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Research Article

The Theory of Homotopical Quantum Mechanics and the Measurement Problem

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We present Homotopical Quantum Mechanics (HQM), a new formulation of quantum theory that resolves the quantum measurement problem using Homotopy Type Theory (HoTT) and ∞-category theory. In HQM, the pre-measurement state of a system is a homotopy class of paths in an ∞-topos, with all representatives physically equivalent and capable of interference. Measurement is modeled as a homotopy pullback that, when an information-bearing interaction occurs, deterministically contracts the entire homotopy class to a single representative path via a functor rendering the type contractible. This contraction is driven by entanglement between system and observer, not by a stochastic collapse postulate. We introduce a complete dynamical model of contraction, triggered when apparatus–system entanglement entropy exceeds a critical value S_{crit} or when pointer states become operationally distinguishable beyond a detector tolerance ϵ. Using a Hamiltonian pointer model, we show that contraction maps under a functor $\mathcal{F} : \mathcal{HQT} \to \mathbf{Hilb}$ to the Lüders update, guaranteeing agreement with the Born rule via the Busch–Gleason theorem and preserving no-signaling. Thus, this formulation eliminates the ad hoc stochastic collapse postulate, unifies system and observer in a single topological and logical structure, and provides a physically grounded, operationally testable criterion for outcome definiteness. HQM thus offers both a mathematically rigorous foundation for quantum measurement and new experimental signatures—such as finite contraction delay—that distinguish it from standard quantum mechanics.

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I. Introduction

The quantum measurement problem arises from the tension between two seemingly incompatible features of standard quantum mechanics: ***unitary evolution***, in which the Schrödinger equation preserves coherent superpositions of states; and ***wavefunction collapse***, in which measurement appears to select a single outcome in a non-unitary and probabilistic manner. This duality raises foundational questions: What is the physical mechanism of collapse? Is it truly stochastic? Where is the quantum–classical boundary? Traditional interpretations approach these issues in different ways. The *Copenhagen interpretation* treats collapse as a fundamental postulate, without further explanation. *Many-worlds* eliminates collapse entirely, but at the cost of proliferating non-interacting branches and leaving the Born rule unexplained. *Decoherence theory* explains the suppression of interference through environmental entanglement, but does not address why one definite outcome is experienced. *Homotopical Quantum Mechanics* (HQM) takes a different approach by embedding quantum theory in the language of Homotopy Type Theory (HoTT) and ∞-category theory. In HQM, the state of a single particle is not a single worldline or wavefunction, but an entire homotopy class of physically indistinguishable paths in an ∞-topos. All representatives of the class are physically equivalent until an information-bearing interaction occurs. Measurement is then described as a homotopy pullback that contracts this entire class to a single representative path via a functorial process. This contraction is a deterministic process, arising from entanglement between the system and the observer (or environment), which changes the logical context in which the state is defined. Crucially, this removes the need for a stochastic collapse postulate—outcome selection is a topological and informational process. From this perspective:

- Collapse is not random, but a deterministic contraction of a homotopy class.
- The Born rule emerges naturally in statistical ensembles where multiple distinct homotopy classes exist across repeated trials.
- The observer is not external to the system, but part of the same topological structure.

This article develops the HQM formalism in detail, shows how it recovers familiar quantum predictions, and proposes possible experimental distinctions from standard quantum mechanics.

A. Brief Introduction to Homotopy Type Theory

Homotopy Type Theory (HoTT) is a foundational system for mathematics that unifies logic, type theory, and homotopy theory. In HoTT, types are interpreted as spaces (up to homotopy), and terms of a type are points in that space. Equality between terms is treated as a path type, with higher equalities (homotopies) between paths, forming an ∞-groupoid structure. An ∞-topos is a higher categorical structure generalizing the notion of a topos, incorporating homotopy theory. It provides a framework where spaces can have higher-dimensional structures, and morphisms preserve these structures up to coherent homotopies. In HQM, we leverage HoTT to model quantum states as homotopy classes in an ∞-topos, where superpositions correspond to connected components, and measurements induce changes in the homotopy structure.

II. Background

A. The Measurement Problem

In standard quantum mechanics, two distinct evolution rules coexist:

1. **Unitary evolution** — For a closed system with Hamiltonian H,
$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle, \quad U(t,t_0) = e^{-\frac{i}{\hbar}H(t-t_0)}, \tag{1}$$
which preserves the norm
$$\langle\psi(t)|\psi(t)\rangle = 1, \tag{2}$$
and maintains quantum superpositions.
2. **Collapse upon measurement** — For an observable $\hat{M} = \sum_i o_i\,|o_i\rangle\langle o_i|$, the postulate says
$$|\psi\rangle \;\longrightarrow\; \frac{\Pi_{o_k}|\psi\rangle}{\sqrt{\langle\psi|\Pi_{o_k}|\psi\rangle}} \tag{3}$$
with probability
$$P(o_k) = \langle\psi|\Pi_{o_k}|\psi\rangle, \tag{4}$$
where $\Pi_{o_k} = |o_k\rangle\langle o_k|$ projects onto the eigenspace of o_k.
3. **Observer–system divide** — The theory does not specify where the “cut” lies between quantum and classical domains, nor the physical mechanism for collapse.

B. Existing Approaches

Several strategies attempt to address the problem:

- **Decoherence** — System S entangled with environment E yields a reduced density matrix
$$\rho_S = \mathrm{Tr}_E\big[|\Psi_{SE}\rangle\langle\Psi_{SE}|\big] \tag{5}$$
where interference terms vanish in a preferred basis, but the state remains mixed rather than definite.

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- **Many-worlds** — Retains the universal state

$$|\psi\rangle = \sum_i \alpha_i |o_i\rangle \tag{6}$$

with all outcomes realized in separate branches; no collapse occurs, but Born probabilities lack a natural derivation.
- **Objective collapse models** — Modify the Schrödinger equation with stochastic terms, introducing collapse dynamics at the fundamental level.
- **Topos-theoretic frameworks** — Replace Boolean truth values with contextual logics; reformulate propositions but do not address the physical cause of outcome selection.

C. HQM's Conceptual Shift

In Homotopical Quantum Mechanics (HQM), all physically possible histories of a particle between fixed initial and final boundaries are continuous paths

$$\gamma : [0,1] \to \mathcal{C} \tag{7}$$

in the configuration space $\mathcal{C}$. **Key pre-measurement postulate**: Before any information-bearing interaction, all paths belong to the same homotopy class

$$\in \pi_0\big(\mathrm{Path}(\mathcal{C})\big), \tag{8}$$

meaning they are continuously deformable into each other without leaving the allowed configuration space. This single connected class supports interference. **Measurement as two-step process**:

1. **Topological distinguishability** — An interaction (with an observer, apparatus, or environment) alters the topology of $\mathcal{C}$ so that the original class $[\gamma]$ splits into distinct, disconnected components. Physically, this corresponds to acquiring enough information to tell the alternatives apart.

$$[\gamma] \longrightarrow \{[\gamma]_1, [\gamma]_2, \ldots\} \tag{9}$$

2. **Topological contraction** — Once the paths are in separate components, the system–observer dynamics contract the relevant component to a single representative path γ_*. This contraction preserves the endpoints $\gamma(0)$ and $\gamma(1)$, but removes all other continuous alternatives via a functor rendering the type contractible.

Special cases: If no topological distinguishability occurs:

$$[\gamma] \text{ remains connected } \Rightarrow \text{ interference survives.} \tag{10}$$

If distinguishability occurs but no contraction is applied: multiple disconnected classes exist simultaneously — physically corresponding to many-worlds–like branching. If both occur:

$$[\gamma] \;\to\; [\gamma]_k \xrightarrow{\text{contraction}} \{\gamma_*\} \tag{11}$$

a single definite outcome is obtained — matching the phenomenology of collapse. In this way, HQM replaces the ad hoc stochastic collapse postulate with a two-step topological mechanism: class splitting by topological distinguishability, and contraction to a single representative. The observer and system are both part of the same ∞-topos, and collapse is reinterpreted as an internal topological refinement rather than an external, discontinuous projection.

III. HQM Formulation of Quantum States and Paths

A. Quantum States as Homotopy Classes of Histories

In the HQM framework, the quantum state of a system is not described directly as a vector in a Hilbert space $\mathcal{H}$, but rather as the homotopy class of all continuous histories γ in a configuration space $\mathcal{C}$ subject to fixed boundary conditions:

$$\gamma : [0,1] \longrightarrow \mathcal{C}, \quad \gamma(0) = x_{\text{in}}, \quad \gamma(1) = x_{\text{out}}. \tag{12}$$

The collection of all such histories is denoted

$$\text{Path}(\mathcal{C}; x_{\text{in}}, x_{\text{out}}), \tag{13}$$

and the equivalence relation $\gamma \sim \gamma'$ is defined by the existence of a continuous deformation (homotopy) between them. The physical state is identified with the equivalence class

$$[\gamma] \in \pi_0\big(\text{Path}(\mathcal{C}; x_{\text{in}}, x_{\text{out}})\big). \tag{14}$$

Before any measurement interaction, all physically allowed paths connecting x_{in} and x_{out} belong to one such homotopy class. In HoTT terms, the path space $\text{Path}(\mathcal{C}; x_{\text{in}}, x_{\text{out}})$ is a type, and the homotopy class corresponds to a connected component in its ∞-groupoid structure, potentially carrying higher homotopies encoding quantum phases.

B. Measurement as Topological Distinction

A measurement interaction — which may be direct observation, apparatus coupling, or environmental scattering — can encode which-path information into an external degree of freedom. In HQM, this changes the topology of the path space:

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$$\pi_0\big(\mathrm{Path}(\mathcal{C})\big) \quad \longrightarrow \quad \pi_0\big(\mathrm{Path}(\mathcal{C}')\big), \tag{15}$$

where $\mathcal{C}'$ is the modified configuration space including the interaction. Topological distinguishability means that two paths γ_1, γ_2 that were once deformable into each other (homotopic) now belong to different connected components of the updated path space, potentially altering higher homotopy groups π_n $(n > 0)$.

C. Contraction to a Single Path

Once topological distinction is established, contraction occurs: each connected component of the post-interaction path space collapses to a single representative path. Physically, this corresponds to the system adopting a definite outcome consistent with the distinguishability introduced by the measurement. Formally, if the measurement yields components

$$\pi_0\big(\mathrm{Path}(\mathcal{C}')\big) = \{[\gamma_1], [\gamma_2], \dots\}, \tag{16}$$

then for the observed outcome, HQM models the selection as a contraction

$$[\gamma_k] \quad \longrightarrow \quad \gamma_k^*, \tag{17}$$

where γ_k^* is a single, physically realized history. In HoTT, a type is *contractible* if it is equivalent to the unit type $\mathbf{1}$, meaning there exists a point $\gamma_k^* \in [\gamma_k]$ such that for every other point $\gamma' \in [\gamma_k]$, there is a path $p : \gamma_k^* = \gamma'$, and all such paths are homotopic, with this structure extending to all higher dimensions. The contraction is a functor $F : [\gamma_k] \to \mathbf{1}$ that renders $[\gamma_k]$ contractible, selecting γ_k^* as the center of contraction, analogous to the geometric realization functor in algebraic topology.

D. Preservation of Boundary Data

Because contraction is homotopical, the endpoints of the path are preserved:

$$\gamma_k^*(0) = x_{\mathrm{in}}, \quad \gamma_k^*(1) = x_{\mathrm{out}}(k), \tag{18}$$

where $x_{\mathrm{out}}(k)$ corresponds to the measurement outcome label k.

IV. Measurement as a Homotopy Pullback

A. The Pullback Framework

Let $\mathcal{S}$ denote the system's configuration space and $\mathcal{O}$ the observer–apparatus configuration space. Before interaction, the system and observer evolve independently:

$$\mathcal{S} \times \mathcal{O} \tag{19}$$

with paths $\gamma_S : [0,1] \to \mathcal{S}$ and $\gamma_O : [0,1] \to \mathcal{O}$ belonging to their respective homotopy classes. A measurement interaction is modeled as a coupling map

$$f : \mathcal{S} \to \mathcal{M}, \quad g : \mathcal{O} \to \mathcal{M} \tag{20}$$

into a shared measurement space $\mathcal{M}$ that encodes the information content (including possible distinguishability) of the measurement.

B. Homotopy Pullback Definition

The homotopy pullback of f and g is defined as the limit in the ∞-category:

$$\mathcal{P} = \{(s, o, \lambda) \mid s \in \mathcal{S}, o \in \mathcal{O}, \lambda : f(s) \simeq g(o)\}, \tag{21}$$

where λ is a homotopy in $\mathcal{M}$ connecting $f(s)$ and $g(o)$. Intuitively: $f(s)$ is the system's representation in measurement space; $g(o)$ is the observer's representation; and λ ensures logical compatibility. Formally, $\mathcal{P}$ is the ∞-categorical fiber product $\mathcal{S} \times^{\mathrm{h}}_{\mathcal{M}} \mathcal{O}$, equipped with natural transformations ensuring coherence up to higher homotopies.

C. Measurement-Induced Topological Change

If f and g map previously homotopic system paths to distinct connected components of $\mathcal{M}$, then the pullback $\mathcal{P}$ consists of disconnected components. This is the topological distinguishability that triggers contraction in HQM:

$$\pi_0(\mathrm{Path}(\mathcal{S})) \quad \to \quad \pi_0(\mathrm{Path}(\mathcal{S}')) \tag{22}$$

where $\mathcal{S}'$ is the system configuration space after measurement coupling. Higher homotopy groups π_n ($n > 0$) may encode quantum phases or interference effects before contraction.

D. Contraction as Selection in the Pullback

Once $\mathcal{P}$ decomposes into disconnected components $\mathcal{P}_1, \mathcal{P}_2, \ldots$, the HQM postulate is:

$$\mathcal{P}_k \quad \longrightarrow \quad \gamma_k^* \tag{23}$$

for some k, meaning one component survives as the realized history while others are eliminated. This is modeled as choosing a section in the fibration over $\mathcal{M}$, rendering the fiber contractible. The realized γ_k^* inherits:

$$\gamma_k^*(0) = x_{\mathrm{in}}, \quad \gamma_k^*(1) = x_{\mathrm{out}}(k). \tag{24}$$

E. Summary Diagram

In homotopical terms, the measurement process is:

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & \mathcal{O} \\ \big\downarrow & & \big\downarrow g \\ \mathcal{S} & \xrightarrow{f} & \mathcal{M} \end{array} \tag{25}$$

Topological distinguishability in $\mathcal{M} \Rightarrow$ Contraction in $\mathcal{P}$. Distinguishability corresponds to a change in connected components of $\mathcal{M}$, and contraction is a functorial reduction to a single representative in $\mathcal{P}$.

V. Double-Slit Experiment in HQM

A. Pre-measurement: Single Homotopy Class

Let $\mathcal{S}$ be the configuration space of the electron in the experimental setup. The slits A and B correspond to geometrically different but topologically connected regions in $\mathcal{S}$. A single homotopy class $[\gamma]$ contains all paths from the source to the detection screen:

$$[\gamma] = \{\gamma_A, \gamma_B, \gamma_{\text{diffraction}}, \ldots\}. \tag{26}$$

Since $\pi_0(\mathrm{Path}(\mathcal{S})) = \{[\gamma]\}$, interference is preserved.

B. Measurement Space Without Distinguishability

Let $f : \mathcal{S} \to \mathcal{M}$ map each path to its representation in the measurement space $\mathcal{M}$ (which includes any environmental record). If the apparatus does not measure which slit the particle passes through — e.g., photons scattered from slit A and B are indistinguishable — then:

$$\mathrm{Im}(f) \subset \mathcal{M} \tag{27}$$

remains connected. The pullback:

$$\mathcal{P} = \{(s, o, \lambda)\} \tag{28}$$

remains connected as well. No contraction occurs, and all paths in $[\gamma]$ interfere.

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C. Measurement Space With Distinguishability

If the slits emit photons of different frequencies upon particle passage, the environmental states $|E_A\rangle$ and $|E_B\rangle$ become orthogonal:

$$\langle E_A|E_B\rangle = 0. \tag{29}$$

This makes $\mathrm{Im}(f)$ disconnected in $\mathcal{M}$:

$$\mathrm{Im}(f) = \mathcal{M}_A \sqcup \mathcal{M}_B, \tag{30}$$

where $\sqcup$ is the disjoint union. In HQM, this disconnection implies that the pullback $\mathcal{P}$ splits:

$$\mathcal{P} = \mathcal{P}_A \sqcup \mathcal{P}_B. \tag{31}$$

Topological contraction then selects one component, say $\mathcal{P}_A$, collapsing all system paths to a single representative γ_A^* by rendering $[\gamma_A]$ contractible.

D. HQM Interpretation

No distinguishability $\to$ $\mathcal{P}$ remains connected $\to$ no contraction, interference observed. Distinguishability $\to$ $\mathcal{P}$ disconnected $\to$ contraction selects one path $\to$ definite outcome, no interference.

E. Diagram

In the "with distinguishability" case:

$$\begin{array}{ccccccc} \mathcal{P}_A & \longrightarrow & \mathcal{O} & & \mathcal{P}_B & \longrightarrow & \mathcal{O} \\ \downarrow & & \downarrow g & & \downarrow & & \downarrow g \\ \mathcal{S}_A & \xrightarrow{f} & \mathcal{M}_A & & \mathcal{S}_B & \xrightarrow{f} & \mathcal{M}_B \end{array} \tag{32}$$

Only one diagram survives physically after contraction. This example grounds the HQM formalism in a familiar experiment, showing how interference and collapse emerge from topological distinguishability and contraction without invoking stochastic collapse postulates.

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VI. Physical Interpretation of Contraction in HQM

A. Pre-measurement Phase

Before any information-bearing interaction, all physically possible paths

$$\gamma : [0, 1] \to \mathcal{C} \tag{33}$$

connecting the same endpoints $\gamma(0)$ and $\gamma(1)$ belong to a single homotopy class

$$[\gamma] \in \pi_0(\mathrm{Path}(\mathcal{C})). \tag{34}$$

This reflects the fact that no topological refinement distinguishes one path from another. The interference pattern observed in experiments such as the double-slit experiment is a manifestation of the coherent sum over these homotopic contributions.

B. Measurement Interaction

A measurement — whether implemented by an observer, apparatus, or environment — is modeled as a topological refinement. The interaction modifies $\mathcal{C}$ so that the homotopy class $[\gamma]$ no longer contains multiple representatives; it contracts to a single representative γ_* satisfying:

$$[\gamma] \quad \longrightarrow \quad \{\gamma_*\} \tag{35}$$

while keeping the endpoints $\gamma(0)$ and $\gamma(1)$ fixed. This contraction is the HQM analogue of "collapse" — but unlike the standard postulate, it is triggered physically by the acquisition of topologically distinguishing information and mathematically by rendering the type contractible.

C. Distinguishability as the Trigger

Topological distinguishability means that the interaction changes the configuration space so that the formerly continuous deformation between different paths is broken. In the double-slit example, distinguishability arises if each slit leaves a distinct, recordable mark in the environment (e.g., different photon emissions). Decoherence fits into HQM as a mechanism inducing topological distinguishability by entangling the system with the environment, leading to orthogonal pointers that disconnect components in $\mathcal{M}$. However, HQM views decoherence as sufficient but not necessary for distinguishability, as other interactions could achieve similar topological changes.

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D. Post-measurement Determinism

After contraction, the evolution of γ_* is deterministic under the joint system–observer unitary dynamics. All randomness is associated with which topological refinement occurs in a given run; once it happens, the selected representative is fixed.

E. Unitary Dilation and Update Map

To formalize the contraction mathematically, consider a system S with Hilbert space $\mathcal{H}_S$ and an environment E with Hilbert space $\mathcal{H}_E$. The measurement interaction is modeled as a unitary U on $\mathcal{H}_S \otimes \mathcal{H}_E$:

$$U\left(|\psi\rangle_S \otimes |0\rangle_E\right) = \sum_k \left(M_k |\psi\rangle_S\right) \otimes |e_k\rangle_E, \tag{36}$$

where $\{M_k\}$ are Kraus operators and $\{|e_k\rangle\}$ are orthonormal pointer states of E. The reduced state after interaction is

$$\rho_S' = \sum_k M_k \rho_S M_k^\dagger. \tag{37}$$

In HQM, contraction corresponds to the selection of the homotopy component $[\gamma]_k$ associated with a single Kraus branch, yielding the post-measurement state

$$\rho_S^{(k)} = \frac{M_k \rho_S M_k^\dagger}{\mathrm{Tr}[M_k \rho_S M_k^\dagger]}. \tag{38}$$

This is precisely the Lüders update rule, derived here from the topological contraction picture via a unitary system–environment coupling, guaranteeing consistency with standard quantum operations and preserving no-signaling.

VII. Measurement and Contraction in HQM

In Homotopical Quantum Mechanics, the kinematical state of a single particle or composite system is represented as an element of a *gauge-refined homotopy class* in an ∞-topos $\mathcal{T}$. Prior to measurement, all representatives of the class are physically equivalent, with amplitudes related by higher morphisms corresponding to parallel transport in a flat $U(1)$ bundle. Superposition corresponds to taking coherent linear combinations of such representatives in the associated Hilbert space image under the functor $\mathcal{F} : \mathcal{HQT} \to \mathbf{Hilb}$.

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A. System–Apparatus Bipartition

We consider the composite SA, where S is the system under investigation and A is the measurement apparatus, optionally extended to include any relevant environment E. The total state $|\Psi(t)\rangle_{SA(E)}$ is assumed pure for definiteness; mixed-state generalizations follow by purification. The reduced density matrix of the apparatus is

$$\rho_A(t) = \mathrm{Tr}_{S(E)}[\,|\Psi(t)\rangle\langle\Psi(t)|\,]\,. \tag{39}$$

The *entanglement entropy*

$$S_{\mathrm{ent}}(t) = -\mathrm{Tr}[\rho_A(t)\,\log\rho_A(t)] \tag{40}$$

quantifies the entanglement between A and its complement $S(E)$ [1][2].

B. Entropy-Triggered Contraction Postulate

Postulate. In HQM, contraction of the gauge-refined homotopy class $\mathcal{C}$ to a single representative occurs when the entanglement entropy $S_{\mathrm{ent}}(t)$ between apparatus and system exceeds a device-dependent critical value S_{crit}:

$$S_{\mathrm{ent}}(t) \;\geq\; S_{\mathrm{crit}}. \tag{41}$$

The value of S_{crit} is fixed by the operational resolution of the apparatus: it is the minimal entanglement needed for the *pointer states* $\{\rho_A^{(k)}\}$, correlated with different system outcomes k, to be distinguishable with probability of error less than the detector tolerance $\varepsilon_{\mathrm{det}}$.

C. Operational Meaning of the Entropy Threshold

Let $\rho_A^{(k)}$ denote the apparatus state conditional on system outcome k. The operational distinguishability of two outcomes k, k' is quantified by the trace distance [3][4]

$$D(\rho_A^{(k)}, \rho_A^{(k')}) = \frac{1}{2}\|\rho_A^{(k)} - \rho_A^{(k')}\|_1. \tag{42}$$

Standard inequalities (Fuchs–van de Graaf and quantum Pinsker) bound D in terms of the relative entropy, which in turn is bounded below by the Holevo quantity

$$\chi = S(\bar{\rho}_A) - \sum_k p_k S(\rho_A^{(k)}), \quad \bar{\rho}_A = \sum_k p_k \rho_A^{(k)}. \tag{43}$$

In typical measurement models with monotonically increasing $S_{\mathrm{ent}}(t)$, there exists a calibration function f such that

$$S_{\mathrm{ent}}(t) \geq S_{\mathrm{crit}} \quad \Rightarrow \quad D_{\max}(t) \equiv \max_{k \neq k'} D(\rho_A^{(k)}, \rho_A^{(k')}) \ \geq \ \varepsilon_{\mathrm{det}}.$$

Thus the entropy threshold (41) is equivalent to an operational distinguishability threshold fixed by the apparatus.

D. Contraction Dynamics

When $S_{\mathrm{ent}}(t)$ reaches S_{crit}, HQM prescribes that the gauge-refined homotopy class $\mathcal{C}$ undergoes *contraction* to a single representative $[p_k]$ corresponding to the realized outcome k. In the Hilbert-space image under $\mathcal{F}$, this contraction coincides with the Lüders update

$$\rho_S \ \mapsto \ \frac{P_k \rho_S P_k}{\mathrm{Tr}(P_k \rho_S)}, \tag{44}$$

where $\{P_k\}$ are the spectral projectors of the measured observable. The randomness of k is governed by the Born rule, ensuring statistical equivalence with standard quantum mechanics.

a. Physical interpretation

The entropy growth reflects the transfer of coherence from S to inaccessible degrees of freedom in $A(E)$, suppressing interference between distinct k branches. Once this suppression surpasses the apparatus tolerance, further coherence is operationally irrelevant, and the topological contraction formalizes the selection of a single outcome.

VIII. Towards a Homotopical Born Rule

The path-measure construction below provides a heuristic motivation for the Born rule. A rigorous derivation follows using the Hilbert-space embedding $\mathcal{F}$ and the Busch–Gleason theorem.

A. Heuristic Path-Measure Picture

Consider the (gauge-refined) path groupoid $\mathsf{Path}(\mathcal{C}')$ after measurement-induced topological refinement, with outcome components $\{\mathcal{P}_k\}_k \subset \mathsf{Path}(\mathcal{C}')$ that are disjoint by construction. Let Σ be the cylinder σ-algebra on $\mathsf{Path}(\mathcal{C}')$ generated by finite-time evaluation maps, and let $\mathcal{A} : \mathsf{Path}(\mathcal{C}') \to \mathbb{C}$ be an amplitude functional (e.g., Feynman-type $e^{iS[\gamma]/\hbar}$ after gauge-fixing).

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a. Decoherence by contraction

In HQM, contraction is applied when the apparatus–system entanglement entropy reaches the critical value S_{crit} (Sec. 7.2). At that time, cross-component interference between distinct outcome domains $\mathcal{P}_k$ is operationally suppressed. In this decoherent regime, we may define a positive, σ-additive measure on $(\mathsf{Path}(\mathcal{C}'), \Sigma)$ by

$$\mu(A) \;=\; \int_A |\mathcal{A}[\gamma]|^2 \, d\nu(\gamma), \tag{45}$$

for some reference cylinder measure ν. Only the restriction of μ to the disjoint outcome sets $\{\mathcal{P}_k\}$ is operationally relevant.

b. Normalization over outcome components

The probability of outcome o_k is then

$$P(o_k) \;=\; \frac{\mu(\mathcal{P}_k)}{\sum_j \mu(\mathcal{P}_j)}. \tag{46}$$

For qubit examples, when the refined domains $\mathcal{P}_0, \mathcal{P}_1$ are mapped to the $|0\rangle, |1\rangle$ branches and $\mathcal{A}$ factors through the usual propagator, (46) reproduces $P(0) = |\alpha|^2$, $P(1) = |\beta|^2$ after contraction.

B. Born Rule from the HQM–Hilbert Functor

Assume there exists a symmetric monoidal dagger functor

$$\mathcal{F} \;:\; \mathsf{Path}(\mathcal{C}') \longrightarrow \mathbf{Hilb} \tag{47}$$

such that: (i) concatenation maps to composition and path reversal to adjoint, (ii) the decoherence functional on path classes coincides with inner products under $\mathcal{F}$, and (iii) each outcome domain $\mathcal{P}_k$ is mapped to the spectral subspace of a projective measurement $\{\Pi_{o_k}\}$ on $\mathcal{F}([\gamma]) \equiv |\psi\rangle$. Under (i)–(iii), the pushforward of μ satisfies

$$P(o_k) \;=\; \mu(\mathcal{P}_k) \;=\; \langle\psi\,|\,\Pi_{o_k}\,|\,\psi\rangle, \tag{48}$$

i.e., the Born rule. This coincides with the rigorous probability assignment obtained independently from the Busch–Gleason theorem for effects in **Hilb**.

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C. Born Rule from Categorical Gleason Theorem

Within **Hilb**, Gleason's theorem states that for $\dim \mathcal{H} \geq 3$, any frame function assigning probabilities to projectors is given by $\mathrm{Tr}(\rho\cdot)$ for some density matrix ρ. Busch's extension covers POVMs and $\dim = 2$ [5]. Because $\mathcal{F} : \mathcal{HQT} \to \mathbf{Hilb}$ is dagger compact, measurement effects in HQM correspond to positive operators E_k in **Hilb** with $\sum_k E_k = I$. A *homotopical probability measure* is a function

$$\mu : \{E_k\} \to [0, 1] \tag{49}$$

that is additive over mutually orthogonal effects and invariant under homotopical equivalences. By the categorical Gleason theorem (Hardy, Wilce 2012), the only such μ is given by

$$\mu(E_k) = \mathrm{Tr}(\rho\, E_k), \tag{50}$$

where ρ is the density matrix corresponding to the initial HQM state via $\mathcal{F}$. In particular, for a pure state $|\psi\rangle$ and a projective measurement $\{\Pi_k\}$,

$$P(k) = \mathrm{Tr}(|\psi\rangle\langle\psi|\,\Pi_k) = |\langle o_k|\psi\rangle|^2. \tag{51}$$

Thus the Born rule follows from the probabilistic consistency of HQM under the functor $\mathcal{F}$, without additional postulates.

IX. Implications for Quantum Foundations

HQM reframes the measurement problem by treating all pre-measurement evolution as occurring within a single homotopy class of paths representing one particle's state. In the absence of which-path information, all geometrically distinct trajectories remain physically indistinguishable and belong to the same equivalence class. The interference pattern is a manifestation of the coherent sum over these homotopic contributions, with higher homotopy groups potentially encoding phase information. When an observer or environment becomes entangled with the system in a way that distinguishes trajectories —such as detecting different photon signatures from two slits—the homotopy equivalence is broken. This induces a contraction, mapping the original class to a single representative path. The probability of this outcome is computed via the homotopical Born rule, ensuring agreement with standard quantum predictions. In this picture:

- Collapse is not fundamental but emerges from the breaking of path equivalence due to entanglement.
- The observer's role is as an agent of entanglement, refining contextual knowledge through the homotopy pullback, not consciously selecting outcomes.

- Nonlocal correlations arise from the global topology of the homotopy class, consistent with Bell-type phenomena.

X. HQM Analysis of EPR/Bell-Type Experiments

A. Pre-measurement Joint Topology

Consider two spatially separated subsystems A and B prepared in an entangled state. The combined configuration space is:

$$\mathcal{C}_{AB} = \mathcal{C}_A \times \mathcal{C}_B \tag{52}$$

with joint paths

$$\Gamma : [0,1] \to \mathcal{C}_{AB}. \tag{53}$$

Before measurement, all joint paths lie in a single homotopy class

$$\in \pi_0(\mathrm{Path}(\mathcal{C}_{AB})), \tag{54}$$

reflecting the entanglement.

B. Measurement on One Side

When a measurement is performed on A, the apparatus–system interaction refines $\mathcal{C}_{AB}$ into disconnected components corresponding to distinct outcomes for A. The contraction sends $[\Gamma]$ to a single representative Γ_*, consistent with the measurement result on A. Since Γ_* is a path in $\mathcal{C}_{AB}$, the contraction determines the correlated outcome for B, reproducing EPR correlations [6].

C. Locality and No-Signaling

Although contraction appears "instantaneous" in the joint configuration space, HQM respects no-signaling: the marginal distribution for B is unchanged unless classical communication is used. The refinement is purely topological in $\mathcal{C}_{AB}$ and does not involve physical signal propagation.

D. Bell Inequalities

Violation of Bell inequalities arises from the global topology of $\mathcal{C}_{AB}$ before contraction, encoding quantum correlations. The "collapse" to Γ_* is replaced by a deterministic contraction triggered by a local interaction, not a stochastic postulate.

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XI. Experimental Signatures of HQM

A. General Strategy

HQM makes the same quantitative predictions as standard quantum mechanics for probabilities, but differs in the mechanism of collapse as topological contraction. Experiments should probe the onset of distinguishability and contraction timing. If contraction requires a minimal entanglement threshold to render a component contractible, HQM predicts a measurable delay in outcome definiteness.

B. Delayed-Choice and Quantum Eraser Tests

In HQM, removing path-distinguishing information before contraction preserves the homotopy class's connectedness, restoring interference. This matches standard quantum eraser predictions but reframes them as prevention of topological refinement. *Prediction*: In ultra-fast eraser setups, if erasure occurs after partial contraction, HQM predicts residual interference with visibility $V = 1 - \epsilon$, where $\epsilon \propto |\mathrm{Aut}([\gamma])|^{-1}$ depends on the homotopy cardinality.

C. Weak Measurements

Weak measurements extract partial information, leading to partial contraction. HQM predicts a continuous degradation of interference visibility $V(\kappa) = e^{-\kappa/\tau}$, where κ is measurement strength and τ is a timescale related to the homotopy cardinality. This matches known results but provides a topological explanation.

D. Macroscopic Superpositions

HQM predicts that isolated macroscopic superpositions persist longer than expected from decoherence if no topological refinement occurs. *Prediction*: In systems like superconducting qubits, coherence time $\tau_c > \tau_d$ (decoherence time) by a factor proportional to the dimension of higher homotopy groups.

E. Entanglement Swapping

HQM interprets entanglement swapping as contraction in an extended joint configuration space. *Prediction*: Time-resolved experiments may reveal a topological delay in correlation onset, measurable in photonic systems, proportional to the entanglement depth.

XII. Resolution of Some Foundational Questions in HQM

A. Characterization of Contraction Dynamics

The question is whether contraction can be described as a continuous-time topological flow, conjectured to be governed by entanglement entropy. We propose modeling contraction using *Moore flows* in HoTT, as defined in the homotopy theory of Moore flows. A Moore flow is a small semicategory enriched over the biclosed semimonoidal category of enriched presheaves over a reparametrization category $\mathcal{R}$, where $\mathcal{R}$ has contractible map spaces and a semigroup structure on objects. This allows for continuous reparametrization of paths, enabling a continuous-time description. Formally, define the contraction dynamics as a Moore flow $M : \mathcal{R} \to [\gamma_k]$, where the flow parameter $t \in [0, 1]$ (reparametrized in $\mathcal{R}$) maps to a homotopy class that progressively contracts to γ_k^*. The contraction functor $F_t : [\gamma_k] \to [\gamma_k]_t$ renders intermediate types partially contractible, with the final $F_1([\gamma_k]) = \mathbf{1}$. The rate is governed by entanglement entropy $S = -\mathrm{Tr}(\rho \log \rho)$, where ρ is the reduced density matrix of the system-observer entanglement. Define the contraction timescale $\tau \propto e^{-S}$, so higher entropy (stronger entanglement) accelerates contraction. This aligns with decoherence timescales but is topological, as entropy measures the "distinguishability" via orthogonalization in $\mathcal{M}$. This model uses the Quillen equivalence between Moore flows and flows, ensuring consistency with HQM's ∞-topos structure.

B. Operational Distinguishability as the Contraction Trigger

While $S_{\mathrm{ent}}(t)$ provides a qualitative indicator of measurement strength, a physically grounded trigger for contraction is obtained from the *operational distinguishability* of apparatus pointer states. Let $\rho_A^{(k)}(t)$ be the apparatus state conditioned on the system being in $|o_k\rangle$. Define the maximal pairwise trace distance:

$$D_{\max}(t) \;=\; \max_{k \neq k'} \tfrac{1}{2} \left\| \rho_A^{(k)}(t) - \rho_A^{(k')}(t) \right\|_1. \tag{55}$$

HQM trigger postulate (operational form): Contraction occurs at the earliest time t_{crit} such that

$$D_{\max}(t_{\mathrm{crit}}) \;\geq\; \epsilon, \tag{56}$$

where ϵ is the minimal resolvable distinguishability of the apparatus (determined by detector noise, thermal fluctuations, or engineering constraints). In this form, the contraction condition is directly measurable and independent of the entropy functional. The link to S_{ent} follows from the inequality

$$D_{\max}(t) \leq \sqrt{\frac{\ln d_A - S_{\mathrm{ent}}(t)}{2 \ln 2}}, \tag{57}$$

for apparatus dimension d_A, ensuring consistency with the entanglement-entropy picture.

C. Contraction Dynamics from Entanglement: A Fully Specified Model

In the HQM framework, contraction is triggered by *topological distinguishability* induced by entanglement between system and apparatus (or environment). To make this fully predictive, we introduce an explicit open-system model in which the onset of contraction is quantitatively defined.

1. Hamiltonian model

Let S denote the system, A the apparatus. The total Hamiltonian is

$$H = H_S \otimes I_A + I_S \otimes H_A + H_{\text{int}}, \tag{58}$$

with measurement interaction

$$H_{\text{int}} = \sum_k |o_k\rangle\langle o_k| \otimes B_k, \tag{59}$$

where $\{|o_k\rangle\}$ is the pointer basis and B_k are Hermitian operators on A producing distinguishable pointer states. This form ensures that if the system is initially in

$$|\psi\rangle = \sum_k \alpha_k |o_k\rangle, \tag{60}$$

the interaction generates an entangled state

$$|\Psi(t)\rangle = \sum_k \alpha_k |o_k\rangle \otimes |A_k(t)\rangle, \tag{61}$$

with $\langle A_k(t)|A_{k'}(t)\rangle$ decreasing towards $\delta_{kk'}$ as the apparatus states become orthogonal.

2. Trigger condition

Let $\rho_S(t) = \mathrm{Tr}_A\big[|\Psi(t)\rangle\langle\Psi(t)|\big]$ be the reduced system state and

$$S_{\text{ent}}(t) = -\mathrm{Tr}\big[\rho_S(t)\log\rho_S(t)\big] \tag{62}$$

its von Neumann entanglement entropy. **HQM contraction postulate:** When

$$S_{\text{ent}}(t) \geq S_{\text{crit}}, \tag{63}$$

with S_{crit} a fixed threshold (e.g. $\log d_{\text{pointer}}$ for a d-dimensional pointer), the homotopy class $[\gamma_k]$ associated with the realized outcome is contracted to a single representative γ_k^*.

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3. Contraction as a CP map

To preserve no-signaling, contraction is implemented as a completely positive (CP) map:

$$\text{Non-selective:} \quad \rho \mapsto \sum_k \Pi_k \rho \Pi_k, \tag{64}$$

$$\text{Selective outcome } k\text{:} \quad \rho \mapsto \frac{\Pi_k \rho \Pi_k}{\mathrm{Tr}(\Pi_k \rho)}, \tag{65}$$

where $\Pi_k = |o_k\rangle\langle o_k|$. In HQM terms, the pullback $\mathcal{P}$ decomposes into components $[\gamma_k]$, and contraction replaces the selected $[\gamma_k]$ by the contractible singleton $\{\gamma_k^*\}$.

4. Born rule derivation

The amplitude functor $\mathcal{A}$ assigns to each path γ an amplitude $\mathcal{A}[\gamma] \propto e^{iS[\gamma]/\hbar}$. For outcome k,

$$P(k) = \frac{\int_{[\psi \to o_k]} |\mathcal{A}[\gamma]|^2 \, d\mu(\gamma)}{\sum_j \int_{[\psi \to o_j]} |\mathcal{A}[\gamma]|^2 \, d\mu(\gamma)}, \tag{66}$$

which, by construction of H_{int}, reduces to

$$P(k) = |\langle o_k|\psi\rangle|^2. \tag{67}$$

Thus the Born rule emerges from the amplitude structure and the CP contraction dynamics.

5. No-signaling

Because the non-selective map is CPTP and local, tracing over S leaves the apparatus' marginal state unchanged. This guarantees that contraction triggered by entanglement cannot be used for superluminal signaling, even though it is deterministic in the HQM formalism.

6. Experimental prediction

If S_{crit} is nonzero, there is a finite delay between the onset of decoherence and full contraction. In ultra-fast weak-measurement experiments, HQM predicts residual interference visibility

$$V(t) \approx e^{-(S_{\text{crit}} - S_{\text{ent}}(t))_+ / \Delta S}, \tag{68}$$

where $(x)_+ = \max(x, 0)$. Standard quantum theory without such a threshold predicts V to drop as soon as decoherence sets in, providing a potential experimental discriminator.

D. Functorial Embedding of HQM in Hilbert Space Quantum Mechanics

Let $\mathcal{HQT}$ be the dagger compact ∞-category of homotopy quantum types, whose objects are homotopy classes of paths with higher morphisms encoding phases and gauge data. Let **Hilb** denote the category of finite-dimensional Hilbert spaces with linear maps. We define a symmetric monoidal dagger functor

$$\mathcal{F} : \mathcal{HQT} \longrightarrow \mathbf{Hilb} \tag{69}$$

with the following properties:

- On objects: $\mathcal{F}([\gamma]) = \mathbb{C}^n$, where n is the number of distinct measurement outcomes reachable from $[\gamma]$ after topological refinement.
- On morphisms: 1-morphisms map to unitary operators; 2-morphisms (homotopies between paths) map to multiplication by $U(1)$ phase factors, encoding gauge holonomies (e.g. Aharonov–Bohm)[7].
- The monoidal product $\otimes$ corresponds to composition of independent systems.

This embedding ensures that any topological contraction in $\mathcal{HQT}$ has a well-defined image under $\mathcal{F}$ as the Lüders update in **Hilb**, preserving the probabilistic structure and reproducing standard interference effects.

E. No-Signaling from HQM Contraction

Let the system S be entangled with a remote system R. A measurement on S in HQM is implemented by:

1. Unitary dilation: $U_{SA} : \mathcal{H}_S \otimes \mathcal{H}_A \to \mathcal{H}_S \otimes \mathcal{H}_A$, with $\mathcal{H}_A$ the apparatus Hilbert space from $\mathcal{F}(\mathcal{HQT})$.
2. Registration: store the pointer value in a classical register C via an isometry V_{AC}.
3. Selective contraction: conditioning on register value k implements $\rho \mapsto \Pi_k \rho \Pi_k / \mathrm{Tr}(\Pi_k \rho)$.

The non-selective map (averaging over k) is

$$\Lambda(\rho_{SR}) = \sum_k (\Pi_k \otimes I_R)\rho_{SR}(\Pi_k \otimes I_R), \tag{70}$$

which is CPTP and satisfies

$$\mathrm{Tr}_S[\Lambda(\rho_{SR})] = \mathrm{Tr}_S[\rho_{SR}], \tag{71}$$

ensuring the marginal state of R is unchanged. Therefore, HQM contraction—when realized via such a dilation—cannot be used for superluminal signaling.

F. Experimental Discrimination

To test HQM's contraction vs. standard QM, adapt the unconscious observer experiment. Setup: Two qubits in superposition, measured by an "unconscious" observer (e.g., quantum dot) that sets another state without full collapse. Apply unitary rotation and measure interference. In standard QM (collapse), probability = 0.5; in unitary interpretations, deviation due to interference. In HQM, contraction delay $\tau \propto e^{-S}$ predicts partial interference if measurement time $< \tau$, leading to visibility $V = 1 - e^{-t/\tau}$. Test in ion traps or photonic systems: Measure coherence persistence beyond decoherence times, or visibility curves in weak measurements. If observed, distinguishes HQM's topological threshold from instantaneous collapse.

G. Relativistic Consistency via Tomonaga–Schwinger Formalism

In relativistic HQM, states are functionals $\Psi[\Sigma]$ on spacelike hypersurfaces Σ [8][9], evolving via the Tomonaga–Schwinger equation

$$i\hbar \frac{\delta}{\delta \Sigma(x)} \Psi[\Sigma] = \mathcal{H}_{\text{int}}(x) \Psi[\Sigma]. \tag{72}$$

Measurement regions are spacetime domains $\mathcal{R}$ with boundary $\partial\mathcal{R} \subset \Sigma$. Contraction is applied locally within $\mathcal{R}$ when the operational distinguishability threshold is met in $\mathcal{R}$, yielding a new functional $\Psi'[\Sigma]$ that differs from $\Psi[\Sigma]$ only inside the causal future of $\mathcal{R}$. This formulation avoids a preferred foliation and guarantees that spacelike-separated contractions commute, preserving Lorentz invariance and no-signaling.

H. Future Directions

- Develop a field-theoretic HQM with infinite-dimensional configuration spaces.
- Apply HQM to quantum gravity, where topological changes are natural.
- Unify HQM with topos-theoretic quantum logic via ∞-categorical methods.

XIII. Discussion and Outlook

A. Summary of HQM's Core Contributions

HQM reframes quantum measurement as a topological contraction of a homotopy class, retaining unitary evolution, attributing collapse to topological distinguishability, and eliminating the stochastic postulate. It extends to multipartite systems via joint configuration spaces and homotopy pullbacks.

B. Measurement Problem Revisited

The measurement problem becomes the question of what triggers topological refinement. Distinguishability is the essential trigger, with contraction ensuring deterministic outcomes and interference requiring connectedness.

C. Relation to Other Approaches

- **Decoherence** — HQM incorporates decoherence as a mechanism for distinguishability but views it as sufficient, not necessary; contraction provides definiteness.
- **Many-worlds** — HQM avoids branching by selecting a single path after contraction.
- **Objective collapse** — HQM's contraction is deterministic; randomness arises from the choice of refinement.

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