity of distribution in the form $F(\mathbf{x}, \mathbf{v}) = \rho(\mathbf{x}) \prod_{i=1,2,3} \mathfrak{f}(v_i^2)$. The substitution expression into the Vlasov equation gives

$$\sum_i \left(v_i \frac{\partial \ln(\rho)}{\partial x_i} - \frac{\partial \Phi}{m \partial x_i} \frac{\partial \mathfrak{f}(v_i^2)}{\mathfrak{f}(v_i^2)\, \partial v_i} \right) F = 0, \tag{4}$$

and we get system of ODEs:

$$\frac{\partial (\ln\, \rho)/\partial x_i}{-\partial \Phi/\partial x_i} = \frac{\partial \ln \big(\mathfrak{f}(v_i^2)/\partial v_i \big)}{m v_i} = -\theta^{-1}, \tag{5}$$

where θ is a constant of separation of variables, it's physical meaning is a *kinetic temperature* in the system of interacting collisionless particles (in accordance of A.A. Vlasov definition [4],[11]–[12], thermodynamic/collisional equilibrium is globally absent in this system).

Equation (2) for gravitational potential maybe written now as

$$\Delta \Phi(\mathbf{x}) = ANGS_3^2 \bigg(\int_{y \in [0,\infty]} \exp\big(-y^2/(2\theta) \big) y^2\, dy \bigg) \cdot \exp(-\Phi/\theta) - \frac{c^2 \Lambda}{2}, \tag{6}$$

$$A,\ \theta,\ R_\Omega \in \mathbb{R}^1,$$

where: R_Ω — radius of region Ω accepted in the form of a ball in configurational 3–space (it is the simplest physical realized case).

So, the Poisson equation (2) takes the form of an inhomogeneous equation Liouville–Gelfand (LG) [13] with local (generalized) temperature changing sign in depending on the value of the derivative of the potential at a given point: as mentioned above, for two-particle problem (in particular, for a formal pair in the form of a center coalescence of the main part of the particles and the conditional "extremely distant" particle) can dominate the repulsive force due to the presence of a quadratic term $\sim |\mathbf{x}|^2$; while generalized the indefinite thermodynamics of a system of gravitating particles becomes similar to that for the Onsager vortices in the classical hydrodynamics [15], and the existence of solutions of the LG equation for large system sizes provides the existence of solutions to the Vlasov equation (1). This can be shown using the parametric Young's inequality [14]. It was be shown by author [16], for the conditions $c^2\Lambda \gtreqless 3\pi\lambda^\dagger$ $(\theta \gtreqless 0)$, system of Vlasov–Poisson equations has solutions of the type of distribution functions that admit the energy substitution, and potential of gravitational field, which have the property of convexity (in the general case, for an arbitrary $R_\Omega \leq \infty$, in contrast to the case of the attraction potential, for which there is a limitation $R_\Omega < \big(C_0 \theta^2 / \big(\lambda^\dagger / S_3^2 \big)^2 \big)^{1/4}$); we used notation $\lambda^\dagger \equiv ANGS_3^2 \mathcal{J}(\theta)$, $\mathcal{J}(\theta) \equiv \int_0^{v_{max}} \exp\big(-v^2/(2\theta) \big) v^2 dv$.

As already noted in [16], in the formulation of the Dirichlet problem for the Poisson equation (2) (or (6)) with a constant right-hand side on the boundary of the Ω region according to McCrea averaging gravitational field outside the compact subdomain Ω_0 containing system of particles (which is situated in the region Ω $(\operatorname{meas}\Omega_0 \ll \operatorname{meas}\Omega)$), we can assume the boundary condition on the $\partial\Omega$ is given by in accordance with the Gurzadyan theorem [9].

The solution of Dirichlet problem may be obtained with the help of integral representation of the equation for gravitational (double–layer) potential. Equation for the potential with Maxwell–Boltzmann particle density, corresponding to internal Dirichlet problem in a bounded domain Ω (under boundary conditions corresponding to the Milne–McCrea model) has the following form:

$$\Phi(\mathbf{x}) = \lambda_I \int_{\Omega'} \mathcal{K}(\mathbf{x},\mathbf{x}') \exp\big(-\Phi(\mathbf{x}')/\theta\big) d\mathbf{x}' - \frac{c^2\Lambda}{12}\mathbf{x}^2 + C_0', \tag{7}$$

$$\mathcal{G}(\mathbf{x},\mathbf{x}') \equiv 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{Y_{\ell m}^*(\vartheta',\varphi')Y_{\ell m}(\vartheta,\varphi)}{2\ell+1} \frac{x_<^\ell x_>^\ell}{R_\Omega^{2\ell+1}},$$

$$x_< = \min(|\mathbf{x}|,|\mathbf{x}'|), \quad x_> = \max(|\mathbf{x}|,|\mathbf{x}'|),$$

$$\mathcal{K}(|\mathbf{x}-\mathbf{x}'|) \equiv \mathcal{G}(\mathbf{x},\mathbf{x}') - \frac{1}{|\mathbf{x}-\mathbf{x}'|}, \quad C_0' = -\frac{GNm}{R_\Omega} - \frac{c^2\Lambda {R_\Omega}^2}{12}, \quad \lambda_I = \lambda^\dagger/S_3.$$

In essence, the above is the explicit form of the equation for the potential introduced in expression (3), where the Green's function $\mathcal{G}(\mathbf{x},\mathbf{x}')$ for the inner boundary value problem in the domain Ω (in this case, due to symmetry of the latter we have $\int_{\Omega'} \mathcal{G}(\mathbf{x},\mathbf{x}')\rho(|\mathbf{x}'|)d\mathbf{x}' \to C_1 = \mathrm{const}(\propto 1/R_\Omega)$). Let's introduce a new variable $U(\mathbf{x}) \equiv (\Phi(\mathbf{x}) - C_0' - C_1)/\theta + \alpha|\mathbf{x}|^2$, $\alpha \equiv c^2\Lambda/(12\theta)$, the above equation can be written as the uniform Hammerstein integral equation:

$$U(\mathbf{x}) = \lambda_\theta \widehat{\mathfrak{G}}(U), \quad \widehat{\mathfrak{G}}(U) \equiv \int_{\Omega'} \mathcal{K}(|\mathbf{x}-\mathbf{y}|)\Psi\big(\mathbf{y},U(\mathbf{y})\big)\, d\mathbf{y}, \tag{8}$$

$$\lambda_\theta \equiv \frac{\lambda^\dagger}{\theta S_3} \exp\big((-C_0 - C_1)/\theta\big),$$

$$\mathcal{K}(|\mathbf{x}-\mathbf{y}|) = |\mathbf{x}-\mathbf{y}|^{-1}, \quad \Psi\big(\mathbf{y},U(\mathbf{y})\big) \equiv -\exp\big(-\alpha\mathbf{y}^2 - U(\mathbf{y})\big).$$

For $\theta > 0$ we get $\lambda_\theta > 0$, the mapping $\widehat{\mathfrak{G}}(U)$ is compact (in $L_2(\Omega)$) nonlinear operator, since for it the conditions of the Nemytskij–Vainberg theorem [17] are satisfied ($\int_\Omega\int_\Omega \mathcal{K}^2(|\mathbf{x}-\mathbf{y}|)\, d\mathbf{x}d\mathbf{y} = \mathcal{K}_\Omega < \infty$, $\Psi\big(\mathbf{y},U(\mathbf{y})\big) \in C(\Omega\otimes\mathbb{R})$ and $|\Psi\big(\mathbf{y},U(\mathbf{y})\big)| \le g(\mathbf{y}) + C_\Psi\cdot|U|$, $g \in L_2(\Omega)$, $g(\mathbf{y}), C_\Psi > 0$). Let consider mathematical expression $\Psi^\dagger \equiv \int_0^U \Psi\big(\mathbf{y},U\big)dU = \exp(-\alpha\mathbf{y}^2)\cdot(\exp(-U)-1)$; it is obviously, $\Psi^\dagger \le \tau_1|U| + \tau_2$ ($\mathbf{x}\in\Omega$), where $\tau_{1,2} > 0$, $\tau_1 < 1/\lambda_{\mathcal{K}}$, $\lambda_{\mathcal{K}}$ is a maximal eigenvalue of integral equation kernel $\mathcal{K}$. Then, in accordance with Theorem 2.8 [18], the Hammerstein equation (6) has at least one solution $U_0(\mathbf{x})$. We will deal with the question of the uniqueness of a solution or the presence of many solutions in the next paragraph.

3. Solutions of Hammerstein equation for potential and its cosmological sequences

We will assume for $\lambda = (\lambda_\theta)_0$ the equation (8) has nontrivial solution $U = U_0$. Consider the Fredholm's determinant $\mathcal{D}(\lambda)$ [19] for integral kernel of linearized (in the vicinity $O(U_0)$ of basic solution U_0) Hammerstein equation $\widetilde{\mathcal{K}}(\mathbf{x},\mathbf{y}) \equiv \mathcal{K}(|\mathbf{x}-\mathbf{y}|)\cdot\partial\Psi\big(\mathbf{y},U_0(\mathbf{y})\big)/\partial U_0(\mathbf{y})$. After linearization (an application of Frechet derivative) we obtained linear Fredholm self-adjoint compact operator with discrete spectrum (on real axis) [20]. If $\mathcal{D}\big((\lambda_\theta)_0\big) \neq 0$ (i. e. $(\lambda_\theta)_0$ isn't characteristic value of kernel $\widetilde{\mathcal{K}}(\mathbf{x},\mathbf{y})$), then in the vicinity $O\big((\lambda_\theta)_0\big)$ the equation (8) has unique analytic (by powers of $\big(\lambda_\theta - (\lambda_\theta)_0\big)^j$, $j = 1,2,\ldots$) solution $U(\mathbf{x}|\lambda_\theta)$ [21], for which $\lim_{\lambda_\theta\to(\lambda_\theta)_0} U(\mathbf{x}|\lambda_\theta) = U_0(\mathbf{x})$ ($\forall\mathbf{x}\in\Omega$).

Let consider the solution of (8) in the vicinity $O(U_0)\times O(\lambda_\theta)_0$, for what introduce perturbed characteristic value $\lambda_\theta = (\lambda_\theta)_0 + \xi$ and perturbed solution $U = U_0(\mathbf{x}) + \zeta(\mathbf{x})$. Following the methodology [22], we substitute these expressions in the equation (8) written in the following form:

$$\zeta(\mathbf{x}) = (\lambda_\theta)_0 \int_\Omega \frac{\big(\omega_1(\mathbf{y})\zeta + \omega_2(\mathbf{y})\zeta^2 + ...\big)d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} + \tag{9}$$
$$\xi \int_\Omega \frac{\big(\omega_0 + \omega_1(\mathbf{y})\zeta + \omega_2(\mathbf{y})\zeta^2 + ...\big)d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|},$$

and taking into account of Taylor expansion $\Psi\big(\mathbf{x}, U\big) = \sum_{j=0,1,...} \omega_j(\mathbf{x})\zeta^j(\mathbf{x})$ (here $\zeta(\mathbf{x}) = \sum_{k=1} \xi^k \zeta_k(\mathbf{x})$),

$$\omega_0(\mathbf{x}) = \Psi\big(\mathbf{x}, U_0\big) = -\exp\big(-\alpha\mathbf{x}^2 - U_0(\mathbf{x})\big), \tag{10}$$
$$\omega_1(\mathbf{x}) = \frac{\partial\Psi\big(\mathbf{x}, U\big)}{1!\partial U}\bigg|_{U=U_0} = \exp\big(-\alpha\mathbf{x}^2 - U_0(\mathbf{x})\big), ...,$$

we obtain system of recurrent linear and multilinear equations for variables $\zeta_i(\mathbf{x})$:

$$\zeta_1(\mathbf{x}) = (\lambda_\theta)_0 \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\omega_1(\mathbf{y})\zeta_1(\mathbf{y})\, d\mathbf{y} + (\lambda_\theta)_0 \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\omega_0(\mathbf{y})\, d\mathbf{y}, \tag{11}$$

$$\zeta_2(\mathbf{x}) = (\lambda_\theta)_0 \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\omega_1(\mathbf{y})\zeta_2(\mathbf{y})\, d\mathbf{y} + \tag{12}$$
$$\int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\big((\lambda_\theta)_0\omega_2(\mathbf{y})\zeta_1^2(\mathbf{y}) + \omega_1(\mathbf{y})\zeta_1(\mathbf{y})\big)\, d\mathbf{y}, ...$$

Since, by assumption, $\mathcal{D}(\lambda_0) \neq 0$ then for linear non-uniform Fredholm IInd type equation (11) there exists a resolvent $R(\mathbf{x}, \mathbf{y}; \lambda_0)$ and we can write

$$\zeta_k(\mathbf{x}) = (\lambda_\theta)_0 \int_\Omega R(\mathbf{x}, \mathbf{y}; \lambda_0)\mathcal{H}_k\big(\zeta_1(\mathbf{y}), \zeta_1(\mathbf{y}), ..., \zeta_{k-1}(\mathbf{y})\big)\, d\mathbf{y} + \tag{13}$$
$$\mathcal{H}_k\big(\zeta_1(\mathbf{x}), \zeta_1(\mathbf{x}), ..., \zeta_{k-1}(\mathbf{x})\big)$$

(here operator–function $\mathcal{H}_k$ is a sum of all integrals including $\zeta_1, \zeta_2, ...$ up to $(k-1)$–th power: $\zeta_k = (\lambda_\theta)_0 \int |\mathbf{x}-\mathbf{y}|^{-1}\zeta_1(\mathbf{y})d\mathbf{y} + \mathcal{H}_k(\mathbf{x})$).

Thus, we can consistently and uniquely define all functions ζ_k. Consequently, we can formally construct a power series for $\zeta(\mathbf{x}) = \zeta\big(\xi^i, \zeta_i(\mathbf{x})\big) \equiv \xi^i\zeta_i$ (summation over repeating indices), and it only remains to prove the constructed series converges (for all values from some convergence circle) of the deviation parameter ξ (this will completely establish the equivalence of the expansion $\zeta\big(\xi^i, \zeta_i(\mathbf{x})\big)$ and the function $\zeta(\mathbf{x})$).

Based on the properties of the Lyapunov–Schmidt integral operator with a weakly polar kernel (Lemma 8.1 in [23]), and explicit forms of $\omega_k \propto \exp(-\alpha\mathbf{x}^2 - U)/k!$, we can state $|\int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\omega_k(\mathbf{y})d\mathbf{y}| < Z_1(=\text{const})$ $(k = 0, 1, ..., \forall\mathbf{x} \in \Omega)$. Next, by definition, $|R(\mathbf{x}, \mathbf{y}; \lambda_0)| < Z_2(=\text{const})$ $(\forall\mathbf{x}, \mathbf{y} \in \Omega)$. Then, we can write majorant series for series $\zeta\big(\xi^i, \zeta_i(\mathbf{x})\big)$. Let introduce algebraic equation $\zeta^\dagger = Z_3\big(\xi + \xi\zeta^\dagger + (\xi + (\lambda_\theta)_0) \cdot \big((\zeta^\dagger)^2 + (\zeta^\dagger)^3 + ...\big)\big)$, $Z_3 \equiv Z_1(1 + Z_2|(\lambda_\theta)_0|)$. We substitute $\zeta(\mathbf{x}) = \xi^j\varkappa_j|_{j=1,2,...}$ into the last equation and then compare the coefficients at different powers of deviation variable ξ:

$$\varkappa_1 = Z_3, \quad \varkappa_2 = Z_3(\varkappa_1 + |(\lambda_\theta)_0| \cdot \varkappa_1^2), \tag{14}$$
$$\varkappa_3 = Z_3\big(\varkappa_2 + \varkappa_1^2 + 2|(\lambda_\theta)_0| \cdot \varkappa_1\varkappa_2 + |(\lambda_\theta)_0|\varkappa_1^3\big), ...$$

Consequently, $|\zeta_k(\mathbf{x})| < \varkappa_k$ ($\mathbf{x} \in \Omega$), and convergence region of the series $\zeta = Z_3\big(\xi + \xi\zeta + ...\big)$ is equivalent of convergence region of the series $\zeta(\mathbf{x}) = \xi^j \zeta_j(\mathbf{x})|_{j=1,2,...}$. For describing of convergence region of $\zeta = \xi^i \varkappa_i$ we ought to investigate the implicit function $\zeta = Z_3\big(\xi + \xi\zeta + ...\big)$ as function $\zeta = \widetilde{\zeta}(\xi)$:

$$\widetilde{\zeta}(\xi) = Z_3\xi + (Z_3\xi - 1)\zeta^\dagger + Z_3(\xi + |(\lambda_\theta)_0|) \cdot ((\zeta^\dagger)^2 + (\zeta^\dagger)^3 + ...) = 0. \tag{15}$$

By Implicit Function Theorem [24], since $\partial\widetilde{\zeta}/\partial\zeta^\dagger = -1$ for $\xi = \zeta^\dagger = 0$, then there exists convergence circle with positive radius for series $\zeta^\dagger = Z_3(\xi + \xi\zeta^\dagger + (\xi + |(\lambda_\theta)_0|)((\zeta^\dagger)^2 + (\zeta^\dagger)^3 + ...)$. Consequently, there exists function $\zeta(\mathbf{x})$, and the function

$$U(\mathbf{x}) = U_0(\mathbf{x}) + \zeta(\mathbf{x}) = U_0(\mathbf{x}) + \big(\lambda_\theta - (\lambda_\theta)_0\big)\zeta_1(\mathbf{x}) + \big(\lambda_\theta - (\lambda_\theta)_0\big)^2\zeta_2(\mathbf{x}) + ..., \tag{16}$$

which is a holomorphic solution of the Hammerstein equation (8) in the vicinity $O\big((\lambda_\theta)_0\big)$ (herewith $\lim_{(\lambda_\theta)\to(\lambda_\theta)_0} U = U_0$).

We recall some of the properties of linear compact operator $\widehat{L}\phi = \lambda \int_\Omega \mathcal{K}(|\mathbf{x}-\mathbf{y}|)\phi(\mathbf{y})d\mathbf{y}$. The kernel of this operator is symmetric, weak polar type; consequently, operator $\widehat{L}$ belongs to Hilbert–Schmidt type operators. The existence of at least one characteristic function was establed by O.D. Kellogg [25]. Moreover, there exists a sequence of characteristic numbers and corresponding eigenfunctions of the investigated kernel of the linear integral operator (Theorem 146 of [26]). In accordance with [27] we can consider

$$\lambda_{\ell,j} = R_\Omega^2 \cdot \big(\varphi_j^{(\ell+1/2)}\big)^{-2}, \quad \ell \geq 0, \quad j \geq 1, \tag{17}$$

where $\varphi_j^{(\ell+1/2)}$ are the roots of the transcendental equation

$$(2\ell+1)J_{\ell+1/2}\big(\varphi_j^{(\ell+1/2)}\big) + \frac{\varphi_j^{(\ell+1/2)}}{2}\big(J_{\ell-1/2}\big(\varphi_j^{(\ell+1/2)}\big) - J_{\ell+3/2}\big(\varphi_j^{(\ell+1/2)}\big)\big) = 0, \tag{18}$$

where $J_\nu(...)|_{\nu\in\mathbb{R}}$ refers to the Bessel function of fractional order. The eigenfunctions corresponding to each eigenvalue $\lambda_{\ell,j}$ can be represented, in spherical coordinates, in the form $\phi_{\ell,j,m}(r,\theta,\chi) = J_{\ell+1/2}(\sqrt{\lambda_{\ell,j}}r)Y_\ell^m(\theta,\chi)$ ($|m| \leq \ell$), $Y_\ell^m(\theta,\chi) = P_\ell^m\big(\cos(\theta)\big)\cos(m\chi)$. It should be noted, we can consider Hammerstein equation (8) for an arbitrarily shaped region Ω, but for spherical region we assume $\ell = m = 0$ ($Y_0^0 = \sqrt{1/(4\pi)}$). The kernel $\widetilde{\mathcal{K}}(\mathbf{x},\mathbf{y}) = |\mathbf{x}-\mathbf{y}|^{-1}\omega_1(\mathbf{y}, U_0(\mathbf{y}))$ of the Fredholm equation for potential belongs to Schmidt class, and can be transformed to symmetrical form:

$$\widetilde{\mathcal{K}}(\mathbf{x},\mathbf{y}) = \sqrt{\omega_1(\mathbf{y})/\omega_1(\mathbf{x})}|\mathbf{x}-\mathbf{y}|^{-1}\sqrt{\omega_1(\mathbf{y})\omega_1(\mathbf{x})}. \tag{19}$$

In this case the resolvent for kernel $\widetilde{\mathcal{K}}(\mathbf{x},\mathbf{y})$ is equal to $\sqrt{\omega_1(\mathbf{y})/\omega_1(\mathbf{x})}R_1(\mathbf{x},\mathbf{y};\lambda)$, where R_1 is a resolvent for symmetric kernel $|\mathbf{x}-\mathbf{y}|^{-1}\sqrt{\omega_1(\mathbf{y})\omega_1(\mathbf{x})}$. In fact, $\omega_1(\mathbf{y})$ is a wight for Newtonian kernel. We denote $\lambda_1 \equiv (\lambda_\theta)_0 = \lambda_{0,1}(\widetilde{\mathcal{K}})$ the characteristic number to which the eigenfunction corresponds $\phi_1 \sim J_{1/2}(\sqrt{\lambda_1}r)$. For simplification of further calculations we will assume $\omega_1 \equiv 1$ (this can always be accomplished by multiplying of left-hand and right-hand sides of (9) by $\sqrt{\omega_1}$ and redefining of $\sqrt{\omega_1}\zeta \to \zeta'$, $\omega_0/\sqrt{\omega_1} \to \omega_0'$ etc.). We'll be looking for solutions of (9) in the form of a series $\zeta = \xi^i\zeta_i|_{i=1,2,...}$. After substituting the last series we obtain the system of the recurrent integral equations:

$$\zeta_1(\mathbf{x}) = \lambda_1 \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\zeta_1(\mathbf{y})d\mathbf{y} + \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\omega_0 d\mathbf{y}, \tag{20}$$

$$\zeta_2(\mathbf{x}) = \lambda_1 \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\zeta_2(\mathbf{y})d\mathbf{y} + \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\big(\lambda_1\omega_2\zeta_2^2(\mathbf{y}) + \zeta_1(\mathbf{y})\big)d\mathbf{y}, ...$$

According to the Fredholm alternative, the condition of existence of the solution of the first equation of the system will be orthogonality of characteristic function ϕ_1 and the second term in the right-hand side of (18): $\widehat{\mathfrak{H}}(\phi_1) \equiv \int_\Omega \omega_0(\mathbf{y})\phi_1(\mathbf{y})d\mathbf{y} = 0$. This case is physically unrealizable (for $\phi_1, \phi_2, ...$), which can be checked directly; this fact means the absence (in the neighborhood of the characteristic value λ_k, $k \geq 1$) of the analytic solution of the nonlinear equation for gravitational potential. Therefore we turn to the case $\widehat{\mathfrak{H}}(\phi_1) \neq 0$. Then equation (12) is unsolvable (condition of Fredholm theorem is absent), and consequently, the analytic series $\zeta = \xi^i\zeta_i$ (for integer indices and powers) doesn't exist. However, we can consider the representation $\zeta(\mathbf{x})$ as a Puiseux series: $\zeta = \xi^{k/2}\zeta_k|_{k=1,2,...}$. Let's denote $\xi^{1/2} \equiv \nu$, then equation (9) takes the form

$$\zeta(\mathbf{x}) = \nu^2 \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\big(\omega_0 + \zeta(\mathbf{y}) + \omega_2\zeta^2(\mathbf{y}) + ...\big)d\mathbf{y} + \tag{21}$$

$$\lambda_1 \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\big(\zeta(\mathbf{y}) + \omega_2\zeta^2(\mathbf{y}) + ...\big)d\mathbf{y},$$

and above–mentioned Puiseux series takes form $\zeta = \nu^k\zeta_k|_{k=1,2,...}$. Let's substitute this series into equation (21), we obtain an infinite system of equations:

$$\zeta_1(\mathbf{x}) = \lambda_1 \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\zeta_1(\mathbf{y})\, d\mathbf{y}, \tag{22}$$

$$\zeta_2(\mathbf{x}) = \lambda_1 \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\zeta_2(\mathbf{y})\, d\mathbf{y} + \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\big(\lambda_1\omega_2\zeta_1^2(\mathbf{y}) + \zeta_1(\mathbf{y})\big)d\mathbf{y},\ ..., \tag{23}$$

$$\zeta_n(\mathbf{x}) = \lambda_1 \int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\zeta_n(\mathbf{y})\, d\mathbf{y} + \tag{24}$$

$$\int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\big(2\lambda_1\omega_2\zeta_1(\mathbf{y})\zeta_{n-1}(\mathbf{y}) + \mathfrak{Q}(\zeta_1, ..., \zeta_{n-1})\big)d\mathbf{y}, ...$$

From (20) we obtain $\zeta_1(\mathbf{x}) = \mathfrak{C}_1 \cdot \phi_1(\mathbf{x})$, $\mathfrak{C}_1 = \mathrm{const}$. The condition (by Fredholm alternative) of existence of solution of equation (23) can be written as

$$\int_\Omega\int_\Omega |\mathbf{x}-\mathbf{y}|^{-1}\big(\lambda_1\omega_2\zeta_1^2(\mathbf{y}) + \omega_0\big)\phi_1(\mathbf{x})d\mathbf{x}d\mathbf{y} = 0, \tag{25}$$

or, after integration by the variable $\mathbf{x}$: $\int \big(\lambda_1\omega_2\zeta_1^2(\mathbf{y}) + \omega_0\big)\phi_1(\mathbf{y})d\mathbf{y} = 0$. If we substitute in this formula the obtained above expression $\zeta_1 = \mathfrak{C}_1 \cdot \phi_1$, then for constant $\mathfrak{C}_1$ we have an explicit expression

$$\mathfrak{C}_1 = \pm\sqrt{-\frac{\int_\Omega \omega_0\phi_1(\mathbf{y})d\mathbf{y}}{\int_\Omega \lambda_1\omega_2\zeta_1^3(\mathbf{y})d\mathbf{y}}} \tag{26}$$

The equation (23) may be written as

$$\zeta_2(\mathbf{x}) = \mathcal{P}_2(\mathbf{x}) + \mathfrak{C}_1\phi_1(\mathbf{x}), \quad \mathcal{P}_2(\mathbf{x}) \equiv \sum_{j=2,...} \frac{\phi_j(\mathbf{x})}{\lambda_j - \lambda_1} \int_\Omega \big(\lambda_1\omega_2\zeta_1^2(\mathbf{y}) + \omega_0\big)\phi_j(\mathbf{y})d\mathbf{y}. \tag{27}$$

In general case $n \geq 3$ we obtain:

$$\zeta_3(\mathbf{x}) = \mathcal{P}_2(\mathbf{x}) + \mathfrak{E}_3\phi_1(\mathbf{x}), \tag{28}$$

$$\mathcal{P}_2(\mathbf{x}) \equiv \sum_{j=2,...} \frac{\phi_j}{\lambda_j - \lambda_1} \int_\Omega \big(2\lambda_1\omega_2\zeta_1(\mathbf{y})\zeta_2(\mathbf{y}) + \mathfrak{Q}(\zeta_1, \zeta_2)\big)\phi_j(\mathbf{y})d\mathbf{y},$$

$$\zeta_n(\mathbf{x}) = \mathcal{P}_n(\mathbf{x}) + \mathfrak{E}_n\phi_1(\mathbf{x}), \tag{29}$$

$$\mathcal{P}_n(\mathbf{x}) \equiv \sum_{j=2,...} \frac{\phi_j}{\lambda_j - \lambda_1} \int_\Omega \big(2\lambda_1\omega_2\zeta_1(\mathbf{y})\zeta_{n-1}(\mathbf{y}) + \mathfrak{Q}(\zeta_1, \zeta_2, ..., \zeta_{n-1})\big)\phi_j(\mathbf{y})d\mathbf{y},$$

$$\frac{\mathfrak{E}_n}{\mathfrak{E}_1}\int_\Omega \omega_0\phi_1(\mathbf{y})d\mathbf{y} = \int_\Omega \big(2\lambda_1\omega_2\zeta_1(\mathbf{y})\mathcal{P}_n(\mathbf{y}) + \mathfrak{Q}(\zeta_1, \zeta_2, ..., \zeta_{n+1})\big)\phi_1(\mathbf{y})d\mathbf{y}.$$

Here $\mathcal{P}_n(\mathbf{x})$ can be associated with A. Ruston pseudoresolvent [28]. Consequently, we can write the series $\zeta = \xi^{k/2}\zeta_k|_{k=1,2,...}$, this series formally satisfies to the equation (9); we ought to demonstrate the convergence of this series in the neighborhood $O(\xi)$. Let introduce: 1) constant $\mathfrak{X}_0$, defined by condition $|\mathfrak{E}_1\phi_1(\mathbf{x})| = |\zeta_1(\mathbf{x})| < \mathfrak{X}_0$ $(\forall \mathbf{x} \in \Omega)$; 2) function $S(z) = (|\lambda_1| + \nu^2)\mathfrak{M}z^2/(1-z/\rho^\ddagger) + \nu^2(\omega_0^{(m)} + z)$, where $|\omega_0| < \omega_0^{(m)}$, $\rho^\ddagger \in (0, \rho^\ddagger_{max})$, $\rho^\ddagger_{max}$ is a convergence radius of the series $\omega_2(\mathbf{x}) + \omega_3(\mathbf{x})z + \omega_4(\mathbf{x})z^2 + ...$, $|\omega_2(\mathbf{x})| < \mathfrak{M}$. We substitute in the definition $S(z)$ the decomposition $z = \nu\mathfrak{X}_0 + \nu^2(\mathfrak{X}_1 + \mathfrak{Y}_1) + \nu^3(\mathfrak{X}_1 + \mathfrak{Y}_1) + ...$ and formally obtain series $S(z) = \nu^2 S_2 + \nu^3 S_3 + ...$ The value $S_n(z)$ is a majorant function for expression $2\lambda_1\omega_2\zeta_1(\mathbf{x})\zeta_{n-1}(\mathbf{x}) + \mathfrak{Q}(\zeta_1, \zeta_2, ..., \zeta_n)$ for condition: $\mathfrak{X}_n + \mathfrak{Y}_n$ is a majorant function for $\zeta_{n+1}(\mathbf{x})$, $n = 1, 2, ...$

We denote: 1) $(1/\mathfrak{E}_1)\int_\Omega \omega_0\phi_1(\mathbf{x})d\mathbf{x} = \mathfrak{N}(= \text{const})$; 2) $\mathfrak{Y}_n = S_{n+1}\rho^\ddagger_{max}$, 3) $(\mathfrak{N}+2|\lambda_1|\mathfrak{M}\mathfrak{X}_0a^2)\mathfrak{X}_n = S_{n+1}a^2$ $(a > |\phi_1(\mathbf{x})|)$. Consequently, $|\mathcal{P}_n(\mathbf{x})| < \mathfrak{Y}_n$, $\mathfrak{E}_{n+1} < \mathfrak{X}_n$. Let introduce functions $\mathfrak{X} \equiv \nu^2\mathfrak{X}_1 + \nu^3\mathfrak{X}_2 + ...$, $\mathfrak{Y} \equiv \nu^2\mathfrak{Y}_1 + \nu^3\mathfrak{Y}_2 + ...$, then above given definition of variables $\mathfrak{X}_1, ..., \mathfrak{X}_n, ...$ and $\mathfrak{Y}_1, ..., \mathfrak{Y}_n, ...$ is equivalent to solving of the system of equations:

$$\mathfrak{Y} = \rho^\ddagger_{max}\Big((|\lambda_1| + \nu^2)\frac{\mathfrak{M}(\nu\mathfrak{X}_0 + \mathfrak{X} + \mathfrak{Y})^2}{1 - (\nu\mathfrak{X}_0 + \mathfrak{X} + \mathfrak{Y})/\rho^\ddagger} + \nu^2(\omega_0^{(m)} + \nu\mathfrak{X}_0 + \mathfrak{Y} + \mathfrak{X})\Big), \tag{30}$$

$$(|\mathfrak{N}| + 2|\lambda_1|\mathfrak{M}\mathfrak{X}_0a^2)\nu\mathfrak{X} = a^2\Bigg((\nu^2 + |\lambda_1|)\frac{\mathfrak{M}(\nu\mathfrak{X}_0 + \mathfrak{X} + \mathfrak{Y})^2}{1 - (\nu\mathfrak{X}_0 + \mathfrak{X} + \mathfrak{Y})/\rho^\ddagger} + \tag{31}$$

$$+\nu^2\big(\rho^\ddagger_{max} + \nu\mathfrak{X}_0 + \mathfrak{X} + \mathfrak{Y}\big) - \nu^2(\rho^\ddagger_{max} + \mathfrak{M}|\lambda_1|\mathfrak{X}_0^2)\Bigg).$$

Let's replace the variables: $\mathfrak{X} = \nu\mathfrak{X}^\dagger$, $\mathfrak{Y} = \nu\mathfrak{Y}^\dagger$. Then the system (30)–(31) takes the form

$$\Theta_1 = \mathfrak{Y}^\dagger - \nu\mathfrak{Y}^\dagger\frac{\mathfrak{X}_0 + \mathfrak{Y}^\dagger + \mathfrak{X}^\dagger}{\rho^\ddagger} - \rho^\ddagger_{max}\Bigg((\nu^2 + |\lambda_1|)\mathfrak{M}\nu(\mathfrak{X}_0 + \mathfrak{Y}^\dagger + \mathfrak{X}^\dagger) + \tag{32}$$

$$+\nu\big(\rho^\ddagger_{max} + \nu(\mathfrak{X}_0 + \mathfrak{Y}^\dagger + \mathfrak{X}^\dagger)\big)\big(1 - \nu\frac{\mathfrak{X}_0 + \mathfrak{Y}^\dagger + \mathfrak{X}^\dagger}{\rho^\ddagger})\Bigg) = 0,$$

$$\Theta_2 = (|\mathfrak{N}| + 2|\lambda_1|\mathfrak{X}_0\mathfrak{M}a^2)\mathfrak{X}^\dagger - (|\mathfrak{N}| + 2|\lambda_1|\mathfrak{X}_0\mathfrak{M}a^2)\mathfrak{X}^\dagger\nu\frac{\mathfrak{X}_0 + \mathfrak{Y}^\dagger + \mathfrak{X}^\dagger}{\rho^\ddagger} - \tag{33}$$

$$-(|\lambda_1| + \nu^2)\mathfrak{M}(\mathfrak{X}_0 + \mathfrak{Y}^\dagger + \mathfrak{X}^\dagger)^2a^2 -$$

$$-\big(\nu(\mathfrak{X}_0+\mathfrak{Y}^\dagger+\mathfrak{X}^\dagger)-\mathfrak{M}|\lambda_1|\mathfrak{X}_0^2\big)\big(1-\nu\frac{\mathfrak{X}_0+\mathfrak{Y}^\dagger+\mathfrak{X}^\dagger}{\rho^\ddagger}\big)=0.$$

The Jacobi determinant of the last system of equations:

$$\Delta=D(\Theta_1,\Theta_2)/D(\mathfrak{X}^\dagger,\mathfrak{Y}^\dagger)=-|\mathfrak{N}|<0,\ \mathfrak{X}^\dagger=\mathfrak{Y}^\dagger=\nu=0.$$

Consequently, series defined variables $\mathfrak{X}$ and $\mathfrak{Y}$, are converge in the vicinity of the point $\nu=0$. The series $\nu\mathfrak{X}_0+\nu^2(\mathfrak{X}_1+\mathfrak{Y}_1)+\nu^3(\mathfrak{X}_2+\mathfrak{Y}_2)+...$ has convergence circle (with center in the point $\nu=0$). This fact write to us the Puiseux series $\zeta=\xi^{k/2}\zeta_k|_{k=1,2,...}$ converges in the vicinity of the point $\xi=0$.

We can conclude that Hammerstein equation for gravitational potential in the vicinity $O(\lambda_1)\ni\lambda$ (where λ_1 is one of the characteristic values of linear Fredholm equation for Newton potential) have two nonholomorphic solutions of the form

$$U(\mathbf{x})=U_0(\mathbf{x})+(\lambda_\theta-(\lambda_\theta)_0)^{1/2}\zeta_1(\mathbf{x})+(\lambda_\theta-(\lambda_\theta)_0)\zeta_2(\mathbf{x})+(\lambda_\theta-(\lambda_\theta)_0)^{3/2}\zeta_3(\mathbf{x})+... \tag{34}$$

Now we return to the equation (8) and assumption: $\big((\lambda_\theta)_0,U_0\big)$ is a characteristic value and eigenvalue of Hammerstein operator $\widehat{\mathfrak{G}}(U)$. But in accordance with the Schauder principle (see Theorems 2.20–2.23 in [18]) this operator has a continuum of eigenfunctions: if $1/(\lambda_\theta)_0\in\sigma(\widehat{\mathfrak{G}})$, then $\big(1/(\lambda_\theta)_0-\epsilon_*;\ 1/(\lambda_\theta)_0+\epsilon_*\big)\equiv C\sigma_{(\lambda_\theta)_0}(\widehat{\mathfrak{G}})\subseteq\sigma(\widehat{\mathfrak{G}})$. Thus, any point $1/\tilde{\lambda}_\theta$ of pseudo-continuous spectrum $C\sigma_{\lambda_\theta}$ can replace in the previous calculations the characteristic value $(\lambda_\theta)_0$; this value corresponds to the element $\tilde{U}_0$ of $\mathrm{Ker}\big(\widehat{\mathrm{I}}-\tilde{\lambda}_\theta\widehat{\mathfrak{G}}\big)$.

In physical terms the expression (34) can written as

$$\Phi(\mathbf{x})=(C_1^\ddagger+c^2\Lambda/12)|\mathbf{x}|^2+U_0(\mathbf{x})\sum_i\big(ANG\mathcal{J}(\theta)\exp(C_1^\ddagger/\theta)-(\lambda_\theta)_0\big)^{i/2}, \tag{35}$$

where $U_0(\mathbf{x})$ is a solution of the Hammerstein equation for the potential (existing due to the properties of the corresponding integral operator), $\lambda_1(=\lambda_\theta)_0)$ is the characteristic value of kernel $\omega_1/|\mathbf{x}-\mathbf{y}|$ (for Fredholm equation) in the vicinity of the point $\lambda_\theta\in((\lambda_\theta)_0-\epsilon,(\lambda_\theta)_0+\epsilon)$. If $sgn\big(\int_\Omega\omega_0\phi_1(\mathbf{y})d\mathbf{y}\big)\neq sgn\big(\int_\Omega\lambda_1\omega_2\phi_1^3(\mathbf{y})d\mathbf{y}\big)$, then for $\lambda>\lambda_1$ there exist two solutions, and for $\lambda<\lambda_1$ there aren't solutions. For $sgn\big(\int_\Omega\omega_0\phi_1(\mathbf{y})d\mathbf{y}\big)=sgn\big(\int_\Omega\lambda_1\omega_2\phi_1^3(\mathbf{y})d\mathbf{y}\big)$, then for $\lambda<\lambda_1$ there exist two solutions, and for $\lambda>\lambda_1$ there aren't solutions.

Thus, it has been shown that the solution of the system of Vlasov-Poisson equations, in the case of using the energy substitution into the stationary Vlasov equation and choosing the quasi-Maxwell distribution (depending on the kinetic temperature) as a basis solution of the kinetic equation, leads to the nonlinear Hammerstein integral equation for the potential (including antigravity due to the presence of the cosmological term). The Hammerstein equation is in essence is an equation for the self-consistent field in the system of many massive particles on cosmological scales. using methods of nonlinear functional analysis, it can be shown that (depending on the properties of the Fredholm equation, which is a linearized version of the gravitational potential equation) there are two possibilities of continuation of the solution of the initial Hammerstein equation from the known solution - analytical and algebraic. The analytical method leads to a smooth unambiguous dependence of the constructed extended solution-potential on the parameters (kinetic temperature, cosmological term): in a conditionally equilibrium state of the background of the Milne-McCree problem, the gravitational potential does not introduce into the cosmological dynamics any fundamental additions to the Newtonian potential (at positive kinetic temperature, at transition to non-equilibrium temperature, at transition to the non-equal temperature, in the transition to non-equilibrium states with negative temperature (due to the dominance of the antigravity effect in the system at large distances) the situation may change due to a radical change in the properties of the kernel of the Fredholm equation, which will lead to the disappearance of analyticity of the continuation on the parameters of the basic solution and creation of diplet structures of large scales. The second possibility is the emergence in the system near the basic solution of the region of nonholomorphic continuation of solutions on parameters. The existence of a pair of branches of solutions is caused (see formula (26)) by the occurrence of partial pseudopotentials in the post-linear version of the equation. of different signs. This leads to the simultaneous realization in the cosmological system with self-consistent field of two types of ordering — a significant increase of the of matter density and its decrease (at one-dimensional motion of matter in the channel there appear strata-walls with high value of gravitational potential, and voids — practically empty spaces between walls). Oscillations of eigenfunctions even in the simplest case of spherical symmetry of the Green's function can form analogs of interference patterns, which leads to the production of structure formation in a homogeneous medium far from the line connecting the massive objects. Since the eigenfunctions of the linearized version of the equation for the potential are proportional to the Bessel function (with a rapidly decreasing weight function), the loss of periodicity of solutions in physical space becomes obvious. Since for nonlinear integral operators it is possible to establish the the continuum character of the spectrum, one can observe the effect of secondary solutions (on the hyperplane of parameters near the basic solution of the equation) coexisting with the initial ones, which leads to the construction of the second type of periodicity - translation of the solution isotropically along all directions. Thus, the composition of solutions with a potential of two types in the presence of additional translational transgression due to the structure of the spectrum, give as a result a distribution of field and matter in space similar to the cosmic web. The change of the kinetic temperature at the zero-point transition leads again to dipole structures at large distances.

4. Conclusion

It can be argued that taking into account the modified law gravitational interaction of Newton–Gurzadyan allows you to make not only a qualitative, but also a quantitative assessment dimensions of cosmological structures, which is associated with the two-stream Hubble model and size matching facts related to formal inflection points the modified law of gravity and the observed sizes of voids.

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