https://doi.org/10.32388/L1CC6H.3

A New Price of the Arithmetic Asian Option: A Simple Formula

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Abstract: We introduce a simple, explicit formula for pricing the arithmetic Asian options. The pricing formula is as simple as the classical Black-Scholes formula.

Keywords: Average of log-normals, arithmetic Asian option, the Black-Scholes formula.

MSC: 60E05

1 Introduction

To our knowledge, there is no explicit formula for pricing Arithmetic Asian options. Recent literature used orthogonal polynomial expansions to approximate the distribution of the arithmetic average. Examples include Willems (2019) and Asmussen et al (2016). Some of the literature used Edgeworth expansions to approximate the distributions (see, for example, Li and Chen (2016)). Gambaro et al (2020) used a tree method for discrete Asian options. Carsaro et al (2019) adopted a computational method. Cui et al (2018) used approximations. Others such as Aprahmiam and Maddah (2015) used the Gamma distribution approach. Some studies relied on Monte Carlo simulations. Examples include Lapeyre et al (2001) and Fu et al (1999). Others adopted a numerical approach. Examples include Linetsky (2004), Cerny and Kyriakou (2011), and Fusai et al (2011). Curran (1994) used the geometric mean to estimate the arithmetic mean.

The literature on pricing the arithmetic Asian options has two main features in common. First, it relies on approximations. Secondly, it largely adopts complex methods. Consequently, this paper overcomes these two limitations. In this paper, we use a pioneering approach to pricing the arithmetic

Asian options in continuous time. In doing so, we present an exact, simple formula. Particularly, we show that the price of the arithmetic Asian option is equivalent to the price of the European option with an earlier known expiry. The pricing formula is as simple as the classical Black-Scholes formula.

2 The method

The arithmetic average of the price underlying asset $S(u)$ over the time interval $[t, T]$ is given by

$$A_t = \frac{\int_t^T S(u)\,du}{T - t}, \tag{1}$$

where t is the initial time and T is the expiry time. So that, using the Black-Scholes assumptions, $EA_t = E\frac{\int_t^T S(u)du}{T-t} = \frac{e^{r(T-t)}-1}{r(T-t)}s$, where $s \equiv S(t)$ and r is the risk-free rate of return.

By the mean value theorem for integrals, $E\frac{\int_t^T S(u)du}{T-t} = ES\left(\hat{t}\right)$, where $\hat{t}$ is a time such that $t < \hat{t} < T$ and $ES\left(\hat{t}\right) = e^{r\left(\hat{t}-t\right)}s$. This implies that $\frac{e^{r(T-t)}-1}{r(T-t)} = e^{r\left(\hat{t}-t\right)}$. We can solve for $\hat{t}$ as follows

$$\hat{t} = t + \frac{\ln\left(\frac{e^{r(T-t)}-1}{r(T-t)}\right)}{r}. \quad (2)$$

Thus, $\hat{t}$ is known. For example, if $T - t = 1$, $t = 0$ and $r = .01$, $\hat{t} = t + \frac{\ln\left(\frac{e^{.01}-1}{.01}\right)}{.01} = .498$. Also, by continuity, there is a constant a such that $Var\left(A_t\right) = s^2 e^{2r\left(\hat{t}-t\right)}\left(e^{a\left(\hat{t}-t\right)} - 1\right)$.

Proposition:

$$C\left(t, s\right) = sN\left(d_1\right) - e^{-r(T-t)}KN\left(d_2\right), \quad (3)$$

where $d_1 = \frac{1}{\sqrt{\check{v}^2(T-t)}}\left[\ln\left(s/K\right) + \left(r + \check{v}^2/2\right)\left(T-t\right)\right]$, $d_2 = d_1 - \sqrt{\check{v}^2\left(T-t\right)}$, and K is the strike price.

Proof. Let $A_t = e^{\ln A_t} = \frac{s}{s}e^{\ln A_t} = se^{c+\ln A_t}$, where c is not random. Consider this transformation

$$A_t = se^{c+\ln A_t} = se^{c+\frac{W_T}{W_T}\ln A_t} = se^{c+VW_T}, \quad (4)$$

where W_T is a Brownian motion. The option price can be expressed as a weighted average of the Black-Scholes prices conditional on V as follows

$$C(t) = \int_v E\left[e^{-r(T-t)} g(A) / V = v\right] dF(v) = \int_v C_{BS}(v)\, dF(v), \tag{5}$$

where g is the payoff, T is the expiry time, F is the cumulative density of V, and C_{BS} is the Black-Scholes price. By the continuity, the expected value is a specific value of C_{BS} denoted by $\hat{C}_{BS} = C_{BS}(\check{v})$, where $\check{v}$ is a value (outcome) of V. Thus, by continuity

$$C(t) = \int_v C_{BS}(v)\, dF(v) = C_{BS}(\check{v}). \tag{6}$$

Thus, the price of the call option is

$$C(t) = sN(d_1) - e^{-r(T-t)} KN(d_2), \tag{7}$$

where $d_1 = \frac{\ln(s/K)+(r+\check{v}^2/2)(T-t)}{\sqrt{\check{v}^2(T-t)}}$ and $d_2 = d_1 - \sqrt{\check{v}^2(T-t)}$. $\square$

Similar to other models, the volatility parameter $\check{v}$ can be estimated.

A verification:

There is a simple way to verify the result. $C(t, s)$ is the true Asian option price, and $C_{BS}(r, s, \sigma, T-t)$ is the Black-Scholes price of the European option.

By continuity, there is a specific value of the volatility parameter such as $\check{\sigma} = \check{v}$, so that $C(t, s) = C_{BS}(r, s, \check{v}, T - t)$. Therefore, the true Asian option price can be expressed as the Black-Scholes formula with volatility $\check{v}$.

Practical example:

If $r = .05$, $T - t = 1$, $\check{v} = .2$, $s = K = \$100$, then the option price is $C(t, s) = \$6.91$.

3 Conclusion

In sum, this paper offers an explicit, simple formula for the price of the arithmetic Asian options. The contribution will have a big impact on statistics since it will have so many applications in the future. Furthermore, there is a big practical advantage. In practice, the choice of the discrete times to be included in the average is arbitrary and controversial. The industry can avoid this problem altogether by trading continuous-average options (using our formula).

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