Research Article

The Spectral Theory Of Periodic Ordinary Differential Equations

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The spectral theory of periodic differential equations is a branch of mathematics that look into and study of the eigenvalues and eigenfunctions of differential operators with periodic coefficients. These equations appear naturally in many scientific and engineering applications in physics, biology, and mechanics, particularly in the modeling of systems with periodic behavior such as oscillatory systems in physics and biology.

One of the key features of periodic differential equations is the periodicity in the coefficients of the differential operators. This periodicity introduces a mathematical structure that leads to unique properties in the spectral theory of these equations. Unlike the case of constant coefficients, where the eigenvalues and eigenfunctions are typically well-defined, the periodic of the coefficients rises to a more complex spectrum.

The study of the spectral theory of periodic differential equations involves understanding the properties of the spectrum, including the existence and uniqueness of eigenvalues, the behavior of eigenfunctions, and the stability of solutions. This theory plays an important role in the analysis and prediction of the behavior of periodic systems, providing insights into their long-term dynamics and stability.

In this paper, we will explore the spectral theory of periodic differential equations such as Floquet's equation, Hill's equation and Mathieu's equation. Also, we will concentrate on properties of the solutions of them.

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1. Introduction and preliminaries

The spectral theory of periodic ordinary differential equations (ODEs) is an essential area of mathematics that deals with the study of the eigenvalues and eigenfunctions of differential operators with periodic coefficients. These equations appear in science and engineering, particularly focusing on the modeling of systems with periodic behavior, such as oscillatory systems in physics and biology. The aim of the paper is to look deeply at some periodic ordinary differential equations and the characterization of their solutions such as Floquet's equation, Hill's equations and The Mathieu's equations.

1.1. Floquet's equation

Let us briefly show some interesting results about the Floquet's equation.

The Floquet's equation is given by

$$c_0(x)y''(x) + c_1(x)y'(x) + c_2(x)y(x) = 0,$$

where c_0, c_1, c_2 are complex valued functions, piecewise continuous and periodic, all with the same period $q \in \mathrm{R}$ and $q \neq 0$. To ensure that there is no singular point, we assume that the left and right-hand limits of $c_0(x)$ at every point are non-zero. It is clear if $\varphi(x)$ is a solution of (1), then $\varphi(x+q)$ is also a solution of (1).

The following theorem is crucial.

Theorem 1.1. *There are a non- zero constant β and a non-trivial solution $\varphi(x)$ of (1) such that*

$$\varphi(x+q) = \beta\varphi(x).$$

Proof. Let $\psi_1(x)$ and $\psi_2(x)$ are two linearly independent solutions of (1) which satisfy the initial conditions:

$$\psi_1(0) = 1, \quad \psi_1'(0) = 0, \quad \psi_2(0) = 0, \quad \psi_2'(0) = 1.$$

Since $\psi_1(x+q), \psi_2(x+q)$ are linearly independent solutions of (1), it follows from the existence and uniqueness theorem for solutions of the initial value problem for ordinary differential equation that any solution of equation (1) is linear combination of ψ_1 and ψ_2 therefore there are constants $A_{ij} (1 \leq i, j \leq 2)$ such that

$$\psi_1(x+q) = A_{11}\psi_1(x) + A_{12}\psi_2(x).$$

$$\psi_2(x+q) = A_{21}\psi_1(x) + A_{22}\psi_2(x).$$

Therefore,

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$$\begin{pmatrix} \psi_1(x+q) \\ \psi_2(x+q) \end{pmatrix} = \begin{pmatrix} A_{11}\psi_1(x) + A_{12}\psi_2(x) \\ A_{21}\psi_1(x) + A_{22}\psi_2(x) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix},$$

where the matrix $A = (A_{ij})_{(1 \le i,j \le 2)}$ is non-singular i.e. $\det(A) \neq 0$ (if $\det(A) = 0$, then $\psi_1(x+a), \psi_2(x+a)$ would be linearly dependent and this is a contradiction).[1]

We know from the essential property of linear second differential equations that every solution $\varphi(x)$ of (1) can be written as a linear combination form of ψ_1, ψ_2. i.e. any solution $\psi(x)$ can be written as

$$\varphi(x) = \alpha_1\psi_1(x) + \alpha_2\psi_2(x)$$

where α_1, α_2 are constants. This property can be proved as follows:

$$\begin{aligned} & c_0(x)\varphi''(x) + c_1(x)\varphi'(x) + c_2(x)\varphi(x) \\ & \quad = c_0(x)\Big(\alpha_1\psi_1(x) + \alpha_2\psi_2(x)\Big)'' + c_1(x)\Big(\alpha_1\psi_1(x) + \alpha_2\psi_2(x)\Big)' + c_2(x)\Big(\alpha_1\psi_1(x) + \alpha_2\psi_2(x)\Big) \\ & \quad = \alpha_1\Big(c_0(x)\psi_1''(x) + c_1(x)\psi_1'(x) + c_2(x)\psi_1(x)\Big) + \alpha_2\Big(c_0(x)\psi_2''(x) + c_1(x)\psi_2'(x) + c_2(x)\psi_2(x)\Big) \\ & \quad = 0, \end{aligned}$$

because $\psi_1(x)$ and $\psi_2(x)$ are solutions of (1). [2]

We need to prove that (2) holds if

$$\begin{aligned} (A_{11} - \beta)\alpha_1 + A_{21}\alpha_2 &= 0, \\ (A_{22} - \beta)\alpha_2 + A_{12}\alpha_1 &= 0. \end{aligned}$$

The proof for that is as follows:

$$\begin{aligned} \varphi(x+q) &= \alpha_1\psi_1(x+q) + \alpha_2\psi_2(x+q) \\ &= \alpha_1 A_{11}\psi_1(x) + \alpha_1 A_{12}\psi_2(x) + \alpha_2 A_{21}\psi_1(x) + \alpha_2 A_{22}\psi_2(x) \\ &= (\alpha_1 A_{11} + \alpha_2 A_{21})\psi_1(x) + (\alpha_1 A_{12} + \alpha_2 A_{22})\psi_2(x) \\ &= \beta\alpha_1\psi_1(x) + \beta\alpha_2\phi_2(x). \end{aligned}$$

This implies that $\alpha_1 A_{11} + \alpha_2 A_{21} = \beta\alpha_1$ and $\alpha_1 A_{12} + \alpha_2 A_{22} = \beta\alpha_2$ as requested. Now, these equations are satisfied by values of α_1 and α_2 , not both zero, if β is such that

$$\begin{vmatrix} A_{11} - \beta & A_{21} \\ A_{12} & A_{22} - \beta \end{vmatrix} = 0.$$

Therefore we obtain quadratic equation for β

$$(A_{11} - \beta)(A_{22} - \beta) - A_{21}A_{12} = 0$$

i.e.

$$\beta^2 - (A_{11} + A_{22})\beta + \det(A) = 0,$$

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where

$$\det(A) = \begin{vmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{vmatrix}.$$

The equation (7) is satisfied by at least one value of β, the value being non-zero since $\det(A) \neq 0$. It follows from the initial conditions (3) and (4) and (5) that

$$\begin{aligned}
\psi_1(0+q) = \psi_1(q) = A_{11}\psi_1(0) + A_{12}\psi_2(0) = A_{11}. \\
\psi_2(0+q) = \psi_2(q) = A_{21}\psi_1(0) + A_{22}\psi_2(0) = A_{21}. \\
\psi_1'(x+q) = A_{11}\psi_1'(x) + A_{12}\psi_2'(x). \\
\psi_2'(x+q) = A_{21}\psi_1'(x) + A_{22}\psi_2'(x). \\
\psi_1'(0+q) = \psi_1'(q) = A_{11}\psi_1'(0) + A_{12}\psi_2'(0) = A_{12}. \\
\psi_2'(0+q) = \psi_2'(q) = A_{21}\psi_1'(0) + A_{22}\psi_2'(0) = A_{22}.
\end{aligned}$$

Hence, with the usual notation for the Wronskian

$$\begin{aligned}
\det(A) &= \begin{vmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{vmatrix} \\
&= \begin{vmatrix} \psi_1(q) & \psi_1'(q) \\ \psi_2(q) & \psi_2'(q) \end{vmatrix} \\
&= W(\psi_1, \psi_2)(q) \\
&= e^{-\int_0^q \frac{c_1(x)}{c_0(x)} dx}.
\end{aligned}$$

by

$$\begin{vmatrix} \psi_1(0) & \psi_1'(0) \\ \psi_2(0) & \psi_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = W(\psi_1, \psi_2)(0) = 1,$$

and Liouville's formula for Wronskian

$$W(\psi_1, \psi_2)(q) = W(\psi_1, \psi_2)(0) e^{-\int_0^q \frac{c_1(x)}{c_0(x)} dx} = e^{-\int_0^q \frac{c_1(x)}{c_0(x)} dx}.$$

Thus, the equation (7) can be written as

$$\beta^2 - (\psi_1(q) + \psi_2'(q))\rho + e^{-\int_0^q \frac{c_1(x)}{c_0(x)} dx} = 0.$$

□

Theorem 1.2. *There are linearly independent solutions $\varphi_1(x)$ and $\varphi_2(x)$ of equation (1) such that either*

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i. $\varphi_1(x) = e^{m_1 x} P_1(x), \varphi_2(x) = e^{m_2 x} P_2(x)$, *where* m_1, m_2 *are constants, not necessarily distinct and* $P_1(x), P_2(x)$ *are periodic with period* q.

ii. $\varphi_1(x) = e^{mx} P_1(x), \varphi_2(x) = e^{mx}(P_2(x) + xP_1(x))$, *where* m *is constant and* $P_1(x), P_2(x)$ *are periodic with period* q.

Proof.

i) Assume that the equation (7) has two distinct solutions $\beta_1 \neq 0, \beta_2 \neq 0$. Then by the previous Theorem 1.1, there exist two non-trivial solutions $\varphi_1(x), \varphi_2(x)$ of the equation (??)such that

$$\varphi_1(x+q) = \beta_1 \varphi_1(x). \qquad (9)$$
$$\varphi_2(x+q) = \beta_2 \varphi_2(x). \qquad (10)$$

These solutions $\varphi_1(x), \varphi_2(x)$ are linearly independent because if they were not, then $\det(A) = 0$ and this would contradict to $\det(A) \neq 0$. Since $\beta_1 \neq 0, \beta_2 \neq 0$, we can find m_1, m_2 so that $\beta_1 = e^{qm_1}, \beta_2 = e^{qm_2}$. Define

$$P_1(x) = e^{-m_1 x} \varphi_1(x).$$

$$P_2(x) = e^{-m_2 x} \varphi_2(x).$$

By (9),(10) we have

$$\begin{aligned} P_1(x+q) &= e^{-m_1(x+q)} \varphi_1(x+q) \\ &= e^{-m_1(x+q)} \beta_1 \varphi_1(x) \\ &= e^{-m_1 x} e^{-m_1 q} \beta_1 \varphi_1(x) \\ &= e^{-m_1 x} e^{-m_1 q} e^{qm_1} \varphi_1(x) \\ &= e^{-m_1 x} \varphi_1(x) = P_1(x). \\ P_2(x+q) &= e^{-m_2(x+q)} \varphi_2(x+q) \\ &= e^{-m_2 x} e^{-m_2 q} \beta_2 \varphi_2(x) \\ &= e^{-m_2 x} e^{-m_2 q} e^{am_2} \varphi_2(x) \\ &= e^{-m_2 x} \varphi_2(x) \\ &= P_2(x). \end{aligned}$$

This means $P_1(x)$ and $P_2(x)$ have a period q. Thus, by (11) and (12), we have

$$\varphi_1(x) = e^{m_1 x} P_1(x),$$

$$\varphi_2(x) = e^{m_2 x} P_2(x),$$

where $P_1(x), P_2(x)$ have period q.

ii) Assume that the equation (7) has a repeated solution $\beta_1 = \beta_2 = \beta$. Then similar to part (i), define m so that $\beta = e^{qm}$. Then, by Theorem 1.1 , there exists a non-trivial solution $\varphi_1(x)$ of equation (1) such that

$$\Psi_1(x+q) = \beta \Psi_1(x).$$

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Assume that $\Psi_2(x)$ is another solution of equation (??) which is linearly independent of $\Psi_1(x)$. $\Psi_2(x+q)$ is also solution of equation (??), therefore there are two constants $d_1 \neq 0, d_2 \neq 0$ such that

$$\Psi_2(x+a) = d_1\Psi_1(x) + d_2\Psi_2(x).$$

Our goal is to compute d_2. From (15),(16), we have

$$W(\Psi_1, \Psi_2)(x) = \begin{vmatrix} \Psi_1(x) & \Psi_1^{'}(x) \\ \Psi_2(x) & \Psi_2^{'}(x) \end{vmatrix}.$$

$$W(\Psi_1, \Psi_2)(x+q) = \begin{vmatrix} \Psi_1(x+q) & \Psi_1^{'}(x+q) \\ \Psi_2(x+q) & \Psi_2^{'}(x+q) \end{vmatrix}$$

$$= \begin{vmatrix} \beta\Psi_1(x) & \beta\Psi_1^{'}(x) \\ d_1\Psi_1(x) + d_2\Psi_2(x) & d_1\Psi_1^{'}(x) + d_2\Psi_2^{'}(x) \end{vmatrix}$$

$$= \beta d_2 W(\Psi_1, \Psi_2)(x).$$

Hence, by Liouville's theorem for the Wronskian

$$W(\Psi_1, \Psi_2)(x+q) = W(\Psi_1, \Psi_2)(x) e^{-\int_x^{x+q} \frac{c_1(x)}{c_0(x)} dx},$$

$$\beta d_2 W(\Psi_1, \Psi_2)(x) = W(\Psi_1, \Psi_2)(x) e^{-\int_x^{x+q} \frac{c_1(x)}{c_0(x)} dx}.$$

Therefore,

$$e^{-\int_x^{x+q} \frac{c_1(x)}{c_0(x)} dx} = e^{-\int_0^{q} \frac{c_1(x)}{c_0(x)} dx} = \beta d_2,$$

since the integrand has period q. But β is a repeated solution for the equation (8) therefore

$$e^{-\int_0^{q} \frac{c_1(x)}{c_0(x)} dx} = \beta^2.$$

Then $\beta d_2 = \beta^2$ .i.e. $d_2 = \beta$. Then (16) becomes

$$\Psi_2(x+q) = d_1\Psi_1(x) + \beta\Psi_2(x).$$

There are now two subcases:

- **Case 1:** If $d_1 = 0$, we have $\Psi_2(x+q) = \beta\Psi_2(x)$. This together with (15) shows that we have the same situation as
  $\varphi_1(x) = e^{m_1 x}P_1(x).$
  $\varphi_2(x) = e^{m_2 x}P_2(x).$

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But $\beta_1 = \beta_2 = \beta$. Hence, as in a part (i) of proof of this theorem with $\varphi_1(x) = \Psi_1(x), \varphi_2(x) = \Psi_2(x)$ and $m_1 = m_2 = m$.

- **Case 2:** If $d_1 \neq 0$. Then, define

  $P_1(x) = e^{-mx}\Psi_1(x).$

  $P_2(x) = e^{-mx}\Psi_2(x) - \frac{d_1}{q\beta}xP_1(x).$

  Then by (9) and (10), we can prove that $P_1(x)$ and $P_2(x)$ have period q as follows:

$$\begin{aligned} P_1(x+q) &= e^{-m(x+q)}\Psi_1(x+q) \\ &= e^{-m(x+q)}\beta\Psi_1(x) \\ &= e^{-mx}e^{-mq}\beta\Psi_1(x) \\ &= e^{-mx}e^{-mq}e^{mq}\Psi_1(x) \\ &= e^{-mx}\Psi_1(x) \\ &= P_1(x). \end{aligned}$$

$$\begin{aligned} P_2(x+q) &= e^{-m(x+q)}\Psi_2(x+q) - \frac{d_1}{q\beta}(x+q)P_1(x+q) \\ &= e^{-mx}e^{-mq}(d_1\Psi_1(x) + \beta\Psi_2(x)) - \frac{d_1}{q\beta}(x+q)P_1(x) \\ &= e^{-mx}e^{-mq}d_1\Psi_1(x) + e^{-mx}e^{-mq}\beta\Psi_2(x) - \frac{d_1}{q\beta}xP_1(x) - \frac{d_1}{\beta}P_1(x) \\ &= e^{-mx}e^{-mq}e^{mq}\Psi_2(x) - \frac{d_1}{q\beta}xP_1(x) + e^{-mq}d_1P_1(x) - \frac{d_1}{\beta}P_1(x) \\ &= e^{-mx}\Psi_2(x) - \frac{d_1}{q\beta}xP_1(x) \\ &= P_2(x). \end{aligned}$$

Therefore

$$\Psi_1(x) = e^{mx}P_1(x).$$

$$\Psi_2(x) = e^{mx}\left(P_2(x) + \frac{d_1}{q\beta}xP_1(x)\right).$$

  This sub-case comes under the second part of the theorem with $\varphi_1(x) = \Psi_1(x)$ and $\varphi_2(x) = \frac{q\beta}{d_1}\Psi_2(x)$.

The proof now is complete. i.e.

$$\begin{aligned} \varphi_1(x) &= \Psi_1(x) \\ &= P_1(x)e^{mx}. \\ \varphi_2(x) &= \frac{q\beta}{d_1}\Psi_2(x) \\ &= \frac{q\beta}{d_1}e^{mx}\left(P_2(x) + \left(\frac{d_1}{q\beta}xP_1(x)\right)\right) \\ &= e^{mx}\left(xP_1(x) + P_2(x)\right). \end{aligned}$$

□

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2. Hill's equation

The name of Hill's equations is given to the equation

$$\left(P(x)y'(x)\right)' + Q(x)y(x) = 0,$$

where $P(x)$, $Q(x)$ are periodic real- valued functions with the same period q. Additionally, it is assumed that $P(x)$ is continuous and nowhere zero and that $P'(x)$ and $Q(x)$ are piecewise continuous. Thus, (18) is a case of equation (1).

We mention here two ways in which equation (1) with real coefficients can be transformed into an equation of the type (18). First, let

$$\int_0^q \frac{c_1(t)}{c_0(t)} dt = 0.$$

If equation (1) is multiplied by

$$c_0^{-1}(x)e^{A(x)} = c_0^{-1}(x)e^{\int_0^x \frac{c_1(t)}{c_0(t)} dt},$$

where $A(x) = \int_0^x \frac{c_1(t)}{c_0(t)} dt$, then

$$e^{A(x)}y''(x) + \frac{c_1(x)}{c_0(x)}e^{A(x)}y'(x) + \frac{c_2(x)}{c_0(x)}e^{A(x)}y(x) = 0$$

has the formula of Hill's equation by assumption

$$P(x) = e^{A(x)}.$$
$$Q(x) = \frac{c_2(x)}{c_0(x)}e^{A(x)}.$$
$$P'(x) = A'(x)e^{A(x)} = \frac{c_1(x)}{c_0(x)}e^{A(x)}.$$

By (19), $A(x)$ has the period q. i.e. $A(x+q) = A(x)$ and therefore $P(x)$ and $Q(x)$ defined by (22) and (23) have the same period q. Second, instead of (19), let $\frac{c_1(x)}{c_0(x)}$ has a piecewise continuous derivative and put

$$y(x) = z(x)e^{-\frac{1}{2}A(x)}.$$

Then

$$y'(x) = \left(z'(x) - \frac{1}{2}A'(x)z(x)\right)e^{-\frac{1}{2}A(x)},$$

$$y''(x) = \left(z''(x) - z'(x)A'(x) + \left(\frac{1}{4}A'^2(x) - \frac{1}{2}A''(x)\right)z(x)\right)e^{-\frac{1}{2}A(x)}.$$

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Subtitute (24), (25) and (26) in the Floquet's equation (1), and by using

$$A'(x) = \frac{c_1(x)}{c_0(x)},$$
$$A''(x) = \frac{c_1'(x)c_0(x) - c_0'(x)c_1(x)}{c_0^2(x)}$$

we have

$$\begin{aligned}
& c_0(x)\left((z''(x) - z'(x)A'(x)) + \left(\frac{1}{4}A'^2(x) - \frac{1}{2}A''(x)\right)z(x)\right) \\
& + c_1(x)\left(z'(x) - \frac{1}{2}A'(x)z(x)\right) + c_2(x)z(x) \\
& = c_0(x)z''(x) + \left(c_1(x) - c_0(x)A'(x)\right)z'(x) \\
& \quad + \left(c_2(x) + c_0(x)\left(\frac{1}{4}A'^2(x) - \frac{1}{2}A''(x)\right) - \frac{1}{2}c_1(x)A'(x)\right)z(x) \\
& = c_0(x)z''(x) + \left(c_1(x) - c_0(x)\frac{c_1(x)}{c_0(x)}\right)z'(x) \\
& \quad + \left(c_2(x) + c_0(x)\left(\frac{1}{4}\left(\frac{c_1(x)}{c_0(x)}\right)^2 - \frac{1}{2}\left(\frac{c_1(x)}{c_0(x)}\right)'\right) - \frac{1}{2}c_1(x)\frac{c_1(x)}{c_0(x)}\right)z(x) \\
& = c_0(x)z''(x) + \left(c_2(x) + \frac{1}{4}\frac{c_1^2(x)}{c_0(x)} - \frac{1}{2}c_0(x)\left(\frac{c_1(x)}{c_0(x)}\right)' - \frac{1}{2}\frac{c_1^2(x)}{c_0(x)}\right)z(x) \\
& = c_0(x)z''(x) + \left(c_2(x) - \frac{1}{4}\frac{c_1^2(x)}{c_0(x)} - \frac{1}{2}c_0(x)\left(\frac{c_1(x)}{c_0(x)}\right)'\right)z(x) = 0.
\end{aligned}$$

Then divide by $c_0(x) \neq 0$, we have

$$z''(x) + \left(\frac{c_2(x)}{c_0(x)} - \frac{1}{4}\left(\frac{c_1(x)}{c_0(x)}\right)^2 - \frac{1}{2}\left(\frac{a_1(x)}{a_0(x)}\right)'\right)z(x) = 0$$

which, because the coefficient of $z(x)$ is periodic, has the form of (18) with $P(x) = 1$ and $z(x)$ in place of $y(x)$.

Now, we examine in more details the solutions $\psi_1(x)$ and $\psi_2(x)$ given by Theorem 1.2 as applied to (18). For (18), since $e^{-\int_0^q \frac{c_1(t)}{c_0(t)}dt} = e^{-A(q)} = 1$ where $A(q) = \int_0^q \frac{c_1(t)}{c_0(t)}dt = 0$ therefore the equation (8) becomes

$$\beta^2 - (\psi_1(q) + \psi_2'(q))\beta + 1 = 0$$

and the characteristic multipliers β_1 and β_2 therefore satisfy $\beta_1\beta_2 = 1$. The solutions $\psi_1(x)$ and $\psi_2(x)$ of (18) which satisfy the initial condition (3) are real-valued because $P(x)$ and $Q(x)$ are real-valued.

Define a real number

$$D = \psi_1(q) + \psi_2'(q),$$

and it is called the discriminant of the equation (18) therefore the equation (27) becomes

$$\beta^2 - D\beta + 1 = 0.$$

We have five cases which can be collected in the following theorem:

Theorem 2.1. *We have the following cases:*

- **Case I:** *If $D > 2$, then the quadratic equation (29) has two positive real distinct solutions $\beta_1 \neq 1$ and $\beta_2 \neq 1$. But $\beta_1\beta_2 = 1$. Thus, there is real a non-zero number m such that $\beta_1 = e^{qm}$, $\beta_2 = e^{-qm}$. Thus, by part(i) of Theorem 1.2 , $\varphi_1(x) = e^{mx}P_1(x)$ and $\varphi_2(x) = e^{-mx}P_2(x)$, where $P_1(x)$ and $P_2(x)$ have period q.*
- **Case II:** *If $D < -2$, then the quadratic equation (29) has two negative real distinct solutions $\beta_1 \neq 1$ and $\beta_2 \neq 1$. But $\beta_1\beta_2 = 1$. Thus, there is real a non-zero number $m + \frac{i\pi}{q}$ such that $\beta_1 = e^{qm+i\pi}$, $\beta_2 = e^{-qm-i\pi}$. Thus, by part(i) of Theorem 1.2, $\varphi_1(x) = e^{(m+\frac{i\pi}{q})x}P_1(x)$ and $\varphi_2(x) = e^{-(m+\frac{i\pi x}{q})}P_2(x)$, where $P_1(x)$ and $P_2(x)$ have period q.*
- **Case III:** *If $-2 < D < 2$, then the quadratic equation (29) has two distinct complex conjugated solutions β_1, β_2. But $\beta_1\beta_2 = 1$, therefore their moduli are unity. Hence, there exists real number α such that $0 < q\alpha < \pi$ or $-\pi < q\alpha < 0$ and $\beta_1 = e^{iq\alpha}, \rho_2 = e^{-iq\alpha}$. Thus, by part(i) of Theorem 1.2, $\varphi_1(x) = e^{i\alpha x}P_1(x)$ and $\varphi_2(x) = e^{-i\alpha x}P_2(x)$, where $P_1(x)$ and $P_2(x)$ have period q.*
- **Case IV:** *If $D = 2$ then the quadratic equation (29) has only one repeated solution $\beta_1 = \beta_2 = 1$. To decide which part of the Theorem 1.2, we need to compute the rank of $A - \mathrm{I}$, where I is the identity matrix and*

  $$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \psi_1(q) & \psi_1'(q) \\ \psi_2(q) & \psi_2'(q) \end{pmatrix}.$$

  Note that there are here two sub-cases:

  i. *If $\psi_2(q) = \psi_1'(q) = 0$, but*

  $$W(\psi_1, \psi_2)(q) = \begin{vmatrix} \psi_1(q) & 0 \\ 0 & \psi_2'(q) \end{vmatrix} = \psi_1(q).\ \psi_2'(q).\ W(\psi_1, \psi_2)(0) = \begin{vmatrix} \psi_1(0) & \psi_1'(0) \\ \psi_2(0) & \psi_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

  where we have used the initial conditions (3). But $W(\psi_1, \psi_2)(q) = W(\psi_1, \psi_2)(0)e^{-\int_0^q \frac{c_1(t)}{c_0(t)}dt} = W(\psi_1, \psi_2)(0)$, where we have used our assumption(19), then $W(\psi_1, \psi_2)(q) = \psi_1(q).\ \psi_2'(q) = W(\psi_1, \psi_2)(0) = 1$.
  Then we have $\psi_1(q).\ \psi_2'(q) = 1$.But $D = \psi_1(q) + \psi_2'(q) = 2q$. Hence $\psi_1(q) = \psi_2'(q) = 1$. Hence $\mathrm{rank}(A - \mathrm{I}) = 0$. This means that the part(i) of Theorem 1.2 applies because $\beta_1 = \beta_2 = 1$. The characteristic exponents m_1, m_2 are both zero and theorem 1.2 gives simply $\varphi_1(x) = P_1(x)$ and $\varphi_2(x) = P_2(x)$, where $P_1(x)$ and $P_2(x)$ have period q.

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ii. *If* $\psi_2(q) \neq 0$ *or* $\psi_1^{'}(q) \neq 0$. *In this sub-case* $\text{rank}(A - \text{I}) \neq 0$ *and the part (ii) of Theorem 1.2 applies and similarly* $m = 0$. *Hence, for this sub-case* $\varphi_1(x) = P_1(x)$ *and* $\varphi_2(x) = xP_1(x) + P_2(x)$*, where* $P_1(x)$ *and* $P_2(x)$ *have period* q.

- **Case V:** *If* $D = -2$*, then the quadratic equation (29) has only one repeated solution* $\beta_1 = \beta_2 = -1$. *To decide which part of the Theorem 1.2, we need to compute the rank of* $(A + \text{I})$*.By the same way in* **Case (IV)** *we have two sub-cases:*

i. If $\psi_2(q) = \psi_1^{'}(q) = 0$, then

$$W(\psi_1, \psi_2)(q) = \begin{vmatrix} \psi_1(q) & 0 \\ 0 & \psi_2^{'}(q) \end{vmatrix} = \psi_1(q).\, \psi_2^{'}(q).\ W(\psi_1, \psi_2)(q) = W(\psi_1, \psi_2)(0)e^{-\int_0^q \frac{c_1(t)}{c_0(t)}dt} = W(\psi_1, \psi_2)(0) = 1,$$

where we have used our assumption (19) and the initial conditions (3). Then, we have $\beta_1\beta_2 = -1$ *and* $D = \psi_1(q) + \psi_2^{'}(q) = -2$. *Hence,* $\psi_1(q) = \psi_2^{'}(q) = 1$. *Hence* $\text{rank}(A + \text{I}) = 0$. *This means that the part (i) of Theorem 1.2 applies because* $\beta_1 = \beta_2 = -1$. *The characteristic exponents* $m_1 = m_2 = \frac{i\pi}{q}$*, and therefore for this sub-case* $\varphi_1(x) = e^{\frac{\pi i}{q}x}P_1(x)$ *and* $\varphi_2(x) = e^{\frac{i\pi}{q}x}P_2(x)$*, where* $P_1(x)$ *and* $P_2(x)$ *have period* q. *It follows from that*

$$\begin{aligned} \varphi_1(x+q) &= e^{\frac{i\pi}{a}(x+q)}P_1(x+q) \\ &= e^{\frac{i\pi}{q}x}e^{i\pi}P_1(x) \\ &= -e^{\frac{i\pi}{q}x}P_1(x) = -\varphi_1(x). \\ \psi_2(x+q) &= e^{\frac{i\pi}{q}(x+)}P_2(x+q) \\ &= e^{\frac{i\pi}{q}x}e^{i\pi}P_2(x) \\ &= -e^{\frac{i\pi}{q}x}P_2(x) = -\varphi_2(x), \end{aligned}$$

and hence all solutions of Hill's equation satisfy $\varphi(x+q) = -\varphi(x)$.

ii. *If* $\beta_2(q) \neq 0$ *or* $\beta_1^{'}(q) \neq 0$. *In this sub-case* $\text{rank}(A + \text{I}) \neq 0$ *and the part (ii) of theorem 1.2 applies and with* $m = \frac{i\pi}{q}$. *Finally, for this sub-case* $\varphi_1(x) = P_1(x)$ *and* $\varphi_2(x) = xP_1(x) + P_2(x)$ *where* $P_1(x) = e^{\frac{i\pi}{q}x}p_1(x)$ *and* $P_2(x) = e^{\frac{i\pi}{q}x}p_2(x)$. *Similarly, it follows from that* $P_1(x+q) = -P_1(x)$ *and* $P_2(x+q) = -P_2(x)$.

Lemma 2.2.[3] *Liouville's formula for for the Wronskian of two solutions* $\psi_1(x)$ *and* $\psi_2(x)$ *of equation (1) is*

$$W(\psi_1, \psi_2)(x_2) = W(\psi_1, \psi_2)(x_1)\exp\left(-\int_{x_1}^{x_2}\frac{c_1(x)}{c_0(x)}dx\right).$$

For equation (18) , this gives

$$\frac{W(\psi_1, \psi_2)(x_2)}{P(x_1)} = \frac{W(\psi_1, \psi_2)(x_1)}{P(x_2)} = \text{constant}.$$

Proof. The Hill's equation (18) can be written as

$$P(x)y''(x) + P'(x)y'(x) + Q(x)y(x) = 0.$$

Then

$$\begin{aligned} W(\psi_1, \phi_2)(x_2) &= W(\psi_1, \psi_2)(x_1)\exp\left(-\int_{x_1}^{x_2} \frac{P'(x)}{P(x)} dx \right) \\ &= W(\psi_1, \psi_2)(x_1)\exp\left(-\ln(P(x_2) + \ln P(x_1) \right) \\ &= W(\psi_1, \psi_2)(x_1)\exp\left(\ln\frac{P(x_1)}{P(x_2)} \right) \\ &= W(\psi_1, \psi_2)(x_1). \frac{P(x_1)}{P(x_2)}. \end{aligned}$$

This implies that

$$\frac{W(\psi_1, \psi_2)(x_2)}{P(x_1)} = \frac{W(\psi_1, \psi_2)(x_1)}{P(x_2)}. \qquad (32)$$

□

3. Boundedness and periodicity of solutions

Theorem 3.1. *We have*

I. *If* $|D| > 2$*, all non-trivial solutions of Hill's equation are unbounded in* $(-\infty, +\infty)$*.*
II. *If* $|D| < 2$*, all solutions of Hill's equation are bounded in* $(-\infty, +\infty)$*.*

Proof.

i. If $D > 2$, we have the **Case (I)** of Theorem 2.1 and $\varphi_1(x) = e^{mx}P_1(x), \varphi_2(x) = e^{-mx}P_2(x)$ hold. Any linear combination of $\varphi_1(x)$ and $\varphi_2(x)$ is unbounded either as $x \to \infty$ or as $x \to -\infty$ (or both).

   If $D < -2$, we have **case (II)** of Theorem 2.1 and $\varphi_1(x) = e^{(m+\frac{i\pi}{q})x}P_1(x), \varphi_2(x) = e^{-(m+\frac{i\pi}{q})x}P_2(x)$ hold. Any linear combination of $\varphi_1(x)$ and $\varphi_2(x)$ is unbounded either as $x \to -\infty$ or $x \to +\infty$ (or both).

1. If $|D| < 2$. i.e. $-2 < D < 2$, we have **case (III)** of Theorem 2.1 and $\varphi_1(x) = e^{i\alpha x}P_1(x), \varphi_2(x) = e^{-i\alpha x}P_2(x)$ hold. Hence $|\varphi_1(x)| = |P_1(x)|$ and $|\varphi_2(x)| = |P_2(x)|$. Now $P_1(x), P_2(x)$ are bounded in $(-\infty, +\infty)$, being periodic. Hence $\psi_1(x), \psi_2(x)$ are bounded in $(-\infty, +\infty)$ therefore so also are all linear combinations of them. This proves part (ii). □

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Definition 3.2. The equation (18) is unstable if all non-trivial solutions are unbounded in $(-\infty, +\infty)$.

Definition 3.3. The equation (18) is conditionally stable if there is a non-trivial solution which is bounded in $(-\infty, +\infty)$.

Definition 3.4. The equation (18) is *stable* if all non-trivial solutions are bounded in $(-\infty, +\infty)$.

Remark 3.5. By the Theorem 3.1, the equation (18) is unstable if $|D| > 2$, and stable if $|D| < 2$, and it is also stable if $|D| = 2$ under the Sub-case (i) of Case (IV) and (V) in the Theorem 2.1, and the equation is conditionally stable, but is not stable if $|D| = 2$ under the second Sub-case (ii) of (IV) and (V) in the Theorem 2.1.

Theorem 3.6. *The Hill's equation (18) has non-trivial solutions with period q if and only if $D = 2$, and with semi-period q if and only if $D = -2$. All solutions of the Hill's equation have period q or semi-period a if and only if, additionally $\psi_2(q) = \psi_1^{'}(q) = 0$.*

Theorem 3.7. *Let k be a positive integer, then the equation (18) has non-trivial solutions with period kq if and only if there exists an integer l such that $R = 2\cos\left(\frac{2l\pi}{k}\right)$.*

Proof. Since periodic solutions are bounded in $(-\infty, +\infty)$, so **Case (I)** and **Case (II)** of the Theorem 2.1 do not arise here.

If $k = 1$, then we have **Case (IV)** of the Theorem 2.1 and choose $l = 0$. If $k = 2$, then case (III) does not occur because no non-trivial linear combination of $\varphi_1(x) = e^{i\alpha x}P_1(x)$ and $\varphi_2(x) = e^{-i\alpha x}P_2(x)$ has a period $2q$ because $q\alpha$ is not multiple of π. Hence the theorem is covered by **cases (IV) and (V)** and by choosing $l = 0$ and $l = 1$.

If $k > 2$ and the solution does not have period q or $2q$, it is **case (III)** that occurs, then a non-trivial linear combination $\alpha_1\varphi_1(x) + \alpha_2\varphi_2(x)$ of $\varphi_1(x), \varphi_2(x)$ in $\varphi_1(x) = e^{i\alpha x}P_1(x), \varphi_2(x) = e^{-i\alpha x}P_2(x)$ has a period kq if and only if $\alpha_1\varphi_1(x)(1 - e^{ikq\alpha}) + \alpha_2\varphi_2(x)(1 - e^{-ikq\alpha}) = 0$, i.e., $1 = e^{ikq\alpha}$, this means $kq\alpha = 2l\pi$ for some integer l and $D = \beta_1\beta_2 = 2\cos(q\alpha) = 2\cos\left(\frac{2\pi l}{k}.\right)$ □

Corollary 3.8. *A non-trivial solution of (18), which has a period $2q$, has either period q or semi-period q.*

Proof. It follows from the fact, when $k = 2$, it is the **Case (IV)** or **Case (V)** of the Theorem 2.1 that occur. □

Corollary 3.9. *If the equation (18) has a non-trivial solution with period ka, where k is a positive integer and $k > 2$, but no solution with period q or $2q$, then all solutions have period kq.*

Proof. As we have seen, it is **Case (III)** of the Theorem 2.1 that occurs with $kq\alpha = 2l\pi$ holding. Hence both $\varphi_1(x) = e^{i\alpha x}P_1(x), \varphi_2(x) = e^{-i\alpha x}P_2(x)$ have a period kq and the corollary follows. □

Remark 3.10. Since $kq\alpha = 2l\pi$ gives $\alpha = \frac{2\pi l}{kq}$, we see from $\varphi_1(x) = e^{i\alpha x}P_1(x), \varphi_2(x) = e^{-i\alpha x}P_2(x)$ that, in same conditions of the Corollary 3.9, so any solution $\varphi(x)$ of (18) has the form

$$\varphi(x) = \alpha_1 \exp\left(\frac{2li\pi x}{kq}\right)P_1(x) + \alpha_2 \exp\left(-\frac{2l\pi ix}{kq}\right)P_2(x),$$

where c_1, c_2 are constants.

Theorem 3.11. *Let $P(x)$, $Q(x)$ be even. Then the equation (18) has a non-trivial solution which is*

I. *Even and with period q if and only if $\psi_1'(\frac{1}{2}q) = 0$.*

II. *Odd and with period q if and only if $\psi_2(\frac{1}{2}q) = 0$.*

III. *Even and with semi-period q if and only if $\psi_1(\frac{1}{2}q) = 0$.*

IV. *Odd and with semi-period q if and only if $\psi_2'(\frac{1}{2}q) = 0$.*

Proof.

$P(x), Q(x)$ are even, $\psi(-x)$ is solution of (18) when $\psi(x)$ is. Particularly $\psi_1(x)$ and $\psi_1(-x)$ are solutions which satisfy the same initial conditions at $x = 0$. Hence $\psi_1(x) = \psi_1(-x)$, so that $\psi_1(x)$ is even. Similarly, $\psi_2(x) = -\psi_2(-x)$, so that '$\psi_2(x)$ is odd. It follows that any even solution of (18) is a multiple of $\psi_1(x)$ while any odd solution is multiple of $\psi_2(x)$. Now $\psi_1(\frac{1}{2}q) = \psi_1(-\frac{1}{2}q)$ and so $\psi_1(x)$ has period q if and only if $\psi_1'(\frac{1}{2}q) = \psi_1'(-\frac{1}{2}q)$. But, since $\psi_1(x)$ is even, $\psi_1'(\frac{1}{2}q) = -\psi_1'(-\frac{1}{2}q)$, and this proves part (I). For part (III), note that $\psi_1(x)$ has semi-period q if and only if $\psi_1'(\frac{1}{2}q) = -\psi_1'(-\frac{1}{2}q)$ and $\psi_1(\frac{1}{2}q) = -\psi_1(-\frac{1}{2})$. But, by $\psi_1(x) = \psi_1(-x)$, we have $\psi_1(\frac{1}{2}q) = \psi_1(-\frac{1}{2}q)$ and this proves part (iii) Parts (II) and (IV) are proved similarly by using $\psi_2(x)$. □

4. Complex-Valued Coefficients

There are one or two places in sequel where we need to refer to an equation of the form (18) but with $P(x), Q(x)$ complex-valued coefficients. The number D is still defined by (28) but it is now complex, and there is one further case in addition to the cases in the Theorem 2.1. This case is denoted by **Case (VI)**.

Case (VI): $D = \psi_1(q) + \psi_2'(q)$ is non- real. Here β_1 and β_2 are non-real and distinct. Also, $|\beta_1| \neq 1$ and $|\beta_2| \neq 1$ because if $\beta_1 = e^{i\theta}$, where θ is real, then, by $\beta_1\beta_2 = 1$, $\beta_2 = e^{-i\theta}$ and hence $D = \beta_1 + \beta_2 = 2\cos\theta$ is a real. Therefore, there is a non-real number m with $\mathrm{Re}m \neq 0$ such that $\beta_1 = e^{qm}, \beta_2 = e^{-qm}$. Then, as for $\varphi_1(x) = e^{mx}P_1(x)$ and $\varphi_2(x) = e^{-mx}P_2(x)$. Because of $\mathrm{Re}m \neq 0$, we have the result that, as the part (I) of the Theorem 3.1 , all non-trivial solutions of (18) are unbounded in this case.

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5. Instability and Stability interval

Let us involve a real parameter λ in the form

$$Q(x) = \lambda s(x) - q(x),$$

where $s(x)$ and $q(x)$ are piecewise continuous with period a and there exists a constant $s > 0$ such that $s(x) \geq s$. If we write $p(x)$ instead of $P(x)$, then the equation (18) becomes

$$\left(p(x)y'(x)\right)' + (\lambda s(x) - q(x))y(x) = 0.$$

In order to indicate the dependence of λ which occurs in the last equation (34), we write $\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$ for the solutions of (34) which satisfy the initial conditions

$$\phi_1(0, \lambda) = 1, \quad \phi_1'(0, \lambda) = 0,$$

$$\phi_2(0, \lambda) = 0, \quad \phi_2'(0, \lambda) = 1.$$

Define the discriminant

$$D(\lambda) = \phi_1(a, \lambda) + \phi_2'(a, \lambda).$$

Although the parameter λ is taken to be real here, it is sometimes necessary to allow it to be complex. Whether λ is real or complex, $\phi_1(x, \lambda)$,$\phi_2(x, \lambda)$ and their $x-$ derivatives are for fixed x, analytic functions of λ. Hence, by (37), $D(\lambda)$ is analytic function of λ.[4]

Since, in particular, $D(\lambda)$ defined by (37) is a continuous function of λ because $\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$ are analytic functions of λ for fixed x, the values of λ for which $|D(\lambda)| < 2$ form an open set on the real $\lambda-$ axis. This set, which as we shall see is not empty, so this set can be written as the union of countable of disjoint open intervals. Thus, the second part of the Theorem 3.1, the equation (34) is stable when λ lies in these intervals, and the intervals are therefore called the stability intervals of (34). Similarly, the intervals in which $|D(\lambda)| > 2$ are called the instability intervals of (34). Finally, the intervals formed by the closure of the stability intervals, as well as, those in which $|D(\lambda)| \leq 2$ are called the conditional stability intervals of (34).

6. The periodic and semi-periodic eigenvalue problems

We introduce two eigenvalue problems associated with (34) and the interval $[0, q]$, where λ is regarded as eigenvalue parameter. These problems are basic in the theory of stability and instability intervals. Some of their properties that we mention here will be used in the investigation of $D(\lambda)$.

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I. The periodic eigenvalue problem includes (34) considered to hold in $[0, q]$, and the periodic boundary conditions $\begin{matrix} y(q) = y(0) \\ y'(q) = y'(0). \end{matrix}$ Note it is a self-adjoint problem and the existence of a countable infinity of eigenvalues can be established by the standard method of constructing the Green's function and defining a compact linear operator in an inner-product space. Here, the inner-product space is that continuous functions in $[0, q]$ with the inner-product

$(f_1, f_2) = \int_0^q f_1(x)\bar{f_2}(x)s(x)dx.$ We shall denote the eigenfunctions throughout by $\phi_n(x)$ and the eigenvalues by $\lambda_n(n \in \mathrm{N})$, where $\lambda_0 \le \lambda_1 \le \lambda_2 \ldots$ and $\lambda_n \to \infty$ as $n \to \infty$.

Any double eigenvalue is counted twice. Choose $\phi_n(x)$ as real-valued and to form an orthonormal set over $[0, q]$ with weight function $s(x)$. Thus $\int_0^q \phi_n(x)\phi_m(x)s(x)dx = \begin{cases} 1 & \text{if} \quad n = m, \\ 0 & \text{if} \quad n \ne m. \end{cases}$

By (38), the $\phi_n(x)$ can be extended to $\mathrm{R} = (-\infty, +\infty)$ as continuously differentiable functions with period q. Hence the λ_n are the values of λ for which (34) has a non-trivial solution with period q. Further, the double eigenvalues are the values of λ for which all solutions of (34) have period q. It follows from previous section **Case(IV)** of the Theorem 2.1) that $D(\lambda_n) - 2 = 0$. i.e. λ_n are the zeros of the functions $D(\lambda) - 2$ and that λ_n is double eigenvalue if and only if $\psi_2(q, \lambda_n) = \psi_1'(q, \lambda_n) = 0$.

II. The semi-periodic eigenvalue problem includes (34), considered to hold in $[0, q]$, and the semi-periodic boundary conditions

$y(q) = -y(0).$

$y'(q) = -y'(0).$ It is also a self-adjoint problem and we shall denote the eigenfunctions by $\xi_n(x)$ and the eigenvalues by $\mu_n(n \in \mathrm{N})$ where $\mu_0 \le \mu_1 \le \mu_2. \ldots$, and $\mu_n \to \infty$ as $n \to \infty$. Any double eigenvalue is counted twice. Choose $\xi_n(x)$ as real-valued and to form an orthonormal set over $[0, q]$ with weight function $s(x)$. By (40), the $\xi_n(x)$ can be extended to $\mathrm{R} = (-\infty, +\infty)$ as continuously differentiable functions with semi-period q.

Hence the μ_n are the values of λ for which (34) has a non-trivial solution with semi-period q. Further, the double eigenvalues are the values of λ for which all solutions of (34) have periodic q. Then by previous section ***Case*(V)** of the Theorem 2.1), we conclude that $D(\mu_n) + 2 = 0$. i.e. μ_n are the zeros of the functions $D(\lambda) + 2$ and that μ_n is double eigenvalue if and only if $\psi_2(q, \mu_n) = \psi_1'(q, \mu_n) = 0$.

In the following sections, we need some consequences of variational nature concerning the λ_n and μ_n. We give proofs here in the case of λ_n only because the results of λ_n and μ_n are similar.

Let $\mathcal{F}$ be the set of all complex-valued functions $f(x)$ which are continuous in $[0, q]$ and have piecewise continuous derivative in $[0, q]$. Then the Dirichlet integral $J(f, g)$ in $\mathcal{F}$ is given by

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$$J(f, g) = \int_0^q \left(p(x) f'(x) \overline{g}'(x) + q(x) f(x) \overline{g}(x) \right) dx$$

$$= \int_0^q p(x) f'(x) \overline{g}'(x) dx + \int_0^q q(x) f(x) \overline{g}(x) dx.$$

Assume that

$$I = \int_0^q p(x) f'(x) \overline{g}'(x) dx.$$

If $g''(x)$ exists and is piecewise continuous in $[0, q]$ and integration by parts

$$I = \left[p(x) f(x) \overline{g}'(x) \right]_0^q - \int_0^q f(x) \left(p(x) \overline{g}'(x) \right)' dx,$$

then

$$J(f, g) = \left[p(x) f(x) \overline{g}'(x) \right]_0^q - \int_0^q f(x) \left(p(x) \overline{g}'(x) \right)' dx + \int_0^q q(x) f(x) \overline{g}(x) dx$$

$$= \left[p(x) f(x) \overline{g}'(x) \right]_0^q + \int_0^q f(x) \left[q(x) \overline{g}(x) - \left(p(x) \overline{g}'(x) \right)' \right] dx. \qquad (43)$$

If $f(x)$ and $g(x)$ satisfy the boundary conditions (38), the integrated terms cancel out. Particularly, if we substitute $g(x) = \phi_n(x)$ in (43), then we have

$$J(f, \phi_n) = \left[p(x) f(x) \overline{\phi}_n'(x) \right]_0^q + \int_0^q f(x) \left[q(x) \overline{\phi}_n(x) - \left(p(x) \overline{\phi}_n'(x) \right)' \right] dx$$

$$= \left[p(x) f(x) \phi_n'(x) \right]_0^q + \int_0^q f(x) \left[q(x) \phi_n(x) - \left(p(x) \phi_n'(x) \right)' \right] dx.$$

But for each $n \in \mathrm{N}$, ϕ_n satisfy the equation (34) with $\lambda = \lambda_n$, therefore

$$J(f, \psi_n) = \left[p(x) f(x) \phi_n'(x) \right]_0^q + \int_0^q f(x) \left[p(x) \phi_n'(x) + \lambda_n s(x) \phi_n(x) - \left(p(x) \phi_n'(x) \right)' \right] dx.$$

Denote the Fourier coefficients by f_n and are given by

$$f_n = \int_0^q f(x) s(x) \phi_n(x) dx. \qquad (44)$$

This gives

$$J(f, \phi_n) = \lambda_n f_n. \qquad (45)$$

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A particular case of (44) is

$$J(\psi_n, \psi_m) = \begin{cases} \lambda_n & \text{if} \quad n = m, \\ 0 & \text{if} \quad n \neq m. \end{cases} \qquad (46)$$

Theorem 6.1. *Let $f(x)$ be in $\mathcal{F}$ and satisfies the boundary condition (38). Then, with the Fourier coefficients f_n defined as (44), then*

$$\sum_{n=0}^{\infty} \lambda_n |f_n|^2 \leq J(f,f) \qquad (47)$$

Proof. Assume first that $q(x) \geq 0$. Then, by (42), we have $J(g,g) \geq 0$ for any $g(x) \in \mathcal{F}$ and particularly, $J(f - \sum_{n=0}^{N} f_n\phi_n, f - \sum_0^N f_n\phi_n)$, where N is a positive integer. Then

$$J(f,f) - J(\sum_{n=0}^{N} f_n\phi_n, f) - J(f, \sum_{n=0}^{N} f_n\phi_n) + J(\sum_{n=0}^{N} f_n\phi_n, \sum_{m=0}^{N} f_m\phi_m)$$
$$J(f,f) - \sum_{n=0}^{N} f_n J(\phi_n, f) - \sum_{n=0}^{N} \bar{f}_n J(f, \phi_n) + \sum_{n=0}^{N} \lambda_n f_n \bar{f}_n \geq 0,$$

where we have used (46) to obtain the last summation. Since

$$J(\phi_n, f) = J(f, \bar{\phi}_n),$$

then, by using (45), we have

$$J(f,f) - \sum_{n=0}^{N} \lambda_n |f_n|^2 \geq 0$$

i.e.

$$J(f,f) \geq \sum_{n=0}^{N} \lambda_n |f_n|^2.$$

Now let $N \to \infty$, we have

$$J(f,f) \geq \sum_{n=0}^{\infty} \lambda_n |f_n|^2.$$

To prove the theorem without assumption that $q(x) \geq 0$. Let q_0 be a constant which is sufficiently large to make

$$q(x) + q_0 s(x) \geq 0$$

in $[0, a]$. Then (34) can be written as

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$$\left(p(x)y'(x)\right)' + (\Lambda s(x) - Q(x))y(x) = 0,$$

where $\Lambda = \lambda + q_0$ and $Q(x) = q(x) + q_0 s(x)$. Since $Q(x) = q(x) + q_0 s(x) \geq 0$, the first part of the proof gives

$$\sum_{n=0}^{\infty} (\lambda_n + q_0)|f_n|^2 \leq \int_0^q \left(p(x)|f'(x)|^2 + \left(q(x) + q_0 s(x)\right)|f(x)|^2\right)dx.$$

By Parseval's formula

$$\sum_{n=0}^{\infty} |f_n|^2 = \int_0^q |f(x)|^2 s(x)dx,$$

and

$$J(f,f) \geq \sum_{n=0}^{\infty} \lambda_n |f_n|^2$$

follows in the general case. Since $\lambda_n \geq \lambda_0$ and by (50) and (51), we have

$$J(f,f) \geq \lambda_0 \sum_{n=0}^{\infty} |f_n|^2 = \lambda_0 \int_0^q |f(x)|^2 s(x)dx.$$

The equality holds here only when $f_n = 0$ for all n such that $\lambda_n > \lambda_0$, i.e. only when $f(x)$ is an eigenfunction corresponding to λ_0. Thus

$$\lambda_0 = \min \left(\frac{J(f,f)}{\int_0^q |f(x)|^2 s(x)dx}\right),$$

where the minimum being taken over all $(f(x) \neq 0) \in \mathcal{F}$ which satisfy (49). The minimum is attained only when $f(x)$ is an eigenfunction corresponding to λ_0. □

Theorem 6.2. *Let* $\lambda_{1,n}(n \geq 0)$ *be the eigenvalues in the periodic problem over* $[0, a]$ *when* $p(x)$, $q(x)$ *and* $s(x)$ *are replaced by* $p_1(x)$, $q_1(x)$ *and* $s_1(x)$, *where*

$$p_1(x) \geq p(x), \quad q_1(x) \geq q(x), \quad s_1(x) \leq s(x)$$

Then

i. *If* $s_1(x) = s(x)$, *we have* $\lambda_{1,n} \geq \lambda_n$ *for all* n.

ii. *Otherwise, we have* $\lambda_{1,n} \geq \lambda_n$ *provided* n *is such that* $\lambda_n \geq 0$.

Proof. Let $\phi_{1,n}(x)$ be the eigenfunction corresponding to $\lambda_{1,n}$ and let $J_1(f, g)$ be the Dirichlet integral given by

$$J_1(f,g)=\int_0^q\left(p_1(x)f'(x)\bar{g}'(x)+q_1(x)f(x)\bar{g}(x)\right)dx.$$

By using our assumption (53), we have $J_1(f,f)\geq J(f,f)$.

To prove the theorem for $n=0$, we consider $f(x)=\phi_{1,0}(x)$. Then, by (52), we have $\lambda_{1,0}=J_1(\phi_{1,0},\phi_{1,0})\geq J(\phi_{1,0},\phi_{1,0})\geq\lambda_0\int_0^q\phi_{1,0}^2(x)s(x)dx$. Now, by (53), we have

$$\int_0^q\phi_{1,0}^2(x)s(x)dx\geq\int_0^q\phi_{1,0}^2(x)s_1(x)dx=1$$

with the equality holding in part(i) of the theorem and strict inequality in part (ii). Hence

$$\lambda_{1,0}=J_1(\psi_{1,0},\psi_{1,0})\geq J(\psi_{1,0},\psi_{1,0})\geq\lambda_0\int_0^q\psi_{1,0}^2(x)s(x)dx$$

gives $\lambda_{1,0}\geq\lambda_0$ in first case, but it only gives $\lambda_{1,0}\geq\lambda_0$ in the second case if $\lambda_0\geq 0$. This proves the theorem for $n=0$.

For $n=1$, consider

$$f(x)=C_0\phi_{1,0}(x)+C_1\phi_{1,1}(x),$$

where C_0 and C_1 are real constants such that

$$C_0^2+C_1^2=1,$$
$$C_0A_0+C_1A_1=0,$$

where

$$A_0=\int_0^q\phi_{1,0}(x)\phi_0(x)s(x)dx$$

$$A_1=\int_0^q\phi_{1,1}(x)\phi_0(x)s(x)dx.$$

Such a choice of C_0 and C_1 is always possible.

The first condition (55) makes

$$\begin{aligned}\int_0^q f^2(x)s_1(x)dx&=\int_0^q C_0^2\phi_{1,0}^2(x)s_1(x)dx\int_0^q C_1^2\phi_{1,1}^2(x)s_1(x)dx\\&\quad+2\int_0^q C_0.\,C_1\phi_{1,0}(x)\phi_{1,1}(x)s_1(x)dx\\&=C_0^2+C_1^2\\&=1,\end{aligned}$$

and the second condition makes

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$$
\begin{aligned}
f_0 &= \int_0^q f(x)\phi_0(x)s(x)dx \\
&= \int_0^q \Big(C_0\phi_{1,0}(x) + C_1\phi_{1,1}(x) \Big)\phi_0(x)s(x)dx \\
&= C_0\int_0^q \phi_{1,0}(x)\phi_0(x)s(x)dx + C_1\int_0^q \phi_{1,1}(x)\phi_0(x)s(x)dx \\
&= C_0A_0 + C_1A_1 \\
&= 0.
\end{aligned}
$$

By (46) applied to J_1, we get

$$J_1(f,f) = \lambda_{1,0}C_0^2 + \lambda_{1,1}C_1^2 \le \lambda_{1,1}(C_0^2 + C_1^2) = \lambda_{1,1}.$$

Also, by (47) and the fact $f_0 = 0$, we have

$$J(f,f) \ge \lambda_0|f_0|^2 + \sum_{n=1}^{\infty}\lambda_n f_n^2 \ge \lambda_1\sum_{n=1}^{\infty} f_n^2 = \lambda_1\int_0^q f^2(x)s(x)dx$$

on using the Parseval Formula (50). Hence, by $J_1(f,f) \ge J(f,f)$, we obtain

$$\lambda_{1,1} \ge \lambda_1\int_0^q f^2(x)s(x)dx.$$

Then the theorem for $n = 1$ follows in similar way for $n = 0$. The argument can be extended to deal with general case for n. Consider

$$f(x) = C_0\phi_{1,0}(x) + \cdots + C_n\phi_{1,n}(x)$$

, where $C_0, \cdots, C_n$ are real constants such that

$$C_0^2 + C_1^2 + \ldots + C_n^2 = 1, \quad f_r = 0$$

and $0 \le r \le n-1$. The latter conditions are n homogeneous linear algebraic equations to be satisfied by the $n+1$ numbers $C_0, \cdots, C_n$ and such numbers always exist which satisfy the normalization condition

$$C_0^2 + C_1^2 + \cdots + C_n^2 = 1.$$

Then, the proof of theorem for general n follows the same lines as the proof for $n = 1$. □

Remark 6.3. We note from the last theorem that

i. $\lambda_{1,n} > \lambda_n$ if $J_1(f,f) > J(f,f)$.

ii. If $p_1(x) = p(x)$, $q_1(x) = q(x)$ and $s_1(x) = s(x)$, then $J_1(f,f) = J(f,f)$.

7. The function $D(\lambda)$

We have defined $D(\lambda)$ by (37) as the discriminant of the equation (49).

Theorem 7.1. *For* $n \in \mathbb{N}$,

I. *The numbers* λ_n *and* μ_n *occur in the order*

$\lambda_0 < \mu_0 \le \mu_1 < \lambda_1 \le \lambda_2 < \mu_2 \le \mu_3 < \lambda_3 \le \lambda_4 < \ldots$

II. *In the intervals* $[\lambda_{2m}, \mu_{2m}]$, $D(\lambda)$ *decrease from* 2 *to* -2.

III. *In the intervals* $[\mu_{2m+1}, \lambda_{2m+1}]$, $D(\lambda)$ *increase from* -2 *to* 2.

IV. *In the intervals* $(-\infty, \lambda_0)$ *and* $(\lambda_{2m+1}, \lambda_{2m+2})$, $D(\lambda) > 2$.

V. *In the intervals* (μ_{2m}, μ_{2m+1}), $D(\lambda) < -2$.

Proof. We prove the theorem in many stages.

a. There is a number Λ such that $D(\lambda) > 2$ for all $\lambda \le \Lambda$. Since $s(x) \ge s > 0$, we can choose Λ negative if necessary, so that

$q(x) - \Lambda s(x) > 0$

in $(-\infty, +\infty)$. Let $y(x)$ be any non-trivial solution of (49) such that $y(0) \ge 0$ and $y'(0) \ge 0$. Then there is an interval $(0, \delta)$ in which $y(x) > 0$. Consider any interval $(0, X)$ in which $y(x) > 0$.

In $(0, X)$ we have

$\left(p(x)y'(x)\right)' = (q(x) - \lambda s(x))y(x) > 0$

for all $\lambda \le \Lambda$, by (60). Hence $p(x)y'(x)$ increasing in $(0, X)$. This gives if $p(x) > 0$, then $y'(x) > 0$ in $(0, X)$ and therefore $y(x)$ is increasing in $(0, X)$. It follows that $y(x)$ has no zero $x = X$ in $(0, \infty)$, and therefore $p(x)y'(x)$ and $y(x)$ are increasing in $(0, \infty)$.

In particular

$\psi_1(q, \lambda) > \psi_1(0, \lambda) = 1,$

$\psi_2'(q, \lambda) > \psi_2'(0, \lambda) = 1,$

where we have used $p(q) = p(0)$ in the second inequality. Hence $D(\lambda) > 2$ for all $\lambda \le \Lambda$.

b. $D'(\lambda)$ is not zero at values of λ such that $|D(\lambda)| < 2$. First differentiate (34) with $y(x) = \psi_1(x, \lambda)$, with respect to λ. This gives

$\frac{d}{dx}p(x)\frac{d}{dx}\left(\frac{\partial\psi_1(x,\lambda)}{\partial\lambda}\right) + (\lambda s(x) - q(x))\frac{\partial\psi_1(x,\lambda)}{\partial\lambda} = -s(x)\psi_1(x, \lambda).$ Also, from the initial conditions (35) we have

$\frac{\partial\psi_1(0,\lambda)}{\partial\lambda} = \frac{d}{dx}\left(\frac{\partial\psi_1(0,\lambda)}{\partial\lambda}\right) = 0.$ The variation of constants formula applied to (61) and (62) gives

$$p(x)\frac{d^2}{dx^2}\left(\frac{\partial\psi_1(x,\lambda)}{\partial\lambda}\right) + p'(x)\frac{d}{dx}\left(\frac{\partial\psi_1(x,\lambda)}{\partial\lambda}\right) + (\lambda s(x) - q(x))\frac{\partial\psi_1(x,\lambda)}{\partial\lambda} + s(x)\psi_1(x, \lambda) = 0.$$

$$\frac{d^2}{dx^2}\left(\frac{\partial\phi_1(x,\lambda)}{\partial\lambda}\right) + \frac{p'(x)}{p(x)}\frac{d}{dx}\left(\frac{\partial\psi_1(x,\lambda)}{\partial\lambda}\right) + \frac{(\lambda s(x) - q(x))}{p(x)}\frac{\partial\psi_1(x,\lambda)}{\partial\lambda} = -\frac{s(x)\psi_1(x,\lambda)}{p(x)}.$$

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$$\frac{\partial \psi_1(x,\lambda)}{\partial \lambda} = -\psi_1(x,\lambda)\int_0^x \frac{\psi_2(t,\lambda)\left(\frac{-s(t)\psi_1(t,\lambda)}{p(t)}\right)}{W\left(\psi_1,\psi_2\right)(t)}dt + \psi_2(x,\lambda)\int_0^x \frac{\psi_1(t,\lambda)\left(-\frac{s(t)\psi_1(t,\lambda)}{p(t)}\right)}{W\left(\psi_1,\psi_2\right)(t)}dt$$

Then $= \psi_1(x,\lambda)\int_0^x \frac{\psi_2(t,\lambda)s(t)\psi_1(t)}{p(t)W\left(\psi_1,\psi_2\right)(t)}dt - \psi_2(x,\lambda)\int_0^x \frac{\psi_1(t,\lambda)s(t)\psi_1(t,\lambda)}{p(t)W\left(\psi_1,\psi_2\right)(t)}dt$ since

$$= (p(0))^{-1}\left(\int_0^x \psi_1(x,\lambda)\psi_2(t,\lambda)s(t)\psi_1(t,\lambda)dt - \int_0^x \psi_2(x,\lambda)\psi_1(t,\lambda)s(t)\psi_1(t,\lambda)dt\right)$$

$$= (p(0))^{-1}\int_0^x \left(\psi_1(x,\lambda)\psi_2(t,\lambda) - \psi_2(x,\lambda)\psi_1(t,\lambda)\right)s(t)\psi_1(t,\lambda)dt$$

$p(x)W(\psi_1,\psi_2)(x)$ is a constant and has the value $p(0)$. Similarly, $\frac{\partial \psi_1(x,\lambda)}{\partial \lambda} = (p(0))^{-1}\int_0^x \left(\psi_1(x,\lambda)\psi_2(t,\lambda) - \psi_2(x,\lambda)\psi_1(t,\lambda)\right)s(t)\psi_1(t,\lambda)dt.$ Now differentiation of both sides of (64) with respect to x gives $\frac{d}{dx}\left(\frac{\partial \psi_2(x,\lambda)}{\partial \lambda}\right) = \frac{d}{dx}\left((p(0))^{-1}\int_0^x \left(\psi_1(x,\lambda)\psi_2(t,\lambda) - \psi_2(x,\lambda)\psi_1(t,\lambda)\right)s(t)\psi_2(t,\lambda)dt\right)$

$$= (p(0))^{-1}\int_0^x \left(\psi_1'(x,\lambda)\psi_2(t,\lambda) - \psi_2'(x,\lambda)\psi_1(t,\lambda)\right)s(t)\psi_2(t,\lambda)dt.$$

This together with (63) gives

$$D'(\lambda) = \psi_1'(q,\lambda) + \psi_2''(q,\lambda)$$
$$= (p(0))^{-1}\int_0^q \left(\psi_1'\psi_2^2(t,\lambda) + \left(\psi_1 - \psi_2'\right)\psi_1(t,\lambda)\psi_2(t,\lambda) - \psi_2\psi_1^2(t,\lambda)\right)s(t)dt.$$

where we have substituted $x = q$, and we have written for $(n = 1, 2)$

$$\psi_n(q,\lambda) = \psi_n, \qquad \psi_n'(q,\lambda) = \psi_n'.$$

Since by Liouville's formula for Wronskian

$$\psi_1\psi_2' - \psi_2\psi_1' = W(\psi_1,\psi_2)(q) = \frac{p(0)}{p(q)} = 1,$$

we have

$$\begin{aligned} D^2(\lambda) &= (\psi_1 + \psi_2')^2 = \psi_1^2 + (\psi_2')^2 + 2\psi_1\phi_2' \\ &= 4\psi_1\psi_2' - 4\psi_2\psi_1' + \psi_1^2 + (\psi_2')^2 - 2\psi_1\psi_2' + 4\psi_2\psi_1' \\ &= 4(\psi_1\psi_2' - \psi_2\psi_1') + \psi_1^2 + (\psi_2')^2 - 2\psi_1\psi_2' + 4\psi_2\psi_1' \\ &= 4 + (\psi_1 - \psi_2')^2 + 4\psi_2\psi_1'. \end{aligned}$$

Hence the equality (65) can be written as

$$\begin{aligned} 4\psi_2 p(0)D'(\lambda) = &-\int_0^q \left(2\psi_2\psi_1(t,\lambda) - \left(\psi_1 - \psi_2'\right)\psi_2(t,\lambda)\right)^2 s(t)dt \\ &-\left(4 - D^2(\lambda)\right)\int_0^q \psi_2^2(t,\lambda)s(t)dt. \end{aligned}$$

Now assume that $|D(\lambda)| < 2$. Then, by the equality (66), we have $\psi_2 D'(\lambda) < 0$ and, in particular $D'(\lambda) \neq 0$ as required.

c. At a zero λ_n of $D(\lambda) - 2$, $D'(\lambda_n) = 0$ if and only if

$$\psi_2(q,\lambda_n) = \psi_1'(q,\lambda_n) = 0$$

Also, if $D'(\lambda_n) = 0,$ then $D''(\lambda_n) < 0.$ If $\psi_2(q,\lambda_n) = \phi_1'(q,\lambda_n) = 0$ holds, we also have $\psi_1(q,\lambda_n) = \psi_2'(q,\lambda_n) = 1$ as in the first Subcase (i) of **case (IV)** of theorem 2.1. Then $D'(\lambda_n) = 0$, by (65).

Conversely, if $D'(\lambda_n) = 0$, then

$$\int_0^q \Big(\psi_1' \psi_2^2(t, \lambda) + \Big(\psi_1 - \psi_2'\Big)\psi_1(t, \lambda)\psi_2(t, \lambda) - \psi_2 \psi_1^2(t, \lambda)\Big)s(t)dt = 0.$$

Since $\psi_1(t, \lambda)$, $\psi_2(t, \lambda)$ are linearly independent, this implies to $\psi_2(q, \lambda_n) = 0$ and $\psi_1(q, \lambda_n) = \psi_2'(q, \lambda_n)$. Then, from (65), we obtain $\psi_1'(q, \lambda_n) = 0$ as required. To prove that result about $D''(\lambda_n)$, we differentiate (65), with respect to λ to express $D''(\lambda)$ in terms of the λ-derivatives of ψ_1, ψ_2, ψ_1' and ψ_2', we then put $\lambda = \lambda_n$ and substitute for the $\lambda -$ the derivatives using (63) and (64), we get

$$D''(\lambda_n) = 2(p(0))^{-2}\left(\left(\int_0^q \psi_1(t, \lambda_n)\psi_2(t, \lambda_n)s(t)dt\right)^2 - \int_0^q \psi_1^2(t, \lambda_n)s(t)dt \int_0^q \psi_2^2(t, \lambda_n)s(t)dt\right).$$

Hence $D''(\lambda_n) \leq 0$ by Cauchy-Schwarz inequality and, further, the case of equality is ruled out because $\psi_1(t, \lambda_n)$, $\psi_2(t, \lambda_n)$ are linearly independent. There is a corresponding result to (c) for the zeros μ_n of $D(\lambda) + 2$ the only difference being that $D''(\mu_n) > 0$ if $D'(\mu_n) = 0$.

It follows from what we have just proved that no zero of $D(\lambda) \mp 2$ is of higher order than the second. Also, a zero λ_n of $D(\lambda) - 2$ is of order 2 only if $D(\lambda)$ assumes a maximum at λ_n, while a zero μ_n of $D(\lambda) + 2$ is of order 2 only if $D(\lambda)$ assumes a minimum at μ_n.

d. In this stage of the proof, we use the preceding results (a)-(c) to determine the behaviour of $D(\lambda)$ as λ increases from $-\infty$ to $+\infty$. When λ is large and negative, $D(\lambda) > 2$ by (a). Hence, as λ increases from $-\infty$, $D(\lambda)$ remains greater than 2 until λ reaches the first zero λ_0 of $D(\lambda) - 2$.

Since $D(\lambda)$ is not assuming a maximum at λ_0, λ_0 is a simple zero of $D(\lambda) - 2$, and it follows that $D(\lambda) < 2$ immediately to the right of λ_0. Then, as λ increases from λ_0, $D(\lambda)$ decreases by (b) until λ reaches the first zero μ_0 of $D(\lambda) + 2$. In the interval $(-\infty, \lambda_0)$, therefore, $D(\lambda) > 2$, and in (λ_0, μ_0), $D(\lambda)$ decreases from 2 to -2.

In general, μ_0 will be a simple zero of $D(\lambda) + 2$ and so $D(\lambda) < -2$ immediately to the right of μ_0. As λ from μ_0, $D(\lambda)$ will remain less than -2 until λ reaches the next zero μ_1 of $D(\lambda) + 2$.

Since $D(\lambda)$ is not assuming a minimum at μ_1, μ_1 is simple zero of $D(\lambda) + 2$, and it follows that $D(\lambda) > -2$ immediately to the right of μ_1. Then, as λ increases from μ_1, $D(\lambda)$ increases by (b) until λ reaches the next zero λ_1 of $D(\lambda) - 2$. In (μ_0, μ_1), therefore, $D(\lambda) < -2$ and in (μ_1, λ_1), $D(\lambda)$ increases from -2 to 2.

In general, λ_1 will be a simple zero of $D(\lambda) - 2$ and so $D(\lambda) > 2$ immediately to the right of λ_1. As λ increases from λ_1, $D(\lambda)$ remains greater than 2 until λ reaches the next zero λ_2 of $D(\lambda) - 2$.

The argument used above starting with λ_0 can now be repeated starting with λ_2 and it will continue to be repeated as $\lambda \to \infty$.

This proves the theorem except when $D(\lambda) \pm 2$ has double zeros. If, for example, μ_0 is a double zero of $D(\lambda) + 2$, then $D(\lambda) > -2$ immediately to the right of μ_0 and the previous analysis of $D(\lambda)$ continues to hold except that the interval (μ_0, μ_1) no longer figures in the argument. What we must show to prove $\lambda_0 < \mu_0 \le \mu_1 < \lambda_1 \le \lambda_2 < \mu_2 \le \mu_3 < \lambda_3 \le \lambda_4 < \cdots$ in the present situation is that $\mu_0 = \mu_1$ or in other words, that μ_0 is double eigenvalue in the semi-periodic problem. That this is the case follows immediately from $\phi_1(a, \mu_n) = \phi_1'(a, \mu_n) = 0$ and the condition corresponding to $\phi_1(q, \lambda_n) = \phi_1'(q, \lambda_n) = 0$ for $D(\lambda) + 2$. □

8. The Mathieu's Equation

The Mathieu equation is

$$y''(x) + (\lambda - 2v\cos 2x)y(x) = 0,$$

where $v \in \mathrm{R}$ and $v \neq 0$. This equation is a case of (34) with the period $q = \pi$.

Theorem 8.1. *For no values of λ and $v \neq 0$ do the solutions of (67) either all have period π or all have semi-period π.*

Proof. We write out the proof for the case of the period π. The case of semi-period π is similar.

We suppose that all solutions of (67) have period π and to get contradiction.

Let $\psi_1(x, \lambda), \psi_2(x, \lambda)$ have period π. $\cos 2x$ is an even function of x and therefore, as in the proof of Theorem 3.11, $\psi_1(x, \lambda)$ is an even function of x while $\psi_2(x, \lambda)$ is an odd function of x. Hence the Fourier expansions

$$\begin{cases} \psi_1(x, \lambda) = \frac{1}{2}a_0 + \sum_{r=1}^{\infty} a_r \cos 2rx \\ \psi_2(x, \lambda) = \sum_{r=1}^{\infty} d_r \sin 2rx. \end{cases}$$

Since $\cos 2x$ in (67) is infinitely differentiable, the same is true of $\psi_1(x, \lambda), \psi_2(x, \lambda)$ as functions of x because the expanding series of $\psi_1(x, \lambda)$ and $\psi_2(x, \lambda)$ are infinitely differentiable. Hence their Fourier series can be differentiated many times term by term. Then, substituting (68) we get

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$$
\begin{aligned}
&-\sum_{r=1}^{\infty}(2r)^2a_r\cos 2rx+(\lambda-2v\cos 2x)\left(\frac{1}{2}a_0+\sum_{r=1}^{\infty}a_r\cos 2rx\right)\\
&=-\sum_{r=1}^{\infty}(2r)^2a_r\cos 2rx+\frac{1}{2}a_0(\lambda-2v\cos 2x)+(\lambda-2v\cos 2x)\sum_{r=1}^{\infty}a_r\cos 2rx\\
&=\sum_{r=1}^{\infty}a_r(\lambda-(2r)^2)\cos 2rx+\frac{1}{2}a_0(\lambda-2v\cos 2x)-v\sum_{r=1}^{\infty}2a_r\cos 2x\cos 2rx\\
&=\sum_{r=1}^{\infty}a_r(\lambda-(2r)^2)\cos 2rx+\frac{1}{2}a_0(\lambda-2v\cos 2x)-v\sum_{r=1}^{\infty}a_r(\cos 2(r+1)x+\cos 2(r-1)x)\\
&=\sum_{r=1}^{\infty}a_r(\lambda-(2r)^2)\cos 2rx+\frac{1}{2}a_0\lambda-v\left(a_0\cos 2x+\sum_{r=1}^{\infty}a_r(\cos 2(r+1)x+\cos 2(r-1)x)\right)\\
&=0,
\end{aligned}
$$

where we have used $\cos 2(r+1)x+\cos 2(r-1)x=2\cos 2rx\cos x$. Similarly, we get

$$
\sum_{r=1}^{\infty}d_r(\lambda-(2r)^2)\sin 2rx-v\left(\sum_{r=1}^{\infty}d_r(\sin 2(r+1)x+\sin 2(r-1)x)\right)=0,
$$

but by using $\sin 2(r+1)x+\sin 2(r-1)x=2\sin 2rx\cos 2x$. These give $\frac{1}{2}\lambda a_0-va_1=0$ and, for $r\geq 1$,

$$
(\lambda-(2r)^2)a_r-v(c_{r-1}+a_{r+1})=0. \qquad (70)
$$

$$
(\lambda-(2r)^2)d_r-v(d_{r-1}+d_{r+1})=0, \qquad (71)
$$

where d_0 is defined to be zero. On eliminating $\left(\lambda-(2r)^2\right)$ between the last two equations we obtain

$$
v\left(a_r\left(d_{r-1}+d_{r+1}\right)-d_r\left(a_{r-1}+a_{r+1}\right)\right)=0
$$

or, since $v\neq 0$ and for for $r\geq 1$,

$$
a_rd_{r+1}-d_ra_{r+1}=a_{r-1}d_r-d_{r-1}a_r. \qquad (72)
$$

The right hand side of the equality (72) is the same as the left hand side but with $r-1$ in place of r. Hence, for $r\geq 1$,

$$
\begin{aligned}
a_rd_{r+1}-d_ra_{r+1}&=a_0d_1-d_0a_1\\
&=a_0d_1
\end{aligned}
$$

since $d_0=0$. Now $a_0\neq 0$ because if $a_0=0$, then, $\frac{1}{2}\lambda a_0-qa_1=0$, and

$$
\left(\lambda-(2r)^2\right)a_r-v(a_{r-1}+a_{r+1})=0
$$

would imply that all $a_r=0$, q being non-zero. Similarly, by (71), we get $d_1\neq 0$. But (68) are convergent infinite series, and so $a_r\to 0$ and $d_r\to 0$ as $r\to\infty$. Hence, let $r\to\infty$ in (8), We obtain the contradiction that $a_0d_1=0$, and this proves the theorem. □

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Notes

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