Computer Analysis of Stochastic Aging According to the Gompertz-Makeham Mortality Law

Marcin Kamiński
Faculty of Civil Engineering, Architecture and Environmental Engineering
Łódź University of Technology, Al. Politechniki 6, 93-590 Łódź, POLAND
Email: Marcin.Kaminski@p.lodz.pl

Abstract – The main aim is to present stochastic computer analysis of the Gompertz-Makeham mortality law applied frequently in biology to approximate mortality rates in various species populations. The exponential time series with three different coefficients defined as the Gaussian uncorrelated random variables is analyzed and its first four central probabilistic moments are derived analytically from the definition as the functions of expectations and standard deviations of these coefficients. They are used further in the visualization of time fluctuations of the expectations, coefficients of variation, skewness, and kurtosis of the mortality rate. Computational experiments performed in the computer algebra system MAPLE compare all these characteristics for various combinations of the input coefficients of variation of the input randomness level. They document that probabilistic characteristics of the mortality rate highly depend upon the input probabilistic parameters combination, where Gaussian uncertainty within the exponent seems to be the most influential. The numerical approach explored in this work may be further extended towards some other probabilistic methods like simulation or perturbation-based algorithms, other probability distributions in time series coefficients, power or polynomial mortality laws with random coefficients as well as more advanced modeling of the mortality rate defined as some stochastic process using probability of transition in time.

Keywords: Gompertz-Makeham mortality law; time series; probabilistic moments; symbolic analysis;

1. Introduction

Stochastic analysis of mortality in biological populations and durability in engineering systems are similar from the mathematical point of view and can be efficiently characterized by the specific time series with random coefficients. There are polynomial, power, and exponential time series representations available in this research area, cf. Charlesworth (2001)

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[1], Jodrá (2009) [2], and van Noortwijk and Pandey (2004) [3] as well as, of course, various stochastic processes with the given transition probability function of survival in time employed by Gavrilov and Gavrilova (2001) [4], Gonos, Trougakos and Chondrogianni (2007) [5] and Sobczyk (1991) [6]. It should be mentioned that the aging phenomenon common for all populations in biology has been numerically studied also in nonlinear models in engineering in terms of the corrosion process proposed by Kamiński (2013) [7] or with the homogenization method applicable for composite materials designing process, cf. Kamiński (2008) [8]. A time series representation is more convenient in computational analysis as it may be directly derived from the statistical data widely accessible in biology, medicine, and engineering and it results in large time and computer power savings and also with high accuracy. It offers a direct analytical derivation of the basic probabilistic characteristics of these series when their coefficients have uniquely defined specific probability density functions, i.e. Gaussian or lognormal, for instance. Full and unique characterization of stochastic aging by its probabilistic characteristics of time fluctuations enables for a reliable prediction of the given population probability of survival or, analogously in engineering, stochastic time-dependent reliability and durability (prediction) of the analyzed technical structure. It should be underlined that computer simulation of the mortality phenomenon and progress in some populations by the stochastic process with renewal, although theoretically very sound, cannot be directly associated with the experimental data. That is why computer analysis of the mortality is provided here with the use of time series with Gaussian coefficients and has been implemented into the symbolic algebra system MAPLE – it is related to the most popular model in this area called Gompertz-Makeham exponential law, see Gompertz (1825) [9], Makeham (1860) [10] and Wrycza (2014) [11]. It is studied including uncertainty in both time-dependent and independent components that are fully uncorrelated (according to the lack of experimental data). Time fluctuations of the first four central probabilistic moments of the mortality rate are analytically derived from the classical integral definitions as some functions of the expectations and standard deviations of these Gaussian coefficients. Next, these moments are numerically studied in the context of their stochastic sensitivity concerning various input uncertainties in the mortality law coefficients. Their visualization and detailed discussion enable us to predict whether mortality distribution at the given time may remain Gaussian or cannot, which is of paramount importance in survival probability estimation for various populations. The paper concludes with a listing of further possible extensions of this model towards some other probability distributions in the coefficients, other frequently explored analytical representations of the mortality itself as well

as related concurrent stochastic computer techniques enabling faster and more accurate analysis of these phenomena.

2. Probabilistic analysis

We assume the mortality law in the following form:

$$\mu(\omega;t) = A(\omega)\exp(B(\omega)\,t) + C(\omega), \tag{1}$$

where $A(\omega)$, $B(\omega)$, and $C(\omega)$ are uncorrelated Gaussian random variables with uniquely given first two probabilistic moments – E[A], $\sigma(A)$, E[B], $\sigma(B)$ as well as E[C], $\sigma(C)$, correspondingly. Additionally, it is assumed that these variables are fully uncorrelated and this algebraic simplification follows a lack of statistical data rather, as all the cross-correlations can be relatively easily incorporated into this model. We investigate analytically the first four probabilistic moments of this mortality process $\mu(X;\tau)$ at the given time moment $\tau \in [0,\infty)$ depending upon the random variable X with the probability density function g_X via the following classical integral definition of the kth central probabilistic moment:

$$m_k(\mu(X;\tau)) = \int_{-\infty}^{+\infty} (\mu(X;\tau) - E[\mu(X;\tau)])^k g_X(x)dx, \tag{2}$$

where $E[\mu(X;\tau)]$ is the expectation of this series defined traditionally as

$$E[\mu(X;\tau)] = \int_{-\infty}^{+\infty} \mu(X;\tau)\, g_X(x)dx. \tag{3}$$

These are in turn:

- expected values

$$E[\mu(\omega;t)] = E[A]\exp\left(E[B]\,t + \tfrac{1}{2}\sigma^2(B)\,t^2\right) + E[C], \tag{4}$$

- variances

$$
\begin{aligned}
Var(\mu(\omega;t)) = \\
&\left(E^2[A]+\sigma^2(A)\right)\exp\left(2E[B]t+2\sigma^2(B)t^2\right)+ \\
&-E^2[A]\exp\left(2E[B]t+\sigma^2(B)t^2\right)+\sigma^2(C),
\end{aligned}
\tag{5}
$$

- third central probabilistic moments

$$
\begin{aligned}
\mu_3(\mu(\omega;t)) = \\
&\left(E^3[A]+3\sigma^2(A)E[A]\right)\exp\left(3E[B]t+4.5\sigma^2(B)t^2\right)+ \\
&-\left(3\sigma^2(A)E[A]+3E^3[A]\right)\exp\left(3E[B]t+2.5\sigma^2(B)t^2\right)+ \\
&+2E^3[A]\exp\left(3E[B]t+1.5\sigma^2(B)t^2\right),
\end{aligned}
\tag{6}
$$

- fourth central probabilistic moments

$$
\begin{aligned}
\mu_4(\mu(\omega;t)) = \\
&\left(6E^4[A]+6\sigma^2(A)E^2[A]\right)\exp\left(4E[B]t+3\sigma^2(B)t^2\right)+ \\
&-\left(4E^4[A]+12\sigma^2(A)E^2[A]\right)\exp\left(4E[B]t+5\sigma^2(B)t^2\right)+ \\
&+\left(3\sigma^4(A)+6\sigma^2(A)E^2[A]+E^4[A]\right)\exp\left(4E[B]t+8\sigma^2(B)t^2\right)+ \\
&-3E^4[A]\exp\left(4E[B]t+2\sigma^2(B)t^2\right)+ \\
&+\left(6\sigma^2(C)E^2[A]+6\sigma^2(C)\sigma^2(A)\right)\exp\left(2E[B]t+2\sigma^2(B)t^2\right) \\
&-6\sigma^2(C)E^2[A]\exp\left(2E[B]t+\sigma^2(B)t^2\right)+3\sigma^4(C).
\end{aligned}
\tag{7}
$$

These equations may be simplified further if only an uncertainty in one of the coefficients *A*, *B*, or *C* is excluded, nevertheless, the analytical forms obtained here have elegant and relatively simple forms that are easy for further computational implementation, either outside a symbolic algebra program. According to the fact that the mortality rate depends linearly upon the coefficient *C* treated here as the Gaussian random variable, the third central probabilistic moment $\mu_3(\mu(\omega;t))$ (and skewness at the same time) is free from any contribution of the quantity *C*. Let us note that these equations can be derived alternatively by using the complex exponential characteristics function adjacent to the Gaussian random variable. Analogous equations related to the power time series with Gaussian random coefficients have been derived and analyzed numerically in [7]. All the aforementioned moments enable us further to calculate the basic probabilistic coefficients, also in an analytical manner (via symbolic computer analysis), and we have in turn

- coefficient of variation

$$\alpha(\mu(\omega;t)) = \sqrt{\frac{Var(\mu(\omega;t))}{E^2[\mu(\omega;t)]}}, \tag{8}$$

- skewness

$$\beta(\mu(\omega;t)) = \frac{\mu_3(\mu(\omega;t))}{\sigma^3(\mu(\omega;t))}, \tag{9}$$

- as well as kurtosis

$$\kappa(\mu(\omega;t)) = \frac{\mu_4(\mu(\omega;t))}{\sigma^4((\mu(\omega;t)))} - 3, \tag{10}$$

where

$$\sigma(\mu(\omega;t)) = \sqrt{Var(\mu(\omega;t))} \tag{11}$$

stands classically for the standard deviation.

3. Computational analysis

Further numerical analysis is focused on the expectations (Figs. 1-3), coefficients of variation (Figs. 4-6), skewness (Figs. 7-9), and kurtosis (Figs. 10-12) of the mortality rate determined as the function of time and some input coefficient of variation of *A*, *B*, and *C*, separately. All the coefficients in the analyzed time series are randomized due to the Gaussian probability distribution at the same time, but one of the coefficients of variation of these parameters changes additionally in the interval [0.00,0.10]. The input expected values are adopted as deterministic input data from the computational sensitivity analysis carried out by Wrycza (2014). The following probabilistic data are adopted: E[*A*]=0.0001 (left series in computer visualization and upper bound on time-dependent aging) and 0.00001 (right series and the lower bound, correspondingly), E[*B*]=0.1 and E[*C*]=0.1 together with their coefficients of variation α(*A*), α(*B*) and α(*C*) smaller or equal to 0.10. The surfaces of the expected values (Figs. 1-3), coefficients of variations (Figs. 4-6), skewness (Figs. 7-9) as well as kurtosis (Figs. 10-12) varying in time are also presented as the functions of input randomness level. The first surface is obtained with α(*A*)=[0.00,0.10] with constant α(*B*)=α(*C*)=0.10, the second series is computed for α(*B*)=[0.00,0.10] with constant α(*A*)=α(*C*)=0.10, and finally, the third one with α(*C*)=[0.00,0.10] and for the constants α(*A*)=α(*B*)=0.10 to verify stochastic sensitivity of the mortality rate probabilistic characteristics.

The expected values surfaces contained in Figs. 1-3 document very precisely that uncertainty in the exponent constant *B* plays the major role here, while these expectations are almost insensitive to the applied fluctuations in coefficients of variation α(*A*) and α(*C*); their extreme values are not so large as for other computational studies [7,8], where one may find 0.20 or even 0.30. This influence of α(*B*) and the resulting value E[μ] systematically increase both together with the aging time. The ratio of the results obtained in the right column and the left one equals almost 10, i.e. the ratio of the expected values E[*A*] applied in these computer experiments, what is expected after Eqn. (1). A small difference from 10 illustrates that the resulting expected values increase a little bit slower as the parameter E[*A*] is included also in the exponent term, contrary to the deterministic situation. The very specific situation is obtained in the right graph in Fig. 3, where α(*C*)®0 leads to the sudden decrease of mortality expectation for the entire time domain. It agrees with the previous deterministic computational experiments, where the parameter *C* has been discovered as the important factor at the early stage of the entire aging process. The majority in an uncertainty of the parameter *B* is more apparent in Figs. 4-6, where we detect no fluctuations of α(μ) together with any modifications of the coefficients α(*A*) and α(*C*) separately. Nevertheless, the extreme output coefficient of variation in any case remains the same and is a few times larger than the extreme combination of input coefficients α(*A*), α(*B*), and α(*C*) altogether. It means that the mortality rate in the exponential Gompertz-Makeham law is highly sensitive to the uncertainty in its coefficients, and especially to α(*B*). This parameter is so important that α(*B*)=0 reduces about ten times the overall value of α(μ) (see Fig. 5). It should be underlined also that the randomness in this aging process increases exponentially in time and becomes many times larger after the critical age of 60, which usually agrees even with the personal observations. Interestingly, the expected value E[*A*] plays also some role in α(μ) and the ten times increase of its amount leads to about a 20% increase in this coefficient of variation – it is a very unusual situation compared to some other case studies [6]. This is of course because E[*A*] and α(*B*) are combined in Eqn. (5) and, as a result, also in Eqn. (8).

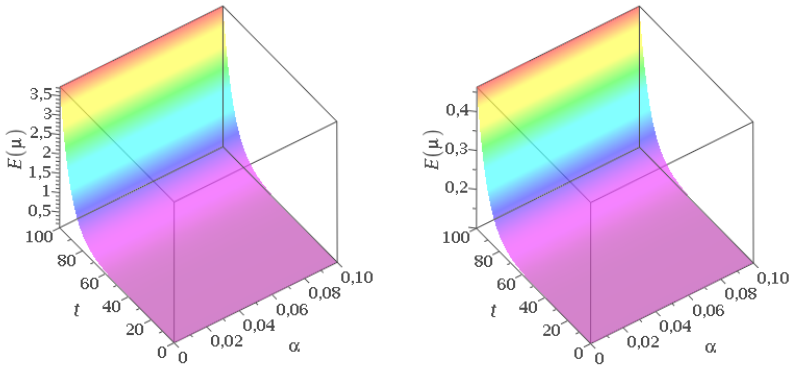

Fig. 1. Expected values of the mortality as the function of time and α(*A*) with α(*B*)=α(*C*)=0.10; E[*A*]=0.0001 (left graph) and E[*A*]=0.00001 (right graph)

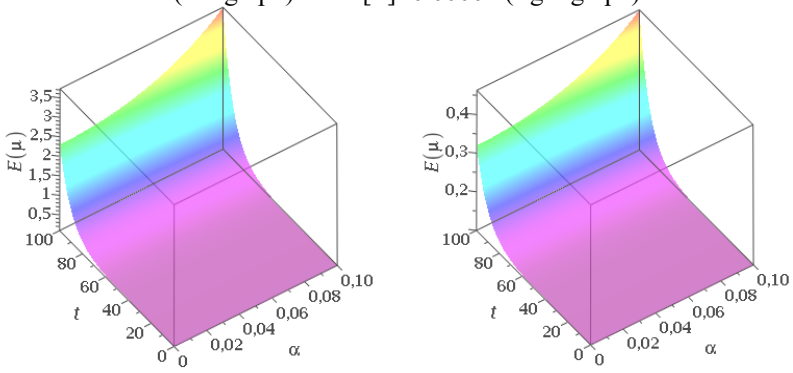

Fig. 2. Expected values of the mortality as the function of time and α(*B*) with α(*A*)=α(*C*)=0.10; E[*A*]=0.0001 (left graph) and E[*A*]=0.00001 (right graph)

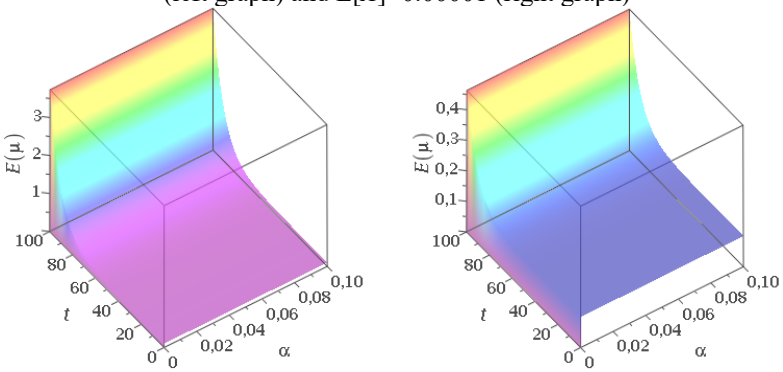

Fig. 3. Expected values of the mortality as the function of time and α(*C*) with α(*A*)=α(*B*)=0.10; E[*A*]=0.0001 (left graph) and E[*A*]=0.00001 (right graph)

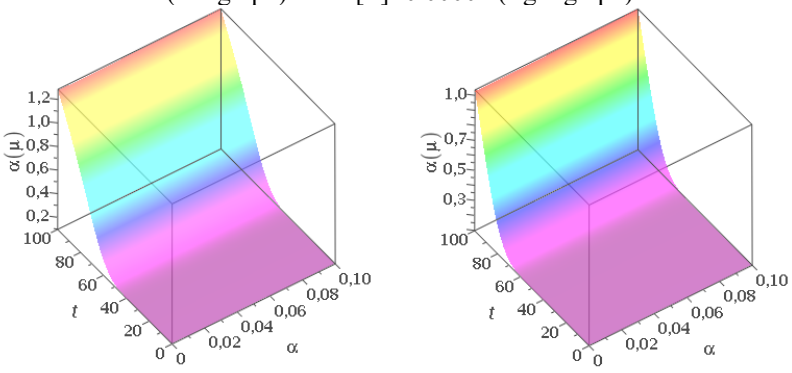

Fig. 4. Coefficients of variations of the mortality as the function of time and α(*A*) with α(*B*)=α(*C*)=0.10; E[*A*]=0.0001 (left graph) and E[*A*]=0.00001 (right graph)

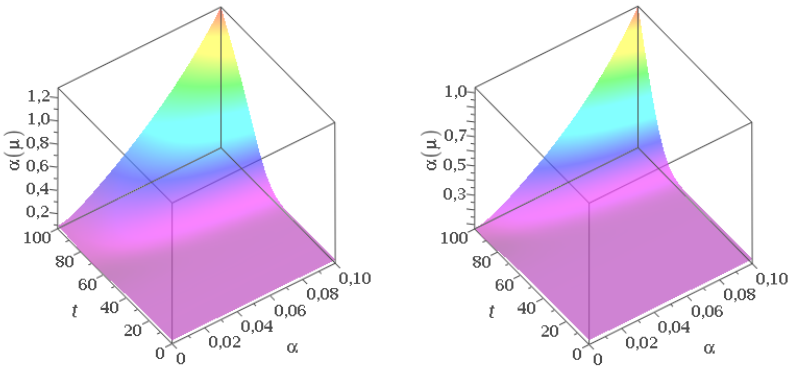

Fig. 5. Coefficients of variations of the mortality as the function of time and α(*B*) with α(*A*)=α(*C*)=0.10; E[*A*]=0.0001 (left graph) and E[*A*]=0.00001 (right graph)

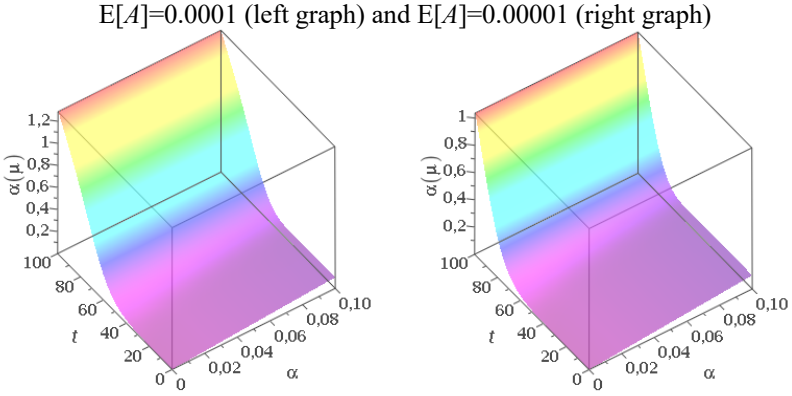

Fig. 6. Coefficients of variations of the mortality as the function of time and α(*C*) with α(*A*)=α(*B*)=0.10; E[*A*]=0.0001 (left graph) and E[*A*]=0.00001 (right graph)

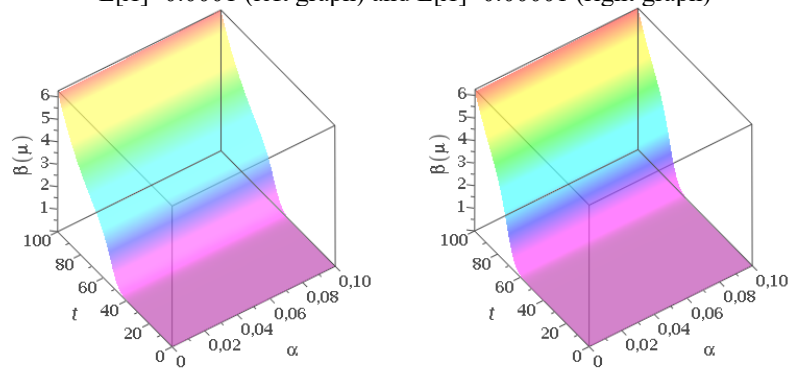

Fig. 7. Skewness of the mortality as the function of time and α(*A*) with α(*B*)=α(*C*)=0.10; E[*A*]=0.0001 (left graph) and E[*A*]=0.00001 (right graph)

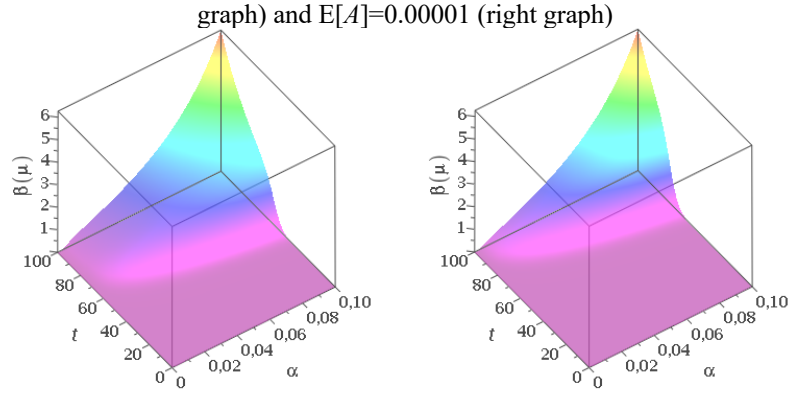

Fig. 8. Skewness of the mortality as the function of time and α(*B*) with α(*A*)=α(*C*)=0.10; E[*A*]=0.0001 (left graph) and E[*A*]=0.00001 (right graph)

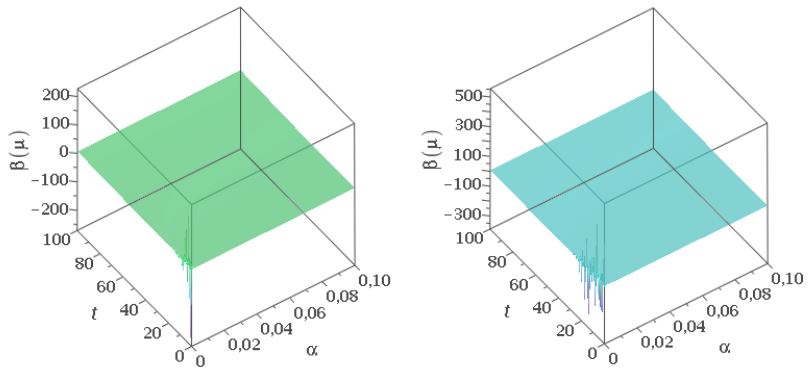

Fig. 9. Skewness of the mortality as the function of time and α(*C*) with α(*A*)=α(*B*)=0.10; E[*A*]=0.0001 (left graph) and E[*A*]=0.00001 (right graph)

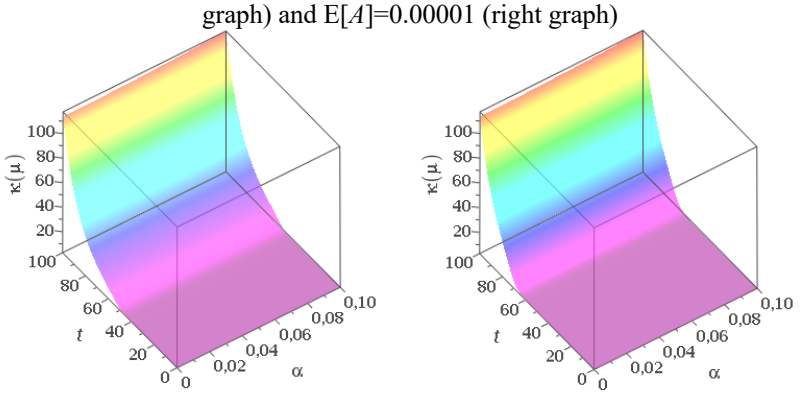

Fig. 10. Kurtosis of the mortality as the function of time and α(*A*) with α(*B*)=α(*C*)=0.10; E[*A*]=0.0001 (left graph) and E[*A*]=0.00001 (right graph)

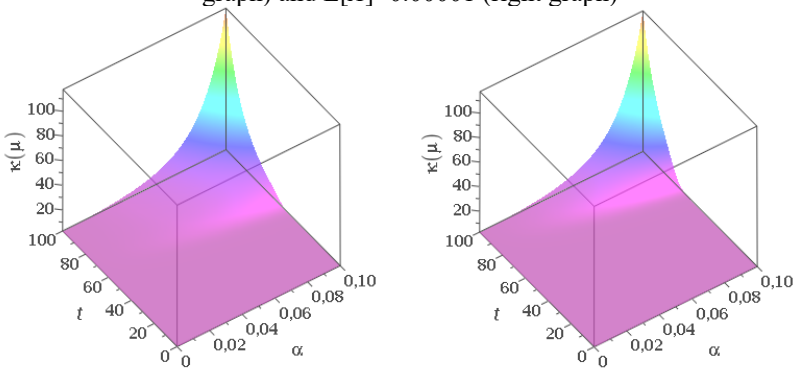

Fig. 11. Kurtosis of the mortality as the function of time and α(*B*) with α(*A*)=α(*C*)=0.10; E[*A*]=0.0001 (left graph) and E[*A*]=0.00001 (right graph)

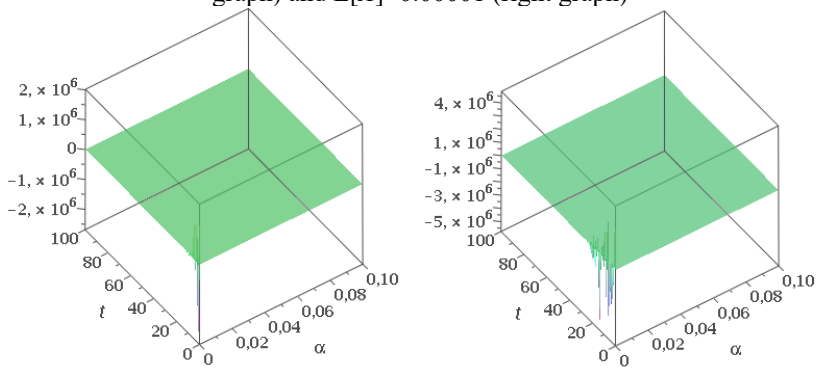

Fig. 12. Kurtosis of the mortality as the function of time and α(*C*) with α(*A*)=α(*B*)=0.10; E[*A*]=0.0001 (left graph) and E[*A*]=0.00001 (right graph)

Finally, we study skewness (Figs. 7-9) and kurtosis (Figs. 10-12) which are unavailable in the literature on this problem. They show whether the final probability distribution of the mortality rate may be symmetric or it is not as well as show the concentration of the probability tale about the expected value (0 is adjacent here to the Gaussian bell-shaped curve and a higher value means of course closer concentration). These coefficients are a little bit different than the results contained in Figs. 1-6. Figs. 7-10 document that an uncertainty in the parameter *A* has no influence on the final results at any time of the aging process. Both parameters start from zeroes at t=0 and reach the values $\beta(\mu)=6$ and $\kappa(\mu)>100$ at the end of the time scale, so that $\mu(t=100)$ cannot have a distribution that is even similar to the Gaussian one (cannot be uniquely defined by its first two moments either). Figs. 8 and 11 confirm the majority of the uncertainty in *B* because $\alpha(B)\to 0$ returns $\beta(\mu)\to 0$ together with $\kappa(\mu)\to 0$ at any time of this process, while the mortality rate is insensitive to this coefficient at time t=0 even in the deterministic case, after Eqn. (1). Both skewness and kurtosis exponentially increase together with time and, independently, with the value of $\alpha(B)$. The extreme combination of skewness and kurtosis corresponds exactly to Figs. 7 and 10, but right now these parameters remain almost zero elsewhere. Numerical results obtained for $\alpha(C)\in[0.00,0.10]$ and given in Figs. 9 and 12 entirely differ from the others. Both skewness and kurtosis equal 0 almost everywhere except some very small computational domains, where $\alpha(C)$ and *t* are close to 0. The very large positive and negative variations of $\beta(\mu)$ and $\kappa(\mu)$ result from numerical discrepancies rather, however, one needs to remember that $\alpha(C)$ is included also in the denominators of Eqns. (9-10) and these denominators are close to 0 at the beginning of both parameters variability scales. This phenomenon coincides a little bit with fluctuations in the corresponding expected values in Fig. 3 (larger effect on the right graph, more smoother transition to 0 on the left graph). The principal conclusion comes from Figs. 7-12 is that the mortality rate can have a distribution similar to the Gaussian probability density function while randomizing *A* and *C* according to the same Gaussian PDF and inserting *B* as the deterministic parameter.

4. Concluding remarks

[1] Gaussian uncertainty in all parameters of the Gompertz-Makeham exponential mortality law enables an analytical derivation of the basic first four probabilistic characteristics of the mortality rate as the functions of the expected values and standard deviations of the constants A, B, and C included in this model. It reflects very well the existing statistical evidence concerning both mortality in biology and medicine as well as durability verification in various areas of modern engineering. The resulting analytical formulas are compact enough to be implemented in any computer program, also out of the computer algebra system like MAPLE or MATLAB.

[2] The major uncertainty source is discovered in the parameter B appearing in this law and it is done through the relevant stochastic sensitivity verification. This randomness together with the uncorrelated uncertainties in A and C may result in the coefficient of variation $\alpha(\mu)$ that is ten times larger than the input ones. Quite naturally and expectedly, this output uncertainty increases exponentially together with aging time and is enormously large after the age of almost 60 years. Further, we notice that a coincidence of all the input extreme uncertainties results in the probability distribution of the mortality rate that is non-symmetric (β close to 6) and has a larger concentration about its expectation than the Gaussian PDF (κ tends to even 100). Smaller stochastic sensitivity of the mortality to the parameters A and especially C is confirmed in the statistical context also.

[3] Further computational studies in this area may concern some other mortality laws like power or polynomial time series with the coefficients having non-Gaussian distributions like at least log-normal one that is justified frequently in the engineering systems durability. It should be mentioned that several probability distributions exclude the availability of an analytical derivation presented above and then one may implement alternatively the traditional Monte-Carlo random sampling in discrete time moments or the generalized stochastic perturbation technique presented in [7].

References

[1] B. Charlesworth, 2001. Patterns of age-specific means and genetic variances of mortality rates predicted by the mutation-accumulation theory of ageing. J. Theor. Biology, 210: 47-65.

[2] P. Jodrá P, 2009. A closed-form expression for the quantile function of the Gompertz–Makeham distribution. Math. Comput. Simul., 79(10): 3069-3075.

[3] J.M. van Noortwijk, M.D. Pandey, 2009. A stochastic deterioration process for time-dependent reliability analysis. In: M.A. Maes, L. Huyse L, edrs. Proc. Eleventh IFIP WG 7.5 Working Conference on Reliability and Optimization of Structural Systems, London, Taylor & Francis, 259-265.

[4] L.A. Gavrilov, S.N. Gavrilova, 2001. The reliability theory of aging and longevity. J. Theor. Biology, 213(4): 527-545.

[5] E.S. Gonos, I.P. Trougakos, N. Chondrogianni, edrs., 2007. Molecular Mechanisms and Models of Aging. New York Acad. Sci. Annals 119, Boston, Blackwell.

[6] K. Sobczyk, 1991. Stochastic differential equations with applications to physics and engineering. Amsterdam, Kluwer Academic Publishers.

[7] M. Kamiński, 2013. The Stochastic Perturbation Method for Computational Mechanics. Chichester, Wiley.

[8] M. Kamiński, 2008. Homogenization of fiber-reinforced composites under the stochastic aging process. Int J Multiscale Comput. Engrg., 6(4): 361-370.

[9] B. Gompertz, 1825. On the nature of the function expressive of the law of the human mortality. Phil. Trans. Roy. Soc., 115: 513–585.

[10] W.M. Makeham, 1860. On the law of mortality and the construction of annuity tables. J. Inst. Actuaries & Assur. Mag., 8: 301–310.

[11] T.F. Wrycza, 2014. Entropy of the Gompertz-Makeham mortality model. Demographic Res., 30: 1397-1404.