Research Article

Nonlinear Representations of Baer Subgeometries of $PG(3,q^2)$

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1. Independent researcher

We study the Bruck–Bose representation in $PG(6,q)$ of Baer subgeometries $PG(3,q)$ of $PG(3,q^2)$ meeting the plane at infinity in a line $PG(1,q)$. Unlike the canonical Baer subgeometry, whose representation is a linear 3-space, a Baer subgeometry meeting the plane at infinity in a line cannot be represented by a linear 3-space in $PG(6,q)$. We construct a family of $q+1$ ruled cubic surfaces with a common affine conic and linear directrices forming a regulus of the Desarguesian spread. We prove that their union, together with the $q+1$ corresponding improper points, represents a Baer subgeometry $PG(3,q)$ whose intersection with the plane at infinity is exactly a line $PG(1,q)$. Moreover, the ruled cubic surfaces represent the $q+1$ Baer subplanes of a pencil of planes of this subgeometry, with the common affine conic representing their common affine line.

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1. Introduction

In the Bruck–Bose representation of $PG(3,q^2)$ in $PG(6,q)$, the canonical Baer subgeometry $PG(3,q)$ is represented by a 3-dimensional subspace whose intersection with the hyperplane at infinity is a plane.

In this paper we consider a fundamentally different situation: Baer subgeometries $PG(3,q)$ of $PG(3,q^2)$ whose intersection with the plane at infinity is exactly a line $PG(1,q)$. Unlike the canonical case, these subgeometries cannot be represented by linear 3-dimensional subspaces of $PG(6,q)$. Our aim is to construct and investigate their nonlinear representations.

In a previous paper [1], a bundle of $q+1$ ruled cubic surfaces V_2^3 in $PG(4,q)$, having a common conic directrix, was considered in connection with the Bruck–Bose representation of $PG(2,q^2)$. The present

paper develops a related construction in the Bruck–Bose representation of $PG(3,q^2)$ in $PG(6,q)$, where a family of ruled cubic surfaces with a common conic directrix is associated with the lines of a regulus.

The present work is related to the representation of Baer subplanes considered in [2] and subsequently studied in [3]. In [2], a Baer subplane $PG(2,q)$ of $PG(2,q^2)$ meeting the line at infinity in a single point is represented in $PG(4,q)$ by a ruled cubic surface. The construction developed here may be viewed as a 3-dimensional analogue of this representation for Baer subgeometries $PG(3,q)$ of $PG(3,q^2)$ meeting the plane at infinity in a line $PG(1,q)$. In both cases, the essential geometric feature is the representation of a Baer subgeometry whose intersection with infinity has smaller dimension than in the canonical case.

Related investigations of subgeometries in the André–Bruck–Bose representation have been carried out in [4][5][6]. These works concern the representation of sublines and subplanes, including their connections with ruled varieties. Although they deal with different ambient geometries, they provide a related geometric framework for the present construction.

We construct a set $\mathscr{W}$ as the union of $q+1$ ruled cubic surfaces of type V_2^3, in the notation of [2], having a common affine conic as conic directrix and whose linear directrices are the lines of a regulus contained in the Desarguesian spread. We determine the relevant plane sections and intersection properties of $\mathscr{W}$ and prove, by means of an explicit projectivity, that $\mathscr{W}$, together with the $q+1$ improper points represented by the lines of the regulus, represents a Baer subgeometry $PG(3,q)$ of $PG(3,q^2)$ whose intersection with the plane at infinity is a line $PG(1,q)$. Moreover, the $q+1$ ruled cubic surfaces represent the $q+1$ Baer subplanes forming a pencil of planes of this subgeometry, with the common affine conic representing their common affine line.

2. Preliminary notes and results

Let $F=GF(q)$ be a finite field, where $q=p^s$ and p is an odd prime, and let $\overline{F}$ denote the algebraic closure of F. We denote by $PG(n,q)$ the n-dimensional projective space over F.

The geometry $PG(n,q)$ is regarded as a subgeometry of the projective geometry over $\overline{F}$ (cf. [7], Definition of subgeometry, p. 25), and its points are called the *rational points*.

A subspace of $PG(n,q)$ of dimension h (an h-space) is denoted by S_h, possibly with a superscript or a subscript when needed to distinguish between different subspaces. A 3-dimensional subspace is also called a *solid*.

Note that every subspace of $PG(n,q)$ is also a subgeometry, whereas the converse is not true in general.

Definition 2.1. *A variety V_u^v of dimension u and order v of $PG(n,q)$ is the set of the rational points of a projective variety $\overline{V}_u^v$ of $\overline{PG(n,q)}$ defined by a finite set of polynomials with coefficients in F.*

Let $K = GF(q^2) = GF(q)(\omega)$ be the quadratic extension of $F = GF(q)$, where ω is a root of the irreducible polynomial $x^2 - s$, with s a non-square of F. Thus $\omega^2 = s$, and every element $a \in GF(q^2)$ can be uniquely written as $a = a_1 + \omega a_2$ (or, equivalently, as (a_1, a_2)), with $a_1, a_2 \in GF(q)$.

The involutory mapping $\sigma : x \mapsto x^q$ is the *Frobenius automorphism* of K, fixing the subfield $GF(q)$ pointwise. For $x \in K$, the element x^q is called the conjugate of x. Since $(\omega^q)^2 = (\omega^2)^q = s$, the element ω^q is also a root of $x^2 - s$. As $\omega \notin GF(q)$, we have $\omega^q \neq \omega$, and hence $\omega^q = -\omega$.

Therefore, for every $a = a_1 + \omega a_2 \in GF(q^2)$, its *conjugate* is $\overline{a} = a^q = a_1 - \omega a_2$.

Let $S = PG(3, q^2)$ be the 3-dimensional projective geometry over K, and let (x, y, z, t) denote a coordinate system C_S for S. Thus $(x, y, z, 1)$, with $x, y, z \in GF(q^2)$, represents a generic *affine point*.

The point $O = (0, 0, 0, 1)$ is the *origin*, and $t = 0$ is the plane at infinity Π_∞.

The Frobenius automorphism σ induces an involutory semilinear collineation of $PG(3, q^2)$, acting on homogeneous coordinates by

$$(x_0, x_1, x_2, x_3)^\sigma = (x_0^q, x_1^q, x_2^q, x_3^q).$$

We shall denote this collineation again by σ.

Definition 2.2. *A line ℓ (respectively, a plane π) of $PG(3, q^2)$ is σ-stable (or σ-invariant) if it is fixed setwise by σ.*

A σ-stable line ℓ contains a Baer subline $PG(1, q)$ consisting precisely of its points fixed by σ; similarly, a σ-stable plane π contains a Baer subplane $PG(2, q)$ consisting precisely of its points fixed by σ.

Thus $\ell \cap \mathrm{Fix}(\sigma) \cong PG(1, q)$ and, respectively, $\pi \cap \mathrm{Fix}(\sigma) \cong PG(2, q)$.

Using the field reduction method underlying the André–Bruck–Bose construction [8][9], we represent the 3-dimensional projective geometry $S = PG(3, q^2)$ in the projective geometry $\Sigma = PG(6, q)$.

To the coordinate system C_S of S corresponds a coordinate system C_Σ of Σ, so that $(x, y, z, 1) \mapsto (x_1, x_2, y_1, y_2, z_1, z_2, 1)$, where $x = x_1 + \omega x_2$, $y = y_1 + \omega y_2$, $z = z_1 + \omega z_2$, with $x_i, y_i, z_i \in GF(q)$. The plane at infinity $\Pi_\infty : t = 0$ of S is represented in Σ by the hyperplane $\Sigma' = PG(5, q)$, also having equation $t = 0$.

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If $P=(x,y,z,1)$, then its *conjugate point* $\sigma P = P^q = (x^q, y^q, z^q, 1)$ is denoted $\overline{P} = (\overline{x}, \overline{y}, \overline{z}, 1)$. Since $(x,y,z,1) = (x_1 + \omega x_2, y_1 + \omega y_2, z_1 + \omega z_2, 1)$, we have $\overline{P} = (x_1 - \omega x_2, y_1 - \omega y_2, z_1 - \omega z_2, 1)$.

The points of the Baer subgeometry $S^0 \cong PG(3,q)$ of S in *canonical position* are precisely the points fixed by σ, that is, the points with coordinates $(x,y,z,1)$, where $x,y,z \in GF(q)$. In the Bruck–Bose representation, these points are therefore characterized by the conditions $x_2 = y_2 = z_2 = 0$.

The subgeometry S^0 intersects the plane at infinity $\Pi_\infty : t = 0$ in a Baer subplane $\pi_\infty \cong PG(2,q)$, whose points have internal coordinates $(x_1, 0, y_1, 0, z_1, 0)$ in Π_∞. In particular, S^0 coincides with its conjugate and is σ-stable.

If α is a plane, its *conjugate* α^q is the plane whose equation is obtained from that of α by conjugating its coefficients.

Similarly, if ℓ is a line, its *conjugate line* ℓ^q is obtained by conjugating its defining equations. A plane α or a line ℓ is σ-stable if $\alpha^q = \alpha$ or $\ell^q = \ell$, respectively. In this case, its points fixed by σ form, respectively, a plane or a line of the canonical Baer subgeometry $S^0 \cong PG(3,q)$.

We denote by $\mathscr{L}$ the complete star of the q^4+q^2+1 lines of $PG(3,q^2)$ through the origin $O=(0,0,0,1)$.

Under the Bruck–Bose representation, the lines of $\mathscr{L}$ correspond to q^4+q^2+1 planes of $\Sigma = PG(6,q)$ through the point $O' = (0,0,0,0,0,0,1)$. Any two of these planes have only the point O' in common, and their intersections with $\Sigma' = PG(5,q)$ form a regular spread $\mathscr{S}$ consisting of q^4+q^2+1 lines.

Recall that a *regulus* $\mathscr{R}$ of lines in $PG(3,q)$ is one of the two families of $q+1$ mutually skew lines ruling a hyperbolic quadric $Q^+(3,q)$. Equivalently, $\mathscr{R}$ is a set of $q+1$ mutually skew lines such that every line meeting three lines of $\mathscr{R}$ meets all the lines of $\mathscr{R}$.

A line spread is called *regular* if the 3-space spanned by any two of its lines is partitioned by lines of the spread and, for any three of its lines contained in a 3-space, the regulus determined by them is entirely contained in the spread.

More generally, every line of $PG(3,q^2)$ is represented by a plane of Σ intersecting Σ' in a line of $\mathscr{S}$.

Let $\alpha = a_1 + \omega a_2$, $\beta = b_1 + \omega b_2$ and $\gamma = c_1 + \omega c_2$, with $a_i, b_i, c_i \in GF(q)$ and $\omega^2 = s$.

The set $\mathscr{L}$ naturally splits into three disjoint families: $\mathscr{L} = \mathscr{L}_0 \cup \mathscr{L}_\gamma \cup \mathscr{L}_{\alpha,\beta}$, where

$$\mathscr{L}_0 = \{x = 0,\ y = 0\},$$
$$\mathscr{L}_\gamma = \{x = 0,\ z = \gamma y \mid \gamma \in GF(q^2)\},$$
$$\mathscr{L}_{\alpha,\beta} = \{y = \alpha x,\ z = \beta x \mid \alpha, \beta \in GF(q^2)\}.$$

Hence $|\mathscr{L}| = |\mathscr{L}_0| + |\mathscr{L}_\gamma| + |\mathscr{L}_{\alpha,\beta}| = 1 + q^2 + q^4$. In particular, the three coordinate axes $x = 0,\ y = 0$; $x = 0,\ z = 0$; and $y = 0,\ z = 0$ belong to $\mathscr{L}$.

The regular spread $\mathscr{S}$ of $\Sigma' = PG(5, q)$ induced by the corresponding planes of Σ has an analogous decomposition:

$\mathscr{S} = \mathscr{S}_0 \cup \mathscr{S}_\gamma \cup \mathscr{S}_{\alpha,\beta}$,

each line of $\mathscr{S}$ corresponding to a line of $\mathscr{L}$, where $\gamma = c_1 + \omega c_2$, $\alpha = a_1 + \omega a_2$ and $\beta = b_1 + \omega b_2$.

The three families of lines of the spread are given by:

$\mathscr{S}_0 = \{(0,0,0,0,z_1,z_2,0) \mid (z_1,z_2) \in GF(q)^2 \setminus \{(0,0)\}\}$,

$\mathscr{S}_\gamma = \{(0,0,y_1,y_2,c_1y_1 + sc_2y_2, c_2y_1 + c_1y_2, 0) \mid (y_1,y_2) \in GF(q)^2 \setminus \{(0,0)\}\}$,

$\mathscr{S}_{\alpha,\beta} = \{(x_1,x_2,a_1x_1 + sa_2x_2, a_2x_1 + a_1x_2, b_1x_1 + sb_2x_2, b_2x_1 + b_1x_2, 0)|$

$(x_1,x_2) \in GF(q)^2 \setminus \{(0,0)\}\}$.

Altogether, $|\mathscr{S}| = |\mathscr{S}_0| + |\mathscr{S}_\gamma| + |\mathscr{S}_{\alpha,\beta}| = 1 + q^2 + q^4$.

The lines of $\mathscr{S}$ represent the points of the plane at infinity Π_∞ of $PG(3, q^2)$.

3. Baer 3-dimensional subgeometries of $PG(3, q^2)$

The following standard characterization of the point set of a subgeometry $PG(3, q)$ of $PG(3, q^2)$ will be useful in describing the geometric properties of the construction; see [7][10].

Lemma 3.1. *Let $H \subseteq PG(3, q^2)$ be a set of $q^3 + q^2 + q + 1$ points. Then H is the point set of a subgeometry $PG(3, q)$ if and only if the following conditions hold:*

1) *H is not contained in any plane of $PG(3, q^2)$;*

2) *every line ℓ of $PG(3, q^2)$ satisfies $|\ell \cap H| \in \{0, 1, q + 1\}$.*

Remark 1. The Frobenius automorphism σ of $GF(q^2)$ over $GF(q)$ induces a semilinear collineation of $PG(3, q^2)$ whose fixed points form the canonical Baer subgeometry $S^0 \cong PG(3, q)$.

A projective subspace of $PG(3, q^2)$ is defined over $GF(q)$ if and only if it is invariant under σ. Notice that setwise invariance does not mean that the subspace is pointwise fixed. If, for instance, ℓ is a line defined

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over $GF(q)$, then $\sigma(\ell) = \ell$, while the points of ℓ fixed by σ are precisely the $q+1$ points of a Baer subline $PG(1,q)$. The remaining points of ℓ are permuted by σ in conjugate pairs.

Lemma 3.2. *Let $S_3^0 \subset \Sigma$ be the solid representing the canonical Baer subgeometry $S^0 \cong PG(3,q)$ of $PG(3,q^2)$ in the Bruck–Bose representation. Then $S_3^0 \cap \Sigma'$ is a plane $PG(2,q)$ meeting q^2+q+1 distinct lines of the spread $\mathscr{S}$, each in exactly one point.*

Proof. The solid S_3^0 has projective dimension 3, while Σ' is a hyperplane of $\Sigma = PG(6,q)$. Since S_3^0 contains affine points, $S_3^0 \not\subset \Sigma'$. Hence $S_3^0 \cap \Sigma'$ is a plane, say $\pi_0 \cong PG(2,q)$.

The canonical Baer subgeometry S^0 meets the plane at infinity Π_∞ of $PG(3,q^2)$ in a subplane $PG(2,q)$. Under the Bruck–Bose representation, its q^2+q+1 points correspond to q^2+q+1 distinct lines of the spread $\mathscr{S}$, and each of these lines meets $\pi_0 = S_3^0 \cap \Sigma'$ in exactly one point.

Let $S^0 \cong PG(3,q)$ be the canonical Baer subgeometry of $PG(3,q^2)$, and let Π be a plane of $PG(3,q^2)$.

Lemma 3.3. *The intersection $\Pi \cap S^0$ is either a Baer subplane $PG(2,q)$ or a Baer subline $PG(1,q)$.*

Moreover:

1. *$\Pi \cap S^0 \cong PG(2,q)$ if and only if Π is σ-stable;*
2. *$\Pi \cap S^0 \cong PG(1,q)$ if and only if Π is not σ-stable.*

Proof. If Π is σ-stable, then $\Pi^\sigma = \Pi$. Hence the fixed points of σ in Π form a Baer subplane $PG(2,q)$, and therefore $\Pi \cap S^0 \cong PG(2,q)$.

Suppose now that Π is not σ-stable. Then $\Pi \neq \Pi^\sigma$, and, since two distinct planes of $PG(3,q^2)$ meet in a line, we have $L = \Pi \cap \Pi^\sigma$. Moreover, $L^\sigma = \Pi^\sigma \cap \Pi = L$, so L is σ-stable and its fixed points form a Baer subline $PG(1,q)$.

If $P \in \Pi \cap S^0$, then $P^\sigma = P$. Since $P \in \Pi$, we also have $P = P^\sigma \in \Pi^\sigma$, and hence $P \in L$. Conversely, every fixed point of σ on L belongs to $\Pi \cap S^0$. Thus $\Pi \cap S^0$ is precisely the fixed subline of L, and therefore $\Pi \cap S^0 \cong PG(1,q)$.

Hence $\Pi \cap S^0$ is a Baer subplane $PG(2,q)$ if Π is σ-stable, and a Baer subline $PG(1,q)$ otherwise.

Proposition 3.4. *There exists a Baer subgeometry $S' \cong PG(3,q)$ of $PG(3,q^2)$ such that $S' \cap \Pi_\infty \cong PG(1,q)$.*

Proof. Let $S^0 \cong PG(3,q)$ be the canonical Baer subgeometry, that is, the fixed point set of the Frobenius collineation σ.

Since not every plane of $PG(3,q^2)$ is defined over $GF(q)$, choose a plane Π which is not σ-stable. By Lemma 3.3, $\Pi \cap S^0 \cong PG(1,q)$.

Let g be a projective collineation of $PG(3,q^2)$ such that $g(\Pi) = \Pi_\infty$, and set $S' = g(S^0)$. Then $S' \cong PG(3,q)$ is a Baer subgeometry of $PG(3,q^2)$. Moreover, $S' \cap \Pi_\infty = g(S^0) \cap g(\Pi) = g(S^0 \cap \Pi)$.

Since $S^0 \cap \Pi \cong PG(1,q)$, it follows that $S' \cap \Pi_\infty \cong PG(1,q)$.

4. The geometry of $PG(3,q^2)$ in $PG(6,q)$

Baer 3-dimensional subgeometries - Consider the canonical Baer subgeometry $S^0 \cong PG(3,q)$ of $PG(3,q^2)$. Its Bruck–Bose representation in $\Sigma = PG(6,q)$ is the solid $\overline{S^0}$ defined by $x_2 = y_2 = z_2 = 0$. The subgeometry S^0 meets the plane at infinity Π_∞ in the Baer subplane $\pi_\infty \cong PG(2,q)$.

The intersection $\alpha = \overline{S^0} \cap \Sigma'$ is a plane. By Lemma 3.2, α contains no line of the spread $\mathscr{S}$ and meets exactly q^2+q+1 distinct lines of $\mathscr{S}$, each in one point.

More generally, a Baer subgeometry $S \cong PG(3,q)$ of $PG(3,q^2)$ meeting Π_∞ in a Baer subplane $PG(2,q)$ is represented in Σ by a solid whose intersection with Σ' is a plane meeting q^2+q+1 distinct lines of $\mathscr{S}$, each in one point.

On the other hand, by Proposition 3.4, there exist Baer subgeometries $S' \cong PG(3,q)$ of $PG(3,q^2)$ such that $S' \cap \Pi_\infty \cong PG(1,q)$. This case will play a central role in the construction developed in the following sections.

Subplanes - Let $\pi \cong PG(2,q^2)$ be a plane of $PG(3,q^2)$. Then $\pi \cap \Pi_\infty$ is a line $PG(1,q^2)$ and hence contains q^2+1 points.

In the Bruck–Bose representation, π corresponds to a 4-dimensional subspace S_4 of Σ, with $S_4 \not\subset \Sigma'$, such that $S_3 = S_4 \cap \Sigma'$ is a 3-dimensional subspace. The q^2+1 points of $\pi \cap \Pi_\infty$ are represented by q^2+1 spread lines contained in S_3. These lines form a subspread $\mathscr{S}_\pi$ of S_3 (for the 4-dimensional case, see [2]). This configuration is described more precisely in Proposition 4.4.

A plane τ of Σ such that $r = \tau \cap \Sigma'$ is a line not belonging to $\mathscr{S}$ will be called a *transversal plane*. Since $\mathscr{S}$ is a spread, the $q+1$ points of r lie on $q+1$ distinct lines of $\mathscr{S}$. Hence the affine points of τ, together with the improper points represented by these spread lines, represent a subplane $PG(2,q)$ of $PG(3,q^2)$.

Consider now a Baer subplane $\beta \cong PG(2,q)$ in canonical position, that is, a subgeometry of the canonical Baer subgeometry $S^0 \cong PG(3,q)$. In Σ, the subplane β is represented by a transversal plane

$\tau \subset \overline{S^0}$. The line $r = \tau \cap \Sigma'$ meets exactly $q+1$ lines of the spread $\mathscr{S}$, representing the $q+1$ improper points of β.

If β is contained in a plane $\pi \cong PG(2,q^2)$ of $PG(3,q^2)$, then $\beta = \pi \cap S^0$. Let S_4 be the 4-space representing π and let $\mathscr{S}_\pi$ be the subspread contained in $S_4 \cap \Sigma'$. In this case $\tau \subset S_4$, and the $q+1$ spread lines meeting r belong to $\mathscr{S}_\pi$.

Remark 2. For the next results, we shall use the following counting observations.

The number of planes of $PG(3,q^2)$ is $q^6+q^4+q^2+1$. Hence, excluding the plane at infinity Π_∞, there are $q^6+q^4+q^2$ planes not contained in Π_∞.

In the Bruck–Bose representation, these planes correspond to 4-dimensional subspaces S_4 of Σ not contained in Σ'. On the other hand, the total number of 4-dimensional subspaces of Σ not contained in Σ' is $\begin{bmatrix}7\\2\end{bmatrix}_q - \begin{bmatrix}6\\1\end{bmatrix}_q$. Consequently, the number of such 4-spaces which do not represent planes of $PG(3,q^2)$ is $\begin{bmatrix}7\\2\end{bmatrix}_q - \begin{bmatrix}6\\1\end{bmatrix}_q - (q^6+q^4+q^2)$, which is a polynomial in q of degree 10.

Similarly, let $O' \in \Sigma \setminus \Sigma'$ be an affine point. The number of 4-dimensional subspaces of Σ through the point O' is $\begin{bmatrix}6\\4\end{bmatrix}_q$, whereas the number of planes of $PG(3,q^2)$ through the affine point represented by O' is q^4+q^2+1. Hence the number of 4-dimensional subspaces through O' which do not represent planes of $PG(3,q^2)$ is $\begin{bmatrix}6\\4\end{bmatrix}_q - (q^4+q^2+1)$, which is a polynomial in q of degree 8.

For the following proposition, see [9], [10], Section 21 and [11], Lemma 2.2(vii).

Proposition 4.1. *Let S_3 be a 3-dimensional subspace of Σ'. Then S_3 contains either 0, 1 or q^2+1 lines of the spread $\mathscr{S}$. In the latter case, these q^2+1 lines form a spread of S_3.*

Proof. If S_3 contains at most one line of $\mathscr{S}$, there is nothing to prove. Suppose that S_3 contains two distinct spread lines ℓ_1 and ℓ_2. Since ℓ_1 and ℓ_2 are disjoint, $S_3 = \langle \ell_1, \ell_2 \rangle$. A Desarguesian spread has the property that the 3-space generated by any two of its lines is partitioned by lines of the spread. Therefore the lines of $\mathscr{S}$ contained in $S_3 = \langle \ell_1, \ell_2 \rangle$ form a spread of S_3. Since a line spread of $PG(3,q)$ consists of q^2+1 lines, S_3 contains exactly q^2+1 lines of $\mathscr{S}$.

Lemma 4.2. *Let $S \cong PG(3,q)$ be a Baer subgeometry of $PG(3,q^2)$ such that $S \cap \Pi_\infty \cong PG(1,q)$. Then S cannot be represented in $\Sigma = PG(6,q)$ by a 3-dimensional subspace L with $L \not\subset \Sigma'$.*

Proof. Since $S \cong PG(3,q)$, it contains q^3+q^2+q+1 points. By assumption, $S \cap \Pi_\infty \cong PG(1,q)$, and hence S has q^3+q^2 affine points.

Suppose that S were represented by a 3-dimensional subspace L of Σ, with $L \not\subset \Sigma'$. Then $L \cap \Sigma'$ is a plane $PG(2,q)$. Therefore the number of affine points of L is $(q^3+q^2+q+1)-(q^2+q+1)=q^3$.

In the Bruck–Bose representation, affine points of $PG(3,q^2)$ are represented by affine points of Σ. Thus L could represent only q^3 affine points, whereas S has q^3+q^2 affine points, a contradiction. Hence S cannot be represented by a 3-dimensional subspace L of Σ with $L \not\subset \Sigma'$.

Lemma 4.3. *Let $S \cong PG(3,q)$ be a Baer subgeometry of $PG(3,q^2)$ such that $S \cap \Pi_\infty = \ell \cong PG(1,q)$.*

Then the $q+1$ lines of the Desarguesian regular spread $\mathscr{S}$ of Σ' corresponding to the points of ℓ form a regulus $\mathscr{R}$ and are contained in a 3-space $S'_3 \subset \Sigma'$.

Proof. The Baer subline $\ell \cong PG(1,q)$ consists of $q+1$ points of Π_∞. Under the Bruck–Bose representation, these points correspond to $q+1$ lines of the Desarguesian spread $\mathscr{S}$.

Under field reduction, a line $PG(1,q^2)$ of Π_∞ corresponds to a 3-space of Σ' carrying a Desarguesian line spread. The $q+1$ points of a Baer subline $PG(1,q)$ of this line correspond to $q+1$ lines of this spread forming a regulus. Hence the spread lines corresponding to the points of ℓ form a regulus $\mathscr{R}$.

Choose two distinct lines $r_1, r_2 \in \mathscr{R}$. They are disjoint and hence generate a 3-space $S'_3 = \langle r_1, r_2 \rangle \subset \Sigma'$. Since all the lines of a regulus lie in the 3-space generated by any two of its lines, $\mathscr{R} \subset S'_3$.

Therefore the $q+1$ spread lines corresponding to the points of ℓ are contained in a 3-space $S'_3 \subset \Sigma'$.

The following proposition characterizes the 4-dimensional subspaces of Σ representing planes of $PG(3,q^2)$.

Proposition 4.4. *Let S_4 be a 4-dimensional subspace of Σ, with $S_4 \not\subset \Sigma'$, and let $S_3 = S_4 \cap \Sigma'$. Then S_4 represents a plane $PG(2,q^2)$ of $PG(3,q^2)$ if and only if S_3 contains q^2+1 lines of the spread $\mathscr{S}$.*

Proof. Suppose that S_4 represents a plane $\Pi \cong PG(2,q^2)$. Then $\Pi \cap \Pi_\infty$ is a line $PG(1,q^2)$, whose q^2+1 points correspond to q^2+1 lines of $\mathscr{S}$ contained in $S_3 = S_4 \cap \Sigma'$.

Conversely, suppose that S_3 contains q^2+1 lines of $\mathscr{S}$. By Proposition 4.1 and the field-reduction correspondence, these lines represent the q^2+1 points of a line $L \cong PG(1,q^2)$ of Π_∞.

Let $P \in S_4 \setminus S_3$ and let P^* be the corresponding affine point of $PG(3,q^2)$. For every spread line $r \subset S_3$ representing $R \in L$, the plane $\langle P, r \rangle \subset S_4$ represents the line P^*R. Hence S_4 represents the plane $\langle P^*, L \rangle \cong PG(2,q^2)$.

From Remark 2 and Propositions 4.1 and 4.4 we obtain the following.

Corollary 4.5. *The 4-dimensional subspaces of $\Sigma = PG(6,q)$ not contained in $\Sigma' = PG(5,q)$ split into two classes.*

Exactly $q^6+q^4+q^2$ of them represent the planes $PG(2,q^2)$ of $PG(3,q^2)$ distinct from Π_∞. For each such 4-space S_4, the 3-space $S_3 = S_4 \cap \Sigma'$ contains exactly q^2+1 lines of the spread $\mathscr{S}$.

The number of the remaining 4-spaces is $\begin{bmatrix}7\\2\end{bmatrix}_q - \begin{bmatrix}6\\1\end{bmatrix}_q - (q^6+q^4+q^2)$. None of them represents a plane of $PG(3,q^2)$, and for each of them the 3-space $S_3 = S_4 \cap \Sigma'$ contains either 0 or 1 line of $\mathscr{S}$.

Let π be a plane of $\Sigma = PG(6,q)$ such that $\pi \cap \Sigma' = r \in \mathscr{S}$.

Proposition 4.6. *If π represents a line of $PG(3,q^2)$, then π is contained in exactly q^2+1 distinct 4-dimensional subspaces S_4 representing planes $PG(2,q^2)$ of $PG(3,q^2)$. For each of them, $S_3 = S_4 \cap \Sigma'$ contains exactly q^2+1 lines of $\mathscr{S}$.*

Proof. A line of $PG(3,q^2)$ is contained in exactly q^2+1 planes $PG(2,q^2)$. Under the Bruck–Bose representation, these planes correspond to q^2+1 distinct 4-dimensional subspaces S_4 of Σ containing the plane π.

By Proposition 4.4, for each such S_4 the 3-space $S_3 = S_4 \cap \Sigma'$ contains exactly q^2+1 lines of the spread $\mathscr{S}$.

5. A family of rational cubic surfaces

Let $\Sigma = PG(6,q)$ with homogeneous coordinates $(x_1,x_2,y_1,y_2,z_1,z_2,t)$, and let $\Sigma' = \{t=0\}$ be the hyperplane at infinity endowed with the Desarguesian spread $\mathscr{S}$ associated with the André–Bruck–Bose representation introduced in Section 2.

Choose a 3-subspace $S_3 \subset \Sigma'$ containing a complete subspread of $\mathscr{S}$, that is, a set of q^2+1 pairwise disjoint spread lines partitioning the points of S_3. Without loss of generality, we may assume $S_3 = \{x_1 = x_2 = t = 0\}$.

The spread lines of $\mathscr{S}$ contained in S_3 are precisely

$$\mathscr{S}_\gamma = \{(0,0,y_1,y_2,c_1y_1 + s c_2 y_2, c_2 y_1 + c_1 y_2, 0) \mid (y_1,y_2) \neq (0,0)\},$$

where $\gamma = c_1 + \omega c_2 \in GF(q^2)$ and $\omega^2 = s$, together with $\mathscr{S}_0 = \{(0,0,0,0,z_1,z_2,0) \mid (z_1,z_2) \neq (0,0)\}$. Hence $\{\mathscr{S}_\gamma \mid \gamma \in GF(q^2)\} \cup \{\mathscr{S}_0\}$ is a complete subspread of S_3 consisting of q^2+1 lines (see Section 2).

Inside this subspread we choose the regulus $\mathscr{R}$ consisting of the following $q+1$ lines: the q lines $\mathscr{S}_\gamma$ with $\gamma = c_1 \in GF(q)$, that is, with $c_2 = 0$, together with the line $\mathscr{S}_0$. Writing $u = c_1$, we denote these lines by

$\mathscr{R} = r_u \mid u \in GF(q) \cup \infty$, where, for $u \in GF(q)$, $r_u = \{(0,0,1,\lambda,u,u\lambda,0) \mid \lambda \in GF(q)\} \cup \{(0,0,0,1,0,u,0)\}$, while $r_\infty = \mathscr{S}_0$, that is, $r_\infty = \{(0,0,0,0,1,\lambda,0) \mid \lambda \in GF(q)\} \cup \{(0,0,0,0,0,1,0)\}$.

Choose the spread line $\ell = \{(x_1,x_2,0,0,0,0,0) \mid (x_1,x_2) \neq (0,0)\}$. Since $\ell \cap S_3 = \varnothing$, we have $\ell \notin \mathscr{R}$.

Let $P = (0,0,0,0,0,0,1) \notin \Sigma'$ and set $\pi = \langle \ell, P \rangle$. Explicitly, $\pi = \{(x_1,x_2,0,0,0,0,t) \mid x_1,x_2,t \in GF(q), (x_1,x_2,t) \neq (0,0,0)\}$.

Fix the affine conic $\mathscr{C} \subset \pi$ defined by $x_1^2 - s x_2^2 - x_1 t = 0$. Its $q+1$ points are all affine and are given by $Q(\lambda) = (1,\lambda,0,0,0,0,1-s\lambda^2)$, with $\lambda \in GF(q)$, together with $Q(\infty) = (0,0,0,0,0,0,1)$. Indeed, since s is a nonsquare in $GF(q)$, $1 - s\lambda^2 \neq 0$ for every $\lambda \in GF(q)$, and $Q(\infty)$ is affine as well.

For every $u \in GF(q) \cup \{\infty\}$, let $\Phi_u : \mathscr{C} \to r_u$ be the following projectivity.

For $u \in GF(q)$, define

$\Phi_u(Q(\lambda)) = (0,0,1,\lambda,u,u\lambda,0)$ for $\lambda \in GF(q)$, and

$\Phi_u(Q(\infty)) = (0,0,0,1,0,u,0)$.

For $u = \infty$, define $\Phi_\infty(Q(\lambda)) = (0,0,0,0,1,\lambda,0)$ for $\lambda \in GF(q)$, and $\Phi_\infty(Q(\infty)) = (0,0,0,0,0,1,0)$.

The corresponding generators are

$$g_u(Q(\lambda)) = \{(1,\lambda,\tau,\tau\lambda,\tau u,\tau u\lambda,1-s\lambda^2) \mid \tau \in GF(q)\} \cup \{(0,0,1,\lambda,u,u\lambda,0)\}$$

and $g_u(Q(\infty)) = \{(0,0,0,\tau,0,\tau u,1) \mid \tau \in GF(q)\} \cup \{(0,0,0,1,0,u,0)\}$.

For $u = \infty$, the corresponding generators are

$$g_\infty(Q(\lambda)) = \{(1,\lambda,0,0,\tau,\tau\lambda,1-s\lambda^2) \mid \tau \in GF(q)\} \cup \{(0,0,0,0,1,\lambda,0)\}$$

and $g_\infty(Q(\infty)) = \{(0,0,0,0,0,\tau,1) \mid \tau \in GF(q)\} \cup \{(0,0,0,0,0,1,0)\}$.

For every $u \in GF(q) \cup \{\infty\}$, the ruled cubic surface $\mathscr{V}_u$ has linear directrix r_u and conic directrix $\mathscr{C}$, and is given by $\mathscr{V}_u = \bigcup_{Q \in \mathscr{C}} g_u(Q)$.

Finally, define $\mathscr{W} = \bigcup_{u \in GF(q) \cup \{\infty\}} \mathscr{V}_u$. Thus the family $\mathscr{W}$ is obtained by keeping the affine conic $\mathscr{C}$ fixed while the linear directrix varies among the lines of the regulus $\mathscr{R}$.

It follows from the above parametrizations that the affine points of $\mathscr{W}$ are precisely the points of the following two types: $X(\lambda,\mu,\nu)=(1,\lambda,\mu,\mu\lambda,\nu,\nu\lambda,1-s\lambda^2)$, with $\lambda,\mu,\nu\in GF(q)$, and $X_\infty(\mu,\nu)=(0,0,0,\mu,0,\nu,1)$, with $\mu,\nu\in GF(q)$.

Indeed, if $\mu\neq 0$, then $X(\lambda,\mu,\nu)\in\mathscr{V}_u$, where $u=\nu/\mu$. If $\mu=0$ and $\nu\neq 0$, then $X(\lambda,0,\nu)\in\mathscr{V}_\infty$, while $X(\lambda,0,0)=Q(\lambda)\in\mathscr{C}$ and hence belongs to every $\mathscr{V}_u$.

Similarly, if $\mu\neq 0$, then $X_\infty(\mu,\nu)\in\mathscr{V}_u$, where $u=\nu/\mu$. If $\mu=0$ and $\nu\neq 0$, then $X_\infty(0,\nu)\in\mathscr{V}_\infty$, while $X_\infty(0,0)=Q(\infty)\in\mathscr{C}$ and hence belongs to every $\mathscr{V}_u$.

Thus every affine point of $\mathscr{W}$ has one of the two forms above, and conversely every point of either form belongs to $\mathscr{W}$. Moreover, the parametrizations show explicitly that the points for which $\mu=\nu=0$ are precisely the points of the common conic $\mathscr{C}$.

In the remainder of the paper we shall use the following geometric interpretation of each ruled surface $\mathscr{V}_u:=V_2^3(u)$.

Remark 3. As proved in [2][12], each ruled surface $\mathscr{V}_u$, considered in the $PG(4,q)$ spanned by its two directrices, represents a Baer subplane $PG(2,q)$ of $PG(2,q^2)$ having a unique point on the line at infinity. Under this representation, the affine conics contained in $\mathscr{V}_u$ represent the affine lines of the Baer subplane, while the generators represent the lines through its unique point at infinity.

Consequently, each affine conic of $\mathscr{V}_u$ is contained in a unique representing plane; in particular, the plane of an affine conic contains neither another affine conic nor a generator of the same ruled surface. Moreover, as shown in [12], Section 3.5, these affine conics arise as affine components of sections of $\mathscr{V}_u$ by 3-spaces T containing a spread line $m_\gamma\subset S_3$, with $m_\gamma\neq r_u$, where r_u is the linear directrix of $\mathscr{V}_u$.

More precisely, $T\cap\mathscr{V}_u$ is the union of an affine conic and a generator of $\mathscr{V}_u$, the latter meeting the conic in one point.

Proposition 5.1. *For any two distinct parameters $u,v\in GF(q)\cup\{\infty\}$, one has $\mathscr{V}_u\cap\mathscr{V}_v=\mathscr{C}$.*

Proof. Since $\mathscr{C}$ is a directrix of every ruled surface $\mathscr{V}_u$, we have $\mathscr{C}\subseteq\mathscr{V}_u\cap\mathscr{V}_v$.

Let $X\in\mathscr{V}_u\cap\mathscr{V}_v$. Then there exist $Q,Q'\in\mathscr{C}$ such that $X\in g_u(Q)$ and $X\in g_v(Q')$.

Suppose first that X is affine and that $Q=Q(\lambda)$, $Q'=Q(\mu)$, with $\lambda,\mu\in GF(q)$. Since every affine point of $g_u(Q(\lambda))$ has second coordinate λ, whereas every affine point of $g_v(Q(\mu))$ has second coordinate μ, we obtain $\lambda=\mu$, and hence $Q=Q'$.

Assume first that $u, v \in GF(q)$. Then $X \in g_u(Q(\lambda)) \cap g_v(Q(\lambda))$ gives $X = (1, \lambda, \tau, \tau\lambda, \tau u, \tau u\lambda, 1 - s\lambda^2)$ and $X = (1, \lambda, \sigma, \sigma\lambda, \sigma v, \sigma v\lambda, 1 - s\lambda^2)$. Thus $\tau = \sigma$ and $\tau(u - v) = 0$. Since $u \neq v$, we have $\tau = 0$, and therefore $X = Q(\lambda) \in \mathscr{C}$.

Suppose now that $u \in GF(q)$ and $v = \infty$. Then $X \in g_u(Q(\lambda))$ has the form $(1, \lambda, \tau, \tau\lambda, \tau u, \tau u\lambda, 1 - s\lambda^2)$, whereas $X \in g_\infty(Q(\lambda))$ has the form $(1, \lambda, 0, 0, \sigma, \sigma\lambda, 1 - s\lambda^2)$. Comparison of the y_1-coordinates gives $\tau = 0$, and comparison of the z_1-coordinates then gives $\sigma = 0$. Hence again $X = Q(\lambda)$. The case $u = \infty$, $v \in GF(q)$ is analogous.

If $Q = Q(\lambda)$, with $\lambda \in GF(q)$, and $Q' = Q(\infty)$, then every affine point of $g_u(Q(\lambda))$ has first coordinate 1, whereas every affine point of $g_v(Q(\infty))$ has first coordinate 0. Hence the two generators have no common affine point. The same argument applies when $Q = Q(\infty)$ and $Q' = Q(\mu)$.

It remains to consider $Q = Q' = Q(\infty)$. If $u, v \in GF(q)$, then $X \in g_u(Q(\infty)) \cap g_v(Q(\infty))$ gives $X = (0, 0, 0, \tau, 0, \tau u, 1)$ and $X = (0, 0, 0, \sigma, 0, \sigma v, 1)$. Hence $\tau = \sigma$ and $\tau(u - v) = 0$. Since $u \neq v$, we obtain $\tau = 0$, and therefore $X = Q(\infty)$.

If $u \in GF(q)$ and $v = \infty$, then $X \in g_u(Q(\infty))$ has the form $(0, 0, 0, \tau, 0, \tau u, 1)$, whereas $X \in g_\infty(Q(\infty))$ has the form $(0, 0, 0, 0, 0, \sigma, 1)$. Comparison of the y_2-coordinates gives $\tau = 0$, and then comparison of the z_2-coordinates gives $\sigma = 0$. Thus $X = Q(\infty)$. Again, the case $u = \infty$, $v \in GF(q)$ is analogous.

Finally, there is no common improper point. The improper points of the generators of $\mathscr{V}_u$ lie on r_u, whereas those of the generators of $\mathscr{V}_v$ lie on r_v, and the lines r_u and r_v are disjoint for $u \neq v$.

Therefore every point of $\mathscr{V}_u \cap \mathscr{V}_v$ belongs to $\mathscr{C}$. Since $\mathscr{C} \subseteq \mathscr{V}_u \cap \mathscr{V}_v$, we conclude that $\mathscr{V}_u \cap \mathscr{V}_v = \mathscr{C}$.

Corollary 5.2. *Let $\mathscr{W} = \bigcup_{u \in GF(q) \cup \{\infty\}} \mathscr{V}_u$. Then $\mathscr{W}$ contains exactly $q^3 + q^2$ affine points.*

Proof. Each ruled cubic surface $\mathscr{V}_u$ contains exactly $q^2 + q$ affine points, while the affine conic $\mathscr{C}$ contains $q + 1$ points.

By the previous Proposition, for any distinct parameters $u, v \in GF(q) \cup \{\infty\}$, we have $\mathscr{V}_u \cap \mathscr{V}_v = \mathscr{C}$. Hence the sets $(\mathscr{V}_u \setminus \Sigma') \setminus \mathscr{C}$ are pairwise disjoint.

Since there are $q + 1$ ruled surfaces, it follows that $|\mathscr{W} \setminus \Sigma'| = (q + 1)\big((q^2 + q) - (q + 1)\big) + (q + 1)$.

Therefore $|\mathscr{W} \setminus \Sigma'| = (q + 1)(q^2 - 1) + (q + 1) = q^2(q + 1) = q^3 + q^2$.

Hence $\mathscr{W}$ contains exactly $q^3 + q^2$ affine points.

Note that, by the previous Proposition, the only affine points common to two distinct ruled surfaces are the points of the affine conic $\mathscr{C}$. Consequently, every affine point of $\mathscr{W} \setminus \mathscr{C}$ belongs to a unique ruled surface $\mathscr{V}_u$.

More precisely, let X be an affine point of $\mathscr{W} \setminus \mathscr{C}$. Then there exists a unique parameter $u \in GF(q) \cup \{\infty\}$ such that $X \in \mathscr{V}_u$. Moreover, by the parametrization of the generators of $\mathscr{V}_u$, there exists a unique point $Q \in \mathscr{C}$ such that $X \in g_u(Q) \setminus \{Q\}$.

Thus every affine point of $\mathscr{W} \setminus \mathscr{C}$ is uniquely determined by the pair (u, Q) together with its position on the corresponding generator $g_u(Q)$.

6. Plane sctions of $\mathscr{W}$ and the associated Baer subgeometry

Let S_3 be the 3-space of Σ' introduced in Section 5, with equations $x_1 = x_2 = t = 0$. Let π be the plane of the conic $\mathscr{C}$ common to all ruled surfaces, with $\pi \cap \Sigma' = \ell \in \mathscr{S}$, where $\ell \cap S_3 = \varnothing$.

Furthermore, let r_u, $u \in GF(q) \cup \{\infty\}$, denote the lines of the regulus $\mathscr{R}$, which are the linear directrices of the ruled surfaces $\mathscr{V}_u$ and belong to the spread $\mathscr{S}$ in S_3.

Our aim is to determine the intersections with the lines of $PG(3, q^2)$ of the point set represented by $\mathscr{W}$, together with the improper points represented by the lines of $\mathscr{R}$. More precisely, we shall prove that every line of $PG(3, q^2)$ meets this point set in 0, 1 or $q+1$ points.

Under the Bruck–Bose representation, a line of $PG(3, q^2)$ is represented by a plane β of Σ such that $\beta \cap \Sigma' = m$, where $m \in \mathscr{S}$. Hence it is enough to determine the possible intersections of $\mathscr{W}$ with such planes.

Let $P, Q \in \mathscr{W}$ be two distinct affine points contained in such a plane β. Then the line PQ meets Σ' in a point $M = PQ \cap \Sigma'$, and M belongs to the spread line $m = \beta \cap \Sigma'$. Equivalently, $\beta = \langle PQ, m \rangle$.

Thus, in order to prove that a line cannot contain a number of represented points different from 0, 1 or $q+1$, it is enough to consider a plane β containing two distinct affine points $P, Q \in \mathscr{W}$ and determine its intersection with $\mathscr{W}$. When $m \in \mathscr{R}$, the spread line m itself represents one improper point and must also be taken into account.

From now on, when necessary, we use the coordinate model of the affine conic $\mathscr{C}$ fixed in Section 5.

The analysis depends on the position of the spread line $m = \beta \cap \Sigma'$ with respect to S_3, the distinguished line ℓ, and the regulus $\mathscr{R}$. We distinguish the following cases:

1. $m \subset \Sigma' \setminus S_3, m \neq \ell$;
2. $m \subset \Sigma' \setminus S_3, m = \ell$;
3. $m \subset S_3$ and $m \notin \mathscr{R}$;
4. $m = r_u$, for some $u \in GF(q) \cup \infty$.

Case 1. Let $m \subset \Sigma' \setminus S_3$, with $m \neq \ell$, and let $\beta = \langle PQ, m \rangle$, where $P, Q \in \mathscr{W}$ are distinct affine points and $PQ \cap \Sigma' \in m$.

Suppose first that $P, Q \in \mathscr{V}_u$, for some $u \in GF(q) \cup \{\infty\}$. Since $m \cap S_3 = \varnothing$, the points P and Q cannot lie on the same generator of $\mathscr{V}_u$, since the point at infinity of every generator of $\mathscr{V}_u$ belongs to $r_u \subset S_3$. Hence, by Property (v) of Section 3.5 in [12], P and Q determine a conic directrix $\mathscr{C}_u$ of $\mathscr{V}_u$. In the present construction $\mathscr{C}_u$ is an affine conic and therefore contains exactly $q+1$ affine points. Its plane is β.

It remains to prove that β contains no further affine points of $\mathscr{W}$. We prove this at the same time as the case in which P and Q do not belong to a common surface $\mathscr{V}_u$.

Recall that every affine point of $\mathscr{W}$ has one of the forms $X(\lambda, \mu, \nu) = (1, \lambda, \mu, \mu\lambda, \nu, \nu\lambda, 1 - s\lambda^2)$, with $\lambda, \mu, \nu \in GF(q)$, or $X_\infty(\mu, \nu) = (0, 0, 0, \mu, 0, \nu, 1)$, with $\mu, \nu \in GF(q)$.

Since $m \subset \Sigma' \setminus S_3$, we may write $m = \{(\xi, \eta, a_1\xi + sa_2\eta, a_2\xi + a_1\eta, b_1\xi + sb_2\eta, b_2\xi + b_1\eta, 0) \mid (\xi, \eta) \neq (0,0)\}$, where $a_1, a_2, b_1, b_2 \in GF(q)$.

Choose an affine point $P_0 = (a, c, d, f, g, h, 1) \in \beta$. Since $\beta = \langle P_0, m \rangle$, the plane β has equations $y_1 = a_1x_1 + sa_2x_2 + Dt$, $y_2 = a_2x_1 + a_1x_2 + Ft$, $z_1 = b_1x_1 + sb_2x_2 + Gt$ and $z_2 = b_2x_1 + b_1x_2 + Ht$, where $D = d - a_1a - sa_2c$, $F = f - a_2a - a_1c$, $G = g - b_1a - sb_2c$ and $H = h - b_2a - b_1c$.

Let $X(\lambda, \mu, \nu) \in \beta \cap \mathscr{W}$. Substitution into the first two equations of β gives $\mu = a_1 + sa_2\lambda + D(1 - s\lambda^2)$ and $\mu\lambda = a_2 + a_1\lambda + F(1 - s\lambda^2)$. Multiplying the first equation by λ and subtracting the second, we obtain $(1 - s\lambda^2)(D\lambda - F - a_2) = 0$. Since s is a nonsquare, $1 - s\lambda^2 \neq 0$, and hence $D\lambda = F + a_2$. Similarly, from the last two equations we obtain $G\lambda = H + b_2$.

Suppose first that $P = X(\lambda, \mu, \nu)$ and $Q = X(\lambda', \mu', \nu')$. Then $\lambda \neq \lambda'$. Indeed, if $\lambda = \lambda'$, the point $PQ \cap \Sigma'$ has $x_1 = x_2 = t = 0$, and hence belongs to S_3, contrary to $PQ \cap \Sigma' \in m$ and $m \cap S_3 = \varnothing$. Consequently, $D\lambda = F + a_2$ and $D\lambda' = F + a_2$, whence $D = 0$ and $F = -a_2$. Similarly, $G = 0$ and $H = -b_2$.

Suppose now that one of the two points, say Q, is of the form $Q = X_\infty(\mu', \nu') = (0,0,0,\mu',0,\nu',1)$. Substitution into the equations of β gives $D = 0$, $F = \mu'$, $G = 0$ and $H = \nu'$. The other point cannot also be of the form $X_\infty(\mu,\nu)$, since otherwise $PQ \cap \Sigma' \in S_3$. Thus $P = X(\lambda,\mu,\nu)$, and the relations $D\lambda = F + a_2$ and $G\lambda = H + b_2$ give $F = -a_2$ and $H = -b_2$.

Therefore, whenever β contains two distinct affine points of $\mathscr{W}$, its equations reduce to $y_1 = a_1x_1 + sa_2x_2$, $y_2 = a_2x_1 + a_1x_2 - a_2t$, $z_1 = b_1x_1 + sb_2x_2$ and $z_2 = b_2x_1 + b_1x_2 - b_2t$.

For each $\theta \in GF(q)$, put $A_\theta = a_1 + sa_2\theta$ and $B_\theta = b_1 + sb_2\theta$, and define $R_\theta = (1, \theta, A_\theta, \theta A_\theta, B_\theta, \theta B_\theta, 1 - s\theta^2)$. Then $R_\theta \in \beta$. Moreover, $R_\theta \in \mathscr{W}$. Indeed, if $A_\theta \neq 0$, then $R_\theta \in \mathscr{V}_{B_\theta/A_\theta}$; if $A_\theta = 0$ and $B_\theta \neq 0$, then $R_\theta \in \mathscr{V}_\infty$; while, if $A_\theta = B_\theta = 0$, then $R_\theta = Q(\theta) \in \mathscr{C}$.

There is also the affine point $R_\infty = (0,0,0,-a_2,0,-b_2,1)$. Indeed, $R_\infty \in \beta$. Moreover, if $a_2 \neq 0$, then $R_\infty \in \mathscr{V}_{b_2/a_2}$; if $a_2 = 0$ and $b_2 \neq 0$, then $R_\infty \in \mathscr{V}_\infty$; while, if $a_2 = b_2 = 0$, then $R_\infty = Q(\infty) \in \mathscr{C}$. Hence $R_\infty \in \mathscr{W}$.

Thus β contains the $q+1$ distinct affine points $\{R_\theta \mid \theta \in GF(q)\} \cup \{R_\infty\}$.

We now prove that these are all the affine points of $\beta \cap \mathscr{W}$. Let $R \in \beta \cap \mathscr{W}$. If $R = X(\theta,\mu,\nu)$, then the first and third equations of β give $\mu = A_\theta$ and $\nu = B_\theta$, and hence $R = R_\theta$. If $R = X_\infty(\mu,\nu)$, then the second and fourth equations of β give $\mu = -a_2$ and $\nu = -b_2$, and hence $R = R_\infty$. Therefore $\beta \cap \mathscr{W} = \{R_\theta \mid \theta \in GF(q)\} \cup \{R_\infty\}$, and consequently $|\beta \cap \mathscr{W}| = q+1$.

In particular, if $P, Q \in \mathscr{V}_u$, the conic $\mathscr{C}_u$ determined by P and Q satisfies $\mathscr{C}_u \subseteq \beta \cap \mathscr{W}$ and $|\mathscr{C}_u| = q+1$. Since $|\beta \cap \mathscr{W}| = q+1$, it follows that $\beta \cap \mathscr{W} = \mathscr{C}_u$. Thus the plane of $\mathscr{C}_u$ contains no further affine points arising from the other surfaces $\mathscr{V}_v$, with $v \neq u$.

We have therefore proved the following.

Lemma 6.1. *Let $P, Q \in \mathscr{W}$ be two distinct affine points, and let $\beta = \langle PQ, m\rangle$, where $PQ \cap \Sigma'$ belongs to a spread line $m \subset \Sigma' \setminus S_3$, with $m \neq \ell$. Then $|\beta \cap \mathscr{W}| = q+1$.*

Corollary 6.2. *Let β be a plane such that $\beta \cap \Sigma' = m$, where $m \subset \Sigma' \setminus S_3$ and $m \neq \ell$. Then $|\beta \cap \mathscr{W}| \in \{0, 1, q+1\}$.*

Proof. If β contains at most one affine point of $\mathscr{W}$, then $|\beta \cap \mathscr{W}| \in \{0,1\}$. Suppose that β contains two distinct affine points $P, Q \in \mathscr{W}$. Since $PQ \subset \beta$, we have $PQ \cap \Sigma' \in \beta \cap \Sigma' = m$. Hence the preceding Lemma gives $|\beta \cap \mathscr{W}| = q+1$. Therefore $|\beta \cap \mathscr{W}| \in \{0, 1, q+1\}$.

Case 2. Let $m = \ell$ and let β be a plane such that $\beta \cap \Sigma' = \ell$.

Suppose first that $P, Q \in \mathscr{V}_u$, for some $u \in GF(q) \cup \{\infty\}$.

The points P and Q cannot lie on the same generator of $\mathscr{V}_u$, since the point at infinity of every generator belongs to $r_u \subset S_3$, whereas $PQ \cap \Sigma' \in \ell$ and $\ell \cap S_3 = \varnothing$.

Hence, by Property (v) of Section 3.5 in [12], the points P and Q determine a conic directrix of $\mathscr{V}_u$. We shall see below that necessarily $\beta = \pi$, and hence this conic is the common directrix $\mathscr{C}$.

Lemma 6.3. *If β contains two distinct affine points of $\mathscr{W}$, then $|\beta \cap \mathscr{W}| = q + 1$.*

Proof. Since $\beta \cap \Sigma' = \ell$, where $\ell = \{(x_1, x_2, 0, 0, 0, 0)\}$, the affine points of β have constant coordinates y_1, y_2, z_1, z_2. Thus, for some $d, f, g, h \in GF(q)$, the affine part of β is given by $y_1 = d$, $y_2 = f$, $z_1 = g$ and $z_2 = h$.

Recall that every affine point of $\mathscr{W}$ has one of the forms $X(\lambda, \mu, \nu) = (1, \lambda, \mu, \mu\lambda, \nu, \nu\lambda, 1 - s\lambda^2)$, with $\lambda, \mu, \nu \in GF(q)$, or $X_\infty(\mu, \nu) = (0, 0, 0, \mu, 0, \nu, 1)$, with $\mu, \nu \in GF(q)$.

Let first $X(\lambda, \mu, \nu) \in \beta \cap \mathscr{W}$. The equations of β give $\mu = d$, $\mu\lambda = f$, $\nu = g$ and $\nu\lambda = h$.

If $d \neq 0$, then $\mu = d$ and $\lambda = f/d$. Moreover, $\nu = g$, and the last equation requires $h = fg/d$. Hence there is at most one point of the form $X(\lambda, \mu, \nu)$ in $\beta \cap \mathscr{W}$. On the other hand, a point $X_\infty(\mu, \nu)$ cannot belong to β, since its y_1-coordinate is zero, whereas $d \neq 0$. Therefore, if $d \neq 0$, the plane β contains at most one affine point of $\mathscr{W}$.

Suppose now that $d = 0$ and $g \neq 0$. For a point $X(\lambda, \mu, \nu) \in \beta \cap \mathscr{W}$, we have $\mu = 0$ and $\nu = g$. Hence $f = 0$ and $\lambda = h/g$, so again there is at most one point of this form. Moreover, no point $X_\infty(\mu, \nu)$ can belong to β, since its z_1-coordinate is zero, whereas $g \neq 0$. Thus also in this case β contains at most one affine point of $\mathscr{W}$.

It remains to consider $d = g = 0$. If $X(\lambda, \mu, \nu) \in \beta \cap \mathscr{W}$, then $\mu = \nu = 0$, and therefore $f = h = 0$. Hence, if either $f \neq 0$ or $h \neq 0$, no point of the form $X(\lambda, \mu, \nu)$ belongs to β.

A point $X_\infty(\mu, \nu)$ belongs to β if and only if $\mu = f$ and $\nu = h$. Thus, if either $f \neq 0$ or $h \neq 0$, there is exactly one affine point of $\mathscr{W}$ in β, namely $X_\infty(f, h)$.

Consequently, a plane β with $\beta \cap \Sigma' = \ell$ can contain two distinct affine points of $\mathscr{W}$ only if $d = f = g = h = 0$. In this case the affine part of β is given by $y_1 = y_2 = z_1 = z_2 = 0$, and hence $\beta = \pi$.

For every $\lambda \in GF(q)$, the point $X(\lambda, 0, 0) = (1, \lambda, 0, 0, 0, 0, 1 - s\lambda^2) = Q(\lambda)$ belongs to $\beta \cap \mathscr{W}$. Moreover, $X_\infty(0, 0) = (0, 0, 0, 0, 0, 0, 1) = Q(\infty)$ also belongs to $\beta \cap \mathscr{W}$. These are precisely the $q + 1$ affine points of the conic $\mathscr{C}$.

Conversely, if $X(\lambda,\mu,\nu) \in \beta \cap \mathscr{W}$, the equations $y_1 = z_1 = 0$ give $\mu = \nu = 0$, and hence $X = Q(\lambda)$. If $X_\infty(\mu,\nu) \in \beta \cap \mathscr{W}$, the equations $y_2 = z_2 = 0$ give $\mu = \nu = 0$, and hence $X = Q(\infty)$. Therefore $\beta \cap \mathscr{W} = \mathscr{C}$, and consequently $|\beta \cap \mathscr{W}| = q+1$.

Corollary 6.4. *Let β be a plane such that $\beta \cap \Sigma' = \ell$. Then $|\beta \cap \mathscr{W}| \in \{0, 1, q+1\}$.*

Proof. If β contains at most one affine point of $\mathscr{W}$, then $|\beta \cap \mathscr{W}| \in \{0,1\}$. If β contains two distinct affine points of $\mathscr{W}$, then, by the preceding Lemma, $|\beta \cap \mathscr{W}| = q+1$. Therefore $|\beta \cap \mathscr{W}| \in \{0,1,q+1\}$.

Case 3. Let $m \subset S_3$ be a spread line with $m \notin \mathscr{R}$, and let $\beta = \langle PQ, m\rangle$, where $P, Q \in \mathscr{W}$ are distinct affine points and $PQ \cap \Sigma' \in m$.

Lemma 6.5. *There do not exist distinct affine points $P, Q \in \mathscr{W}$ such that $PQ \cap \Sigma'$ belongs to a spread line $m \subset S_3$ with $m \notin \mathscr{R}$.*

Proof. Recall that every affine point of $\mathscr{W}$ has one of the forms

$X(\lambda,\mu,\nu) = (1, \lambda, \mu, \mu\lambda, \nu, \nu\lambda, 1 - s\lambda^2)$, with $\lambda, \mu, \nu \in GF(q)$, or $X_\infty(\mu,\nu) = (0,0,0,\mu,0,\nu,1)$, with $\mu, \nu \in GF(q)$.

We consider all possible pairs of distinct points $P, Q \in \mathscr{W}$.

Suppose first that $P = X(\lambda,\mu,\nu)$ and $Q = X(\lambda',\mu',\nu')$. Since $PQ \cap \Sigma' \in S_3$, there is a point of PQ with $x_1 = x_2 = t = 0$. Since the first coordinate of both P and Q is equal to 1, this point is proportional to $P - Q$. Hence its second coordinate gives $\lambda - \lambda' = 0$, and therefore $\lambda = \lambda'$.

Consequently $PQ \cap \Sigma' \sim (0, 0, \mu - \mu', \lambda(\mu - \mu'), \nu - \nu', \lambda(\nu - \nu'), 0)$.

If $\mu \neq \mu'$, put $u = (\nu - \nu')/(\mu - \mu')$. Then $PQ \cap \Sigma' \sim (0,0,1,\lambda,u,u\lambda,0)$, and hence $PQ \cap \Sigma' \in r_u$.

If $\mu = \mu'$, since $P \neq Q$, we have $\nu \neq \nu'$, and hence $PQ \cap \Sigma' \sim (0,0,0,0,1,\lambda,0) \in r_\infty$. Thus, in both cases, $PQ \cap \Sigma'$ belongs to a line of $\mathscr{R}$.

Suppose now that $P = X(\lambda,\mu,\nu)$ and $Q = X_\infty(\mu',\nu')$. A point of PQ has the form $\alpha P + \beta Q$. If such a point belonged to S_3, then its first coordinate would give $\alpha = 0$. Its last coordinate would then give $\beta = 0$, which is impossible. Therefore $PQ \cap \Sigma' \notin S_3$.

It remains to consider $P = X_\infty(\mu,\nu)$ and $Q = X_\infty(\mu',\nu')$. Since P and Q have the same last coordinate, we have $PQ \cap \Sigma' \sim (0,0,0,\mu - \mu', 0, \nu - \nu', 0)$.

If $\mu \neq \mu'$, put $u = (\nu - \nu')/(\mu - \mu')$. Then $PQ \cap \Sigma' \sim (0,0,0,1,0,u,0) \in r_u$. If $\mu = \mu'$, then $\nu \neq \nu'$, since $P \neq Q$, and $PQ \cap \Sigma' \sim (0,0,0,0,0,1,0) \in r_\infty$.

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Hence, whenever $PQ \cap \Sigma'$ belongs to S_3, it lies on a spread line of the regulus $\mathscr{R}$. Therefore it cannot belong to a spread line $m \subset S_3$ with $m \notin \mathscr{R}$, a contradiction.

Corollary 6.6. *Let β be a plane such that $\beta \cap \Sigma' = m$, where $m \subset S_3$ and $m \notin \mathscr{R}$. Then $|\beta \cap \mathscr{W}| \in \{0, 1\}$.*

Proof. If β contained two distinct affine points $P, Q \in \mathscr{W}$, then $PQ \subset \beta$, and hence $PQ \cap \Sigma' \in \beta \cap \Sigma' = m$. This contradicts the preceding Lemma. Therefore β contains at most one affine point of $\mathscr{W}$, and hence $|\beta \cap \mathscr{W}| \in \{0, 1\}$.

Case 4. This case splits into two subcases, according as $a)$ $m = r_u$, with $u \in GF(q)$, or $b)$ $m = r_\infty$.

$a)$ Let $m = r_u$, with $u \in GF(q)$.

Suppose first that $P \in \mathscr{V}_u$ is an affine point, and let g be the generator of $\mathscr{V}_u$ through P. Since g meets r_u, we have $g \subset \beta = \langle r_u, P \rangle$. We show that no further affine points of $\mathscr{W}$ belong to β.

Assume first that $P = X(\lambda, \rho, \rho u)$. A generic affine point of β has the form $X = (1, \lambda, \rho + a, \rho\lambda + b, u(\rho + a), u(\rho\lambda + b), 1 - s\lambda^2)$, with $a, b \in GF(q)$. If $X \in \mathscr{W}$, comparison with $X(\lambda, \mu, \nu)$ gives $\mu = \rho + a$ and $b = \lambda a$. Hence $\nu = u(\rho + a) = u\mu$, and therefore X belongs to the generator g. Thus β contains exactly the q affine points of g.

If $P = X_\infty(\rho, \rho u)$, a generic affine point of β has the form $X = (0, 0, a, \rho + b, ua, u(\rho + b), 1)$. Since every point of $\mathscr{W}$ with $x_1 = x_2 = 0$ is of the form $X_\infty(\mu, \nu)$, necessarily $a = 0$. Hence $X = (0, 0, 0, \rho + b, 0, u(\rho + b), 1)$, with $b \in GF(q)$, and these are precisely the q affine points of the generator g through P.

Thus, if $P \in \mathscr{V}_u$, the plane β contains exactly the q affine points of the generator of $\mathscr{V}_u$ through P.

We may therefore assume $P \in \mathscr{W} \setminus \mathscr{V}_u$.

Lemma 6.7. *Let $r_u \in \mathscr{R}$, with $u \in GF(q)$, and let $P \in \mathscr{W} \setminus \mathscr{V}_u$ be an affine point. Then the plane $\beta = \langle r_u, P \rangle$ contains exactly q affine points of $\mathscr{W}$. Hence, since r_u represents one improper point, the corresponding section contains exactly $q + 1$ represented points.*

Proof. Suppose first that $P = X(\lambda, \rho, \nu)$. Since $P \notin \mathscr{V}_u$, we have $\nu \neq \rho u$. A generic affine point of β has the form $X = (1, \lambda, \rho + a, \rho\lambda + b, \nu + ua, \nu\lambda + ub, 1 - s\lambda^2)$, with $a, b \in GF(q)$.

If $X \in \mathscr{W}$, then necessarily $X = X(\lambda, \mu, \tau)$ for some $\mu, \tau \in GF(q)$. Comparison of the y_1, y_2-coordinates gives $\mu = \rho + a$ and $b = \lambda a$. Conversely, if $b = \lambda a$, then $X = (1, \lambda, \rho + a, (\rho + a)\lambda, \nu + ua, (\nu + ua)\lambda, 1 - s\lambda^2)$, which belongs to $\mathscr{W}$. Hence, for every $a \in GF(q)$, there is exactly one affine point of $\beta \cap \mathscr{W}$. Therefore β contains exactly q affine points of $\mathscr{W}$.

Suppose now that $P = X_\infty(\rho, \nu)$. Since $P \notin \mathscr{V}_u$, we have $\nu \neq \rho u$. A generic affine point of β has the form $X = (0, 0, a, \rho + b, ua, \nu + ub, 1)$, with $a, b \in GF(q)$. Such a point belongs to $\mathscr{W}$ if and only if it is of the form $X_\infty(\mu, \tau)$, and hence if and only if $a = 0$. Therefore $\beta \cap \mathscr{W}$ consists precisely of the q points $(0, 0, 0, \rho + b, 0, \nu + ub, 1)$, with $b \in GF(q)$.

Thus, in both cases, β contains exactly q affine points of $\mathscr{W}$. Since r_u represents one improper point, the corresponding section contains exactly $q + 1$ represented points.

b) Let $m = r_\infty$.

Suppose first that $P \in \mathscr{V}_\infty$ is an affine point, and let g be the generator of $\mathscr{V}_\infty$ through P. Since g meets r_∞, we have $g \subset \beta = \langle r_\infty, P \rangle$. We show that no further affine points of $\mathscr{W}$ belong to β.

Assume first that $P = X(\lambda, 0, \rho)$. A generic affine point of β has the form $X = (1, \lambda, 0, 0, \rho + a, \rho\lambda + b, 1 - s\lambda^2)$, with $a, b \in GF(q)$. Such a point belongs to $\mathscr{W}$ if and only if $b = \lambda a$. Hence $\beta \cap \mathscr{W}$ consists precisely of the q points $(1, \lambda, 0, 0, \rho + a, (\rho + a)\lambda, 1 - s\lambda^2)$, with $a \in GF(q)$, which are the affine points of g.

If $P = X_\infty(0, \rho)$, a generic affine point of β has the form $X = (0, 0, 0, 0, a, \rho + b, 1)$. Such a point belongs to $\mathscr{W}$ if and only if $a = 0$. Hence $\beta \cap \mathscr{W}$ consists precisely of the q points $(0, 0, 0, 0, 0, \rho + b, 1)$, with $b \in GF(q)$, which are again the affine points of g.

Thus, if $P \in \mathscr{V}_\infty$, the plane β contains exactly the q affine points of the generator of $\mathscr{V}_\infty$ through P.

We may therefore assume $P \in \mathscr{W} \setminus \mathscr{V}_\infty$.

Lemma 6.8. *Let $P \in \mathscr{W} \setminus \mathscr{V}_\infty$ be an affine point. Then the plane $\beta = \langle r_\infty, P \rangle$ contains exactly q affine points of $\mathscr{W}$. Hence, since r_∞ represents one improper point, the corresponding section contains exactly $q + 1$ represented points.*

Proof. Suppose first that $P = X(\lambda, \rho, \nu)$. Since $P \notin \mathscr{V}_\infty$, we have $\rho \neq 0$. A generic affine point of β has the form $X = (1, \lambda, \rho, \rho\lambda, \nu + a, \nu\lambda + b, 1 - s\lambda^2)$, with $a, b \in GF(q)$.

If $X \in \mathscr{W}$, comparison with $X(\lambda, \mu, \tau)$ gives $\mu = \rho$ and $b = \lambda a$. Conversely, if $b = \lambda a$, then $X = (1, \lambda, \rho, \rho\lambda, \nu + a, (\nu + a)\lambda, 1 - s\lambda^2)$ belongs to $\mathscr{W}$. Hence, for every $a \in GF(q)$, there is exactly one affine point of $\beta \cap \mathscr{W}$. Thus β contains exactly q affine points of $\mathscr{W}$.

Suppose now that $P = X_\infty(\rho, \nu)$. Since $P \notin \mathscr{V}_\infty$, we have $\rho \neq 0$. A generic affine point of β has the form $X = (0, 0, 0, \rho, a, \nu + b, 1)$, with $a, b \in GF(q)$. Such a point belongs to $\mathscr{W}$ if and only if it is of the form

$X_\infty(\mu,\tau)$, and hence if and only if $a=0$. Therefore $\beta\cap\mathscr{W}$ consists precisely of the q points $(0,0,0,\rho,0,\nu+b,1)$, with $b\in GF(q)$.

Thus, in both cases, β contains exactly q affine points of $\mathscr{W}$. Since r_∞ represents one improper point, the corresponding section contains exactly $q+1$ represented points.

Corollary 6.9. *Let β be a plane such that $\beta\cap\Sigma'=m$, with $m\in\mathscr{R}$. Then, if β contains an affine point of $\mathscr{W}$, it contains exactly q affine points of $\mathscr{W}$. Hence the corresponding section contains exactly $q+1$ represented points.*

Let $\mathscr{H}$ be the point set of $PG(3,q^2)$ whose affine points are represented by the affine points of $\mathscr{W}$ and whose improper points are the $q+1$ points represented by the lines of the regulus $\mathscr{R}$.

Proposition 6.10. *The point set $\mathscr{H}$ is a Baer subgeometry $PG(3,q)$ of $PG(3,q^2)$ meeting the plane at infinity in a line $PG(1,q)$.*

Proof. Let $S^0\cong PG(3,q)$ be the canonical Baer subgeometry of $PG(3,q^2)$, and consider the projectivity $g:(a,b,c,d)\mapsto(a,b,c,a-\omega d)$.

We show that $\mathscr{H}=g(S^0)$.

Let $(1,\mu,\nu,\lambda)$ be a point of S^0. Its image under g is $g(1,\mu,\nu,\lambda)=(1,\mu,\nu,1-\omega\lambda)$. By the Bruck–Bose correspondence described in Section 2, the point $X(\lambda,\mu,\nu)=(1,\lambda,\mu,\mu\lambda,\nu,\nu\lambda,1-s\lambda^2)$ of $\mathscr{W}$ represents the point $(1+\omega\lambda,\mu(1+\omega\lambda),\nu(1+\omega\lambda),1-s\lambda^2)$ of $PG(3,q^2)$. Since $1-s\lambda^2=(1+\omega\lambda)(1-\omega\lambda)$, this point is projectively equal to $(1,\mu,\nu,1-\omega\lambda)$.

If the first coordinate of a point of S^0 is zero and its last coordinate is non-zero, we may normalize it as $(0,b,c,1)$. Then $g(0,b,c,1)=(0,b,c,-\omega)$. After normalization of the last coordinate, this point has the form $(0,\omega\mu,\omega\nu,1)$, with $\mu,\nu\in GF(q)$, and hence, by the Bruck–Bose correspondence, it is represented by $X_\infty(\mu,\nu)=(0,0,0,\mu,0,\nu,1)$.

Finally, a point $(a,b,c,d)\in S^0$ is mapped by g to the plane at infinity if and only if $a-\omega d=0$. Since $a,d\in GF(q)$ and $\omega\notin GF(q)$, this is equivalent to $a=d=0$. Hence the points of S^0 mapped to the plane at infinity form the line $L_0=\{(0,b,c,0)\mid b,c\in GF(q),\ (b,c)\neq(0,0)\}$. Its $q+1$ points are represented in the Bruck–Bose model by the $q+1$ spread lines r_u, $u\in GF(q)\cup\{\infty\}$, of the regulus $\mathscr{R}$.

Therefore $\mathscr{H}=g(S^0)$. Hence $\mathscr{H}$ is a Baer subgeometry $PG(3,q)$ of $PG(3,q^2)$ and $\mathscr{H}\cap\Pi_\infty\cong PG(1,q)$.

Remark 4. The above result also gives a geometric interpretation of the family $\{\mathscr{V}_u\mid u\in GF(q)\cup\{\infty\}\}$. By Remark 3, each ruled cubic surface $\mathscr{V}_u$ represents a Baer subplane $PG(2,q)$ of the Baer subgeometry

$\mathscr{H}$. Moreover, the common affine conic $\mathscr{C}$ represents an affine line L contained in all these Baer subplanes. By Proposition 5.1, for $u \neq v$ one has $\mathscr{V}_u \cap \mathscr{V}_v = \mathscr{C}$, so the corresponding Baer subplanes intersect precisely in L. Hence the $q+1$ Baer subplanes represented by the surfaces $\mathscr{V}_u$ are exactly the $q+1$ planes of the pencil of $\mathscr{H} \cong PG(3,q)$ having axis L.

Notes

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