Pricing the Arithmetic Asian Options: An Explicit, Simple Formula

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Abstract

We introduce a simple, explicit formula for pricing the arithmetic Asian options. The pricing formula is as simple as the classical Black-Scholes formula. Our method is applicable to both discrete and continuous averages.

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1. Introduction

A long-standing obstacle in statistics is the determination of the distribution of the sum of log-normal variables. To our knowledge, there is no explicit formula for pricing Arithmetic Asian options. Recent literature used orthogonal polynomial expansions to approximate the distribution of the arithmetic average. Examples include Willems (2019) and Asmussen et al. (2016). Some of the literature used Edgeworth expansions to approximate the distributions (see, for example, Li and Chen (2016)). Gambaro et al (2020) \ used a tree method for discrete Asian options. Carsaro et al (2019) adopted a computational method. Cui et al (2018) used approximations. Others such as Aprahmiam and Maddah (2015) used the Gamma distribution approach. Some studies relied on Monte Carlo simulations. Examples include Lapeyre et al. (2001) and Fu et al (1999). Others adopted a numerical approach. Examples include Linetsky (2004), Cerny and Kyriakou (2011), and Fusai et al (2011). Curran (1994) used the geometric mean to estimate the arithmetic mean.

The literature on pricing the arithmetic Asian options has two main features in common. First, it relies on approximations.

Secondly, it largely adopts complex methods. Consequently, this paper overcomes these two limitations. In this paper, we use a pioneering approach to pricing the arithmetic Asian options. In doing so, we present an exact method. Particularly, we show that the price of the arithmetic Asian option is exactly equivalent to the price of the European option with an earlier (known) expiry. The pricing formula is as simple as the classical Black-Scholes formula. Our method is applicable to both discrete and continuous averages.

2. The method

The arithmetic average of the price underlying asset $S(u)$ over the time interval $[t, T]$ is given by

$$A_t = \frac{\int_t^T S(u)du}{T-t}, \qquad (1)$$

where t is the initial time and T is the expiry time. So that, using the Black-Scholes assumptions,

$EA_t = E\frac{\int_t^T S(u)du}{T-t} = \frac{e^{r(T-t)}-1}{r(T-t)}s,$ where $s \equiv S(t)$ and r is the risk-free rate of return.

For the discrete case, $EA_t = \frac{\sum_u ES(u)}{n}$.

Let $A_t = e^{\ln A_t} = \frac{s}{s}e^{\ln A_t} = se^{c+\ln A_t},$ where c is a constant. Consider this transformation

$$A_t = se^{c+\ln A_t} = se^{c+\frac{W_T}{W_T}\ln A_t} = se^{c+V_T W_T} \qquad (2)$$

where W_T is a Brownian motion. The option price can be expressed as a weighted average of the Black-Scholes prices conditional on V as follows%

$$C(t) = \int_V E\left[e^{-r(T-t)}g(A)/V_T = v\right]dF(v) = \int_V C_{BS}(v)dF(v), \qquad (3)$$

where g is the payoff, T is the expiry time, F is the cumulative density of V, and C_{BS} is the Black-Scholes price. By the continuity, the expected value is a specific value of C_{BS} denoted by $\hat{C}_{BS} = C_{BS}(\hat{v}),$ where $\hat{v}$ is a value (outcome) of V. Thus,

$$C(t) = \int_V C_{BS}(v)dF(v) = C_{BS}(\hat{V}). \qquad (4)$$

Thus, the price of the call option is

$$C(t) = sN(d_1) - e^{-r(T-t)}KN(d_2), \qquad (5)$$

where $d_1 = \frac{\ln(s/K) + \frac{\hat{v}2}{2}(T-t)}{\sqrt{\hat{v}2(T-t)}}$, $d_2 = d_1 - \sqrt{\hat{v}2(T-t)}$ and K is the strike price.

Even in the classical Black-Scholes model, the volatility parameter needs to be estimated and the estimation method is arbitrary; similarly, the volatility parameter $\check{V}$ can be estimated. Moreover, the implied value of $\check{V}$ can be computed using the price formula, and the implied values can be used to estimate $\check{V}$.

A verification:

A simple way to verify the result is to let $\tilde{C}(t)$ be the true Asian option price, and $C_{BS}(\sigma)$ be the classical Black-Scholes price of the European option. By the continuity of C_{BS}, there is a specific value of the volatility parameter σ, such as $\hat{\sigma}$, so that $\tilde{C}(t) = C_{BS}(\hat{\sigma}) = C_{BS}(\hat{V})$. Therefore, the true Asian option price can be expressed using the Black-Scholes formula.

3. Conclusion

This paper will open a new path for pricing Asian options. This approach enables firms and investors to calculate the implied volatility for Asian options. This is very useful for firms and investors.

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