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NSE CHARACTERIZATION OF THE ORTHOGONAL GROUP $O_7(3)$

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ABSTRACT. Let G be a group and $\omega(G)$ be the set of element orders of G. Let $k \in \omega(G)$, $s_k = |\{g \in G | o(g) = k\}|$ and $nse(G) = \{s_k | k \in \omega(G)\}$. In this paper, we prove that if G is a group and $O_7(3)$ is the Orthogonal simple group over $GF(3)$ such that $nse(G) = nse(O_7(3))$, then $G \cong O_7(3)$.

1. Introduction

Let G be a finite group and $\omega(G)$ be the set of element orders of G. If $k \in \omega(G)$, then s_k is the number of elements of order k in G. Let $nse(G) = \{s_k | k \in \omega(G)\}$. If n is a positive integer, the set of all the prime divisors of n is denoted by $\pi(n)$. The number of the Sylow p-subgroups P_p of G is denoted by n_p or $n_p(G)$. We set $\pi(G) = \pi(|G|)$. To see notations concerning the finite simple group, we refer to reader [1]. A finite group G is called a simple K_n- group, if G is a simple group and $|\pi(G)| = n$. In 1987, J. G. Thompson posed the following problem related to algebraic number fields [13].

Thompson's Problem. Let $T(G) = \{(k, s_k) | k \in \omega(G), s_k \in nse(G)\}$. Suppose that $T(G) = T(H)$ for some finite groups H. If G is a finite solvable group, is it true that H is necessarily solvable?

A finite group G is characterizable by order and nse; if H is a finite group and $|G| = |H|$ and $nse(G) = nse(H)$, then $G \cong H$. However, some groups are characterized by only the set $nse(G)$. The aim of this paper is to prove that the Orthogonal group $O_7(3)$ is characterizable by nse. The following theorems have been appeared so far:

(1) Theorem[11, 12]. Let G be a group and S be a simple K_i -group, where $i = 3, 4$. $G \cong H$ if and only if $|G| = |H|$ and $nse(G) = nse(H)$.
(2) Theorem[4, 5]. The two groups A_{12} and A_{13} are characterizable by order and nse.
(3) Theorem[7]. All sporadic simple groups are characterizable by nse and order.
(4) Theorem[10]. $L_2(2^m)$ with $2^m + 1$ prime or $2^m - 1$ prime, is characterizable by nse and order.

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(5) Theorem[12, 8]. $L_2(q)$, where $q \in \{7, 8, 9, 11, 13\}$ are characterizable by only the *nse*.

(6) Theorem[6]. $G_2(5)$ is characterizable by *nse*.

Main Theorem. Let G be a group such that $nse(G) = nse(O_7(3))$, where $O_7(3)$ is the Orthogonal group over $GF(3)$. Then $G \cong O_7(3)$.

We will give some lemmas which will be used to prove the main theorem.

Lemma 1.1. [2]*: Let G be a finite group and n be a positive integer dividing $|G|$. If $L_n(G) = \{g \in G | g^n = 1\}$, then $n||L_n(G)|$.*

Lemma 1.2. [9]*: Let G be a finite group and $p \in \pi(G)$ be an odd number. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$ with $(p, m) = 1$. If P is not cyclic, then the number of elements of order n is always a multiple of p^s.*

Lemma 1.3. [12]*: Let G be a group containing more than two elements. If the maximum number s of elements of the same order in G is finite, then G is finite and $|G| \leq s(s^2 - 1)$.*

Lemma 1.4. [3]*: Let G be a finite solvable group and $|G| = mn$, where $m = {p_1}^{\alpha_1} \cdots {p_r}^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1, \ldots, p_r\}$ and h_m be the number of Hall π-subgroups of G. Then $h_m = {q_1}^{\beta_1} \cdots {q_s}^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, \ldots, s\}$:*

(1) ${q_i}^{\beta_i} \equiv 1 (mod\ p_j)$ for some p_j;

(2) The order of some chief factor of G is divided by $q_i^{\beta_i}$.

Lemma 1.5. [14]*: Let G be a simple K_4-group. Then G is isomorphic to one of the following groups:*

(1) A_7, A_8, A_9 and A_{10};

(2) M_{11}, M_{12} or J_2;

(3) one of the following:

- (i) *$L_2(r)$, where r is a prime and $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$ with $a, b, c \geq 1$, and v is a prime number greater than 3.*
- (ii) *$L_2(2^m)$, where $2^m - 1 = u, 2^m + 1 = 3t^b$, with $m \geq 2, u, t$ are primes,$t > 3, b \geq 1$.*
- (ii) *$L_2(3^m)$, where $3^m + 1 = 4t, 3^m - 1 = 2u^c$ or $3^m + 1 = 4t^b, 3^m - 1 = 2u, m \geq 2$, u and t are odd primes, $b \geq 1, c \geq 1$.*
- (iv) *One of the following 28 simple groups:*
  $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, ${O_8}^+(2)$, $G_2(3)$, $U_3(4)$,$U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^2D_4(2)$ or ${}^2F_4(2)$.

Lemma 1.6. [5]*: Every simple K_5-group is isomorphic to one of the following simple groups:*

(1) $L_2(q)$ with $|\pi(q^2 - 1)| = 4$;

(2) $L_3(q)$ with $|\pi(q^2 - 1)(q^3 - 1)| = 4$;

(3) $U_3(q)$ with q satisfies $|\pi(q^2 - 1)(q^3 + 1)| = 4$;

(4) $O_5(q)$ with $|\pi(q^4 - 1)| = 4$;

(5) $Sz(2^{2m+1})$ *with* $|\pi(2^{2m+1}-1)(2^{4m+1}+1)|=4$
(6) $R(q)$, *where* q *is an odd power of* 3, $|\pi(q^2-1)|=3$ *and* $|\pi(q^2-q+1)|=1$*;*
(7) *The following* 30 *simple groups:*
A_{11}, A_{12}, M_{22}, J_3, HS, He, McL, $L_4(4)$, $L_4(5)$, $L_4(7)$, $L_5(2)$, $L_5(3)$, $L_6(2)$, $O_7(3)$, $O_9(2)$, $PSP_8(2)$, $U_4(4)$, $U_4(5)$, $U_4(7)$, $U_4(9)$, $U_5(3)$, $U_6(2)$, $O_8{}^+(3)$, $O_8{}^-(2)$, ${}^3D_4(3)$, $G_2(4)$, $G_2(5)$, $G_2(7)$ *or* $G_2(9)$.

Lemma 1.7. [1]*: Let* G *be a simple* K_n*-group with* $n=4,5$ *and* $13||G|$ *and* $|G||2^9\cdot 3^9\cdot 5\cdot 7\cdot 13$. *Then* G *is one of the following groups:* $L_2(13)$,$L_2(25)$, $L_2(27)$, $L_2(64)$, $L_3(9)$, $L_4(3)$, $G_2(3)$, $Sz(8)$, $S_6(3)$ *and* $O_7(3)$.

2. Main Result

The main theorem is proved by representing some lemmas.

Lemma 2.1. *If* s_n *is the number of elements of order* n *in a group* G*, then* $s_n=k\varphi(n)$ *such that* k *is the number of cyclic subgroups of order* n *in* G.

Proof. It is straightforward. □

Lemma 2.2. *If* $n>2$*, then* $\varphi(n)$ *is even.*

Proof. It is straightforward. □

Lemma 2.3. *If* $m\in\omega(G)$*, then* $\varphi(m)|s_m$ *and* $m|\sum_{d|m}s_d$.

Proof. It follows from Lemma1.1. □

Theorem 2.4. *Let* G *be a group such that* $nse(G)=nse(O_7(3))$*, where* $O_7(3)$ *is the Orthogonal group over* $GF(3)$*. Then* $G\cong O_7(3)$.

Proof. Since $nse(G)=nse(O_7(3))$, it can be concluded that G is finite group. We have:

$$nse(O_7(3))=\{1,354159,5307848,41395536,38211264,275871960,327525120,$$

$$573168960,382112640,343901376,732382560,705438720,305690112,$$

$$297198720,229267584,327525120\}.$$

According to Lemma1.1, $\pi(G)\subseteq\{2,3,5,7,13,23,317,401,1009\}$.

In this argument, first we show that $\pi(G)=\{2,3,5,7,13\}$. According to Lemma2.2, $s_2=354159$ and $2\in\pi(G)$. Since s_{23}, s_{317}, s_{401} and s_{1009} are not equal to none of $nse(G)$ values, hence 23, 317, 401 and 1009 $\notin\pi(G)$, so $\pi(G)\subseteq\{2,3,5,7,13\}$. We have the following:

(1) If $2^a\in\omega(G)$, then $2^{a-1}|s_{2^a}$ and $1\le a\le 11$;
(2) If $3^a\in\omega(G)$, then $2\cdot 3^{a-1}|s_{3^a}$ and $1\le a\le 11$;

(3) If $5^a \in \omega(G)$, then $4 \cdot 5^{a-1} | s_{5^a}$ and $1 \leq a \leq 2$;
(4) If $7^a \in \omega(G)$, then $6 \cdot 7^{a-1} | s_{7^a}$ and $1 \leq a \leq 3$;
(5) If $13^a \in \omega(G)$, then $12 \cdot 13^{a-1} | s_{13^a}$ and $1 \leq a \leq 3$.

To show that $\pi(G) = \{2, 3, 5, 7, 13\}$, we investigate several cases.

Case(1) $\pi(G) = \{2\}$.

If $\pi(G) = \{2\}$, then $\omega(G) \subseteq \{1, 2, 2^2, \ldots, 2^{11}\}$ and $4585351680 + 5307848k_1 + 41395536k_2 + 38211264k_3 + 275871960k_4 + 327525120k_5 + 573168960k_6 + 382112640k_7 + 343901376k_8 + 732382560k_9 + 705438720k_{10} + 305690112k_{11} + 297198720k_{12} + 229267584k_{13} + 327525120k_{14} = 2^n$, such that k_1, k_2, …, k_{14} and n are nonnegative integers. But this equation $0 < \Sigma_{i=1}^{14} k_i \leq 12 - 10 = 2$ has no solution in integers.

Case(2) $\pi(G) = \{2, 3\}$.

In this case, $|G| = 2^n \cdot 3^m$ such that $1 \leq n \leq 11$. If $3^3 \in \omega(G)$, then $exp(P_3)$ can be 3, 9 and 27. If $exp(P_3) = 3$, then by Lemma 2.3, $|P_3||1 + s_3(s_3 = 5307848)$ and $|P_3||3^8$. If $|P_3| = 3$, then $n_3 = s_3/\varphi(3) = 5307848/2 = 2653924$ and $[G : N_G(P_3)] = 2653924$. So, $13 \in \pi(G)$. If $|P_3| = 9$, then $|G| = 2^n.3^2$ such that $0 \leq n \leq 11$.

Also, calculations show that:

$4585351680 + 5307848\ \mathrm{k}_1 + 41395536k_2 + 38211264k_3 + 275871960k_4 + 327525120k_5 + 573168960k_6 + 382112640k_7 + 343901376k_8 + 732382560k_9 + 705438720k_{10} + 305690112k_{11} + 297198720k_{12} + 229267584k_{13} + 327525120k_{14} = 2^n \cdot 3^2$, such that k_1, k_2, $\cdots$,k_{14} and n are nonnegative integers and $0 < \Sigma_{i=1}^{14} k_i \leq 36$. Since $4585351680 \leq 2^n \cdot 3^2 \leq 36 \cdot 732382560$,then $n > 11$. Similarly, it is shown that $|P_3| \neq 3^3, \cdots, 3^7$. Now, it is assumed that $|P_3| = 3^8$ and $|G| = 2^n \cdot 3^8$,$1 \leq n \leq 11$.

We have:

$4585351680 + 5307848\ \mathrm{k}_1 + 41395536k_2 + 38211264k_3 + 275871960k_4 + 327525120k_5 + 573168960k_6 + 382112640k_7 + 343901376k_8 + 732382560k_9 + 705438720k_{10} + 305690112k_{11} + 297198720k_{12} + 229267584k_{13} + 327525120k_{14} = 2^n \cdot 3^2$,

such that k_1, k_2, $\cdots$,k_{14} and n are nonnegative integers and $0 < \Sigma_{i=1}^{14} k_i \leq 108$. But, $4585351680 \leq 2^n \cdot 3^8 \leq 108 \cdot 732382560$ and $n > 11$. If $exp(P_3) = 3^2$, then according to Lemma1.1 $|P_3||1 + s_3 + s_9$. There are several cases for s_9. Namely, 38211264, 275871960, 327525120, 573168960, 382112640, 343901376, 732382560, 705438720, 305690112, 297198720 and 229267584. We have $n_3 = [G : N_G(P_3)] = s_9/\varphi(9)$, it is obtained 5, 7 or $13 \in \pi(G)$. If $exp(P_3) = 3^3$, then $|P_3||1 + s_3 + s_9 + s_{27}$ and s_{27} is one of the following numbers: 41395536, 38211264, 327525120, 573168960, 382112640, 343901376, 732382560, 705438720, 305690112, 297198720 and 229267584. Since $n_3 = [G : N_G(P_3)] = s_{27}/\varphi(27)$, it results that 5, 7, 13, 23, 37, 107, 331 or $2299753 \in \pi(G)$. Hence, $3^3 \notin \omega(G)$ and $exp(P_3) = 3,\ 3^2$. If $exp(P_3) = 3,\ 3^2$, then similar as above, we have a contradiction and $\pi(G) \neq \{2, 3\}$.

Case (3) $\pi(G) = \{2, 5\}$.

If $5^2 \in \omega(G)$, then $s_5 = 38211264$ or 229267584 and $exp(P_5) = 5, 5^2$.

If $exp(P_5) = 5$, then $|P_5||1 + s_5$ and $|P_5| = 5$. Since $n_5 = [G : N_G(P_5)] = s_5/\varphi(5)$, is concluded 3, 7 and $13 \in \pi(G)$. If $exp(P_5) = 5^2$, by Lemma2.3 $|P_5||1 + s_5 + s_{25}(s_{25} = 275871960, 573168960, 732382560)$

and $|P_5||5^2$. It is obvious $|P_5| \neq 5$ and we suppose that $|P_5| = 25$. If $|P_5| = 25$, then 3, 7, 13 or 23 $\in \pi(G)$.

Therefore $5^2 \notin \omega(G)$ and $|G| = 2^n \cdot 5$ such that $1 \leq n \leq 11$. Hence, $4585351680 + 5307848k_1 + 41395536k_2 + 38211264k_3 + 275871960k_4 + 327525120k_5 + 573168960k_6 + 382112640k_7 + 343901376k_8 + 732382560k_9 + 705438720k_{10} + 305690112k_{11} + 297198720k_{12} + 229267584k_{13} + 327525120k_{14} = 2^n \cdot 5$,

such that

$k_1, k_2, \cdots, k_{14}$ and n are nonnegative integers and $0 < \Sigma_{i=1}^{14} k_i \leq 24$. But, $4585351680 \leq 2^n \cdot 5 \leq 24 \cdot 732382560$ and $n > 11$.

Case (4) $\pi(G) = \{2, 7\}$.

We have $7|1 + s_7$ and $s_7 = 327525120$. If $7^2 \in \omega(G)$, then $s_{7^2} = 41395536$ or 732382560. Since $|P_7||1 + s_7 + s_{7^2}$, is derived $|P_7||7^2$. If $|P_7| = 7$, then $7k + 1 = n_7 = s_7/\varphi(7) = 5458520$ and 3, 5, 13 $\in \pi(G)$. If $|P_7| = 7^2$, then $n_7 = s_{7^2}/\varphi(7^2) = [G : N_G(P_7)] = 985608, 17437680$ and 3, 5, 13, 5449 $\in \pi(G)$. Therefore, $7^2 \notin \omega(G)$ and $|G| = 2^n \cdot 7$ such that $1 \leq n \leq 11$. Hence, $4585351680 + 5307848k_1 + 41395536k_2 + 38211264k_3 + 275871960k_4 + 327525120k_5 + 573168960k_6 + 382112640k_7 + 343901376k_8 + 732382560k_9 + 705438720k_{10} + 305690112k_{11} + 297198720k_{12} + 229267584k_{13} + 327525120k_{14} = 2^n \cdot 7$,

such that

$k_1, k_2, \cdots, k_{14}$ and n are nonnegative integers and $0 < \Sigma_{i=1}^{14} k_i \leq 24$. But, $4585351680 \leq 2^n \cdot 7 \leq 24 \cdot 732382560$ and $n > 11$.

Case (5) $\pi(G) = \{2, 13\}$.

We have $13|1+s_{13}$ and $s_{13} = 705438720$. If $13^2 \in \omega(G)$, then $s_{13^2} = 343901376$. Since $|P_{13}||1+s_{13}+s_{13^2}$, is derived $|P_{13}||13^2$. If $|P_{13}| = 13$, then $13k+1 = n_{13} = s_{13}/\varphi(13) = 58786550$ and $3, 5, 39191 \in \pi(G)$. If $|P_{13}| = 13^2$, then $n_{13} = s_{13^2}/\varphi(13^2) = [G : N_G(P_{13})] = 2204496$ and $3, 5 \in \pi(G)$. Therefore, $13^2 \notin \omega(G)$ and $|G| = 2^n \cdot 13$ such that $1 \leq n \leq 11$. Hence, $4585351680 + 5307848k_1 + 41395536k_2 + 38211264k_3 + 275871960k_4 + 327525120k_5 + 573168960k_6 + 382112640k_7 + 343901376k_8 + 732382560k_9 + 705438720k_{10} + 305690112k_{11} + 297198720k_{12} + 229267584k_{13} + 327525120k_{14} = 2^n \cdot 13$, such that

$k_1, k_2, \cdots, k_{14}$ and n are nonnegative integers and $0 < \Sigma_{i=1}^{14} k_i \leq 24$. But, $4585351680 \leq 2^n \cdot 13 \leq 24 \cdot 732382560$ and $n > 11$.

Case (6) $\pi(G) = \{2, 3, 5\}$.

In this case, $|G| = 2^n \cdot 3^m \cdot 5$ and $1 \leq n, m \leq 11$. We have:

$4585351680 + 5307848k_1 + 41395536k_2 + 38211264k_3 + 275871960k_4 + 327525120k_5 + 573168960k_6 + 382112640k_7 + 343901376k_8 + 732382560k_9 + 705438720k_{10} + 305690112k_{11} + 297198720k_{12} + 229267584k_{13} + 327525120k_{14} = 2^n \cdot 3^m \cdot 5$, such that

$k_1, k_2, \cdots, k_{14}$ and n, m are nonnegative integers and $0 < \Sigma_{i=1}^{14} k_i \leq 288$. But, $4585351680 \leq 2^n \cdot 3^m \cdot 5 \leq 288 \cdot 732382560$ and $n, m > 11$.

If $\pi(G) = \{2, 3, 7\}$, then due to the values of nse $3 \cdot 7 \notin \omega(G)$ and $|P_3||s_7 = 2^8 \cdot 3^9 \cdot 5 \cdot 13$ and so $|P_3||3^9$. Similar to case (6), we have a contradiction.

Case (7) $\pi(G) = \{2, 3, 13\}$.

First, we obtain $2 \cdot 13 \notin \omega(G)$ if $2, 13 \in \pi(G)$. If $2 \cdot 13 \in \omega(G)$, set P and Q are Sylow 13−subgroup of G and are conjugate in G. Also $C_G(P)$ and $C_G(Q)$ are conjugate in G. Therefore we have $s_{2 \cdot 13} =$

$\varphi(2\cdot 13)\cdot n_{13}\cdot k$ where k is the number of cyclic subgroups of order 2 in $C_G(P)$. Since $n_{13}=s_{13}/\varphi(13)$, $s_{13}|s_{26}$ and so $s_{13}=s_{26}=705438720$. According to Lemma2.3 $26|1+s_2+s_{13}+s_{26}$, a contradiction. Hence $2\cdot 13\notin\omega(G)$, it follows that the Sylow 2−subgroup of G acts fixed point freely on the set of elements of order 13, $|P_2||s_{13}=2^{10}\cdot 3^9\cdot 5\cdot 7$ and so $|P_2||2^{10}$. If $3\cdot 13\in\omega(G)$, then by Lemma2.3 $3\cdot 13|1+s_3+s_{13}+s_{39}$. Hence $3\cdot 13\notin\omega(G)$, it follows that the Sylow 3−subgroup of G acts fixed point freely on the set of elements of order 13, $|P_3||s_{13}=2^{10}\cdot 3^9\cdot 5\cdot 7$ and so $|P_3||3^9$. In this case, $\pi(G)=\{2,3,13\}$ and $|G|=2^n\cdot 3^m\cdot 13$ and $1\le n\le 10$, $1\le m\le 9$.

So,

$4585351680+5307848k_1+41395536k_2+38211264k_3+275871960k_4+327525120k_5+573168960k_6+382112640k_7+343901376k_8+732382560k_9+705438720k_{10}+305690112k_{11}+297198720k_{12}+229267584k_{13}+327525120k_{14}=2^n\cdot 3^m\cdot 13$,

such that

$k_1,\ k_2,\ \cdots,k_{14}$ and n,m are nonnegative integers and $0<\Sigma_{i=1}^{14}k_i\le 220$. But, $4585351680\le 2^n\cdot 3^m\cdot 13\le 220\cdot 732382560$ and $n>10$, $m>9$.

In the remaining cases, in the same way, we obtain a contradiction. Therefore, $\pi(G)=\{2,3,5,7,13\}$ and $|G|=2^n\cdot 3^m\cdot 5\cdot 7\cdot 13$ such that $1\le n\le 10$,$1\le m\le 9$.

Also we know $2^9\cdot 3^9\cdot 5\cdot 7\cdot 13=4585351680=\sum_{s_k\in nse(G)}s_k\le |G|=2^n\cdot 3^m\cdot 5\cdot 7\cdot 13\le 2^{10}\cdot 3^9\cdot 5\cdot 7\cdot 13$. So, we can assume that $|G|=2^9\cdot 3^9\cdot 5\cdot 7\cdot 13$ or $|G|=2^{10}\cdot 3^9\cdot 5\cdot 7\cdot 13$. The second step is $G\cong O_7(3)$. We show that there is no group such that $|G|=2^{10}\cdot 3^9\cdot 5\cdot 7\cdot 13$ and $nse(G)=nse(O_7(3))$. If G is a solvable group, then $n_{13}=s_{13}/\varphi(13)=2^8\cdot 3^8\cdot 5\cdot 7$ and by Lemma1.4 $3\equiv 1(mod 13)$. So, there is a contradiction, and G is not solvable. Hence, G has a normal series $1\lhd K\lhd L\lhd G$ such that L/K is isomorphic to a simple K_i−group with $i=3,4,5$ and $169\nmid |G|$. If L/K is isomorphic to a simple K_3−group, from [4, 14], $L/K\cong A_5, L_3(2), A_6, L_2(8), L_3(3), U_3(3)$ and $U_4(2)$. By [1] $n_2(L/K)=n_2(A_5)=15$, according to [11], $n_2(G)=15t$,$2\nmid t$ and $n_2=s_2=354159=15t$. There is no solution for t, and for the other groups, we can similarly rule it out.

If L/K is isomorphic to a simple K_n−group with $n=4,5$, then by Lemma1.4, L/K is isomorphic to

$$L_2(13), L_2(25), L_2(27), L_2(64), L_3(9), L_4(3), G_2(3), Sz(8), S_6(3)$$

and $O_7(3)$.

If $L/K\cong L_2(13)$, then $14=n_{13}(L/K)=n_{13}(L_2(13))$ and $n_{13}(G)=14t$ such that $13\nmid t$ for some integer t. Hence, $s_{13}=12\cdot 14$ and $t=4199040=2^7\cdot 3^8\cdot 5$. We have $2^7\cdot 3^8\cdot 5||K||2^7\cdot 3^8\cdot 5$ so, $|K|=2^7\cdot 3^8\cdot 5$ and $N_K(P_{13})=1$. Therefore $K\times P_{13}$ is a Frobenius group and $|P_{13}|||Aut(K)|$, a contradiction.

If $L/K\cong S_6(3)$, then $s_2(L/K)=s_2(S_6(3))=196911$ and $s_2(G)=196911t$. Thus $s_2(G)=196911t=354159$, a contradiction. Similarly, for the other groups $L_2(25), L_2(27), L_2(64), L_3(9), L_4(3), G_2(3)$ and $Sz(8)$, we can obtain a contradiction. The last case is $L/K\cong O_7(3)$. If $\overline{G}=G/K$ and $\overline{L}=L/K$, then

$$\overline{L}\cong\overline{L}(C_{\overline{G}}(\overline{\overline{L}}))/(C_{\overline{G}}(\overline{\overline{L}}))\le(\overline{G}/(C_{\overline{G}}(\overline{\overline{L}}))=(N_{\overline{G}}(\overline{\overline{L}}))/(C_{\overline{G}}(\overline{\overline{L}}))\le Aut(\overline{L}).$$

We define that $M = \{xk | xk \in C_{\overline{G}}(\overline{\overline{L}})\}$. We have: $G/M \cong \overline{G}/C_{\overline{G}}(\overline{\overline{L}})$ and $O_7(3) \leq G/M \leq Aut(O_7(3))$. Also, $G/M \cong O_7(3)$ or $G/M \cong 2 \cdot O_7(3)$. If $G/M \cong O_7(3)$, then according to assumptions $|M| = 2$ and $M = Z(G)$. Hence, G has an element of order $2 \cdot 13$, which is a contradiction. If $G/M \cong 2 \cdot O_7(3)$, $M = 1$ and G has a pure normal subgroup H such that $H \cong O_7(3), nse(G) = nse(O_7(3))$. We know $n_{13}(O_7(3)) = 2^8 \cdot 3^8 \cdot 5 \cdot 7$ so, $n_{13}(O_7(3)) < n_{13}(G)$. According to assumptions $n_{13}(G) = 2^{10} \cdot 3^9 \cdot 5 \cdot 7$ and $s_{13}(G) = 651174204 \notin nse(G)$. Therefore, $|G| = 2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13 = |O_7(3)|$ and according to this argument $G \cong O_7(3)$.

$\square$

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