On n-dimensional Maxwell and Dirac Equations in curved space-time and its applications in SO(p,q) group theoretic image processing

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Abstract

Maxwell equations in p time and q spatial dimensions are formulated. Properties of the Green's function for the associated (p,q)-wave operator are derived. The notions of electric and magnetic fields in (p,q)-dimensions is explored by analogy with four dimensional physics. SO(p,q) invariance properties of the Maxwell equations are deduced and used to formulate SO(p,q)-group theoretic image processing problems in (p,q)-dimensions. Dirac's equation in (p,q)-dimensional space-time is derived using the Clifford algebra of the Dirac gamma matrices. SO(p,q)-invariance of the Dirac equation based on the spinor representation of $SO(p,q)$ is mentioned. The Maxwell's equations in (p,q)-dimensional curved space-time is analyzed using perturbation theory. The Einstein-Maxwell equations for (p,q)-dimensional gravity and electromagnetism is studied and used to derive the equations of motion of point charges carrying mass moving under mutual gravitational and electromagnetic interactions in general relativity. Finally, Dirac's equation in (p,q)-dimensional curved space-time interacting with the (p,q)-dimensional electromagnetic field is looked at. $U(1)$-Gauge, local SO(p,q) Lorentz and diffeomorphism invariance of this equation is analyzed. Local $SO(p,q)$ invariance of the curved space-time Dirac equation is deduced based on transformation properties of the Dirac matrices under the spinor representation.

Keywords: Maxwell equations, $SO(p,q)$ group, Riemannian metric, flat space-time, curved space-time, Clifford algebra in n-dimensions, Dirac equation, Dirac Gamma matrices, Representation of a group, spin group, spin representation, Einstein field equations, Green's function in n-dimensional space, group representations in image field processing.

Introduction

We first formulate the n-dimensional Maxwell equations on a flat space-time manifold having q spatial degrees and p temporal degrees of freedom with $p+q=n$. We explain how to solve these equations by the method of Green's functions and then proceed to generalize these to the situation of curved space-time background with any given Riemannian metric. We derive these Maxwell equations from an action principle and also explain how to determine the motion of charged particles in such an electromagnetic field taking background curvature into account. We adopt the differential geometric approach based on the calculus of differential forms. We also explain the generalized Lorentz gauge invariance of the n-dimensional Maxwell theory and then explain how the n-dimensional Maxwell theory couples to the n-dimensional Dirac equation theory for relativistic quantum mechanics using a Clifford algebra. Finally, we explain the coupling of n-dimensional gravity with n-dimensional Maxwell theory via the Einstein field equations and explain how measurements of the n-dimensional Maxwell field at a discrete set of spatial points can be used in the detection of n-dimensional gravitational waves both in a flat and in a curved background metric of space-time. We discuss an important example of n-dimensional maximally symmetric spaces that are important in cosmology and explain how to formulate the Einstein-Maxwell field equations in such a curved space-time in the presence of a finite set of point charges that generate the current that produces the electromagnetic field and that also execute motion in the electromagnetic field generated by them. Some group theoretic aspects of the n-dimensional Maxwell equations are also discussed especially invariance of the Maxwell equations under the group $SO(p,q)$ and how to solve group theoretic image processing problems like estimating the $SO(p,q)$ generalized Lorentz transformation from measurements of the original electromagnetic field and a noisy version of the transformed electromagnetic field as well as constructing pattern invariants for n-dimensional electromagnetic field images that transform according to the general Lorentz group $SO(p,q)$. There is some discussion on the Green's function in (p,q)-dimensional space-time and how it can be applied to the special case of calculating the energy radiated out by a charges in motion and currents in $(1,n-1)$ space-time. We also explain how the (p,q)-dimensional Green's function can be used to calculate approximately the Maxwell field in (p,q) dimensions when the space-time manifold is given by a curved metric that is a weak perturbation of flat space-time. Finally, we present some discussion on the Clifford algebra generated by Dirac matrices in (p,q)-space-time with applications to analysis of the Maxwell-Dirac equations in (p,q)-dimensions. The spin group as the covering group of $SO(p,q)$ as well as spin representations of $SO(p,q)$ constructed using the Dirac matrices are presented and we explain how the spin representation can be used to deduce the (global) $SO(p,q)$-invariance of the (p,q)-Dirac equation in an external electromagnetic field. We explain how the (p,q)-space time Dirac matrices can be constructed using the creation and annihilation operators in Fermion Fock space and using commutators of these Dirac matrices, how the spin representation of SO(p,q) can be constructed. This discussion of the spin representation of $SO(p,q)$ is based on Lie algebraic methods by showing explicitly that the commutators of the Dirac matrices satisfy the same Lie algebra commutation relations as the standard Lie algebra generators of $SO(p,q)$. Finally, we derive the formula for the (p,q)- spin connection and explain how it can be used to construct the (p,q)-Dirac equation in curved space-time in such a way that this equation has local $SO(p,q)$-invariance just as the conventional curved space-time Dirac equation in (1,3) dimensions has local Lorentz invariance. Some discussion of the Einstein-Maxwell equations in the presence of point charges in (n+1)-dimensional space-time in motion has also been presented. This analysis tells us how in $n+1$-dimensional general relativity, point charges move under their mutual gravitational and electromagnetic interactions and also how to calculate approximately the corrections introduced by gravitational effects on the electromagnetic field generated by point particles in motion when the gravitational field is also that produced by the point particles.

Research highlights:

[1] $SO(p,q)$ invariance of higher dimensional Maxwell equations with applications to group theoretic image processing.

[2] Formulas for the Green's function of the $SO(p,q)$-invariant wave equation in (p,q)-space time dimensions.

[3] Formulation of the basic general relativistic equations describing the motion of N point particles carrying mass and charge in the mutual gravitational and electromagnetic fields generated by them in n-dimensions.

[4] Deriving generalizations of the Maxwell curl equations and the electric and magnetic fields in (p,q)-dimensional space-time starting from the n-dimensional potentials and an action principle.

[5] Formulating Dirac's equation in (p,q)-dimensional curved space-time using tetrad basis for the metric and the spinor connection of the gravitational field.

Problem formulation

1. The generalized Lorentz group SO(p,q) and the n-dimensional Maxwell field

The coordinates for our $n = p + q$-dimensional spacetime are $(x^1, x^2, .., x^n)$ where $x^1, ..., x^p$ are the time coordinates and $x^{p+1}, ..., x^n$ are the spatial coordinates. The metric of space-time is flat:

$$d\tau^2 = \sum_{i=1}^{p}(dx^i)^2 - \sum_{i=p+1}^{n}(dx^i)^2 = \eta_{ij}dx^i dx^j$$

where

$$\eta = ((\eta_{ij})) = diag[I_p, -I_q]$$

The n-dimensional electromagnetic potential is A_i or equivalently as a differential one form,

$$A(x) = A_i(x)dx^i$$

The n-dimensional electromagnetic field tensor associated with this potential is

$$F = dA = A_{i,j}dx^j \wedge dx^i = (1/2)(A_{j,i} - A_{i,j})dx^i \wedge dx^j = (1/2)F_{ij}dx^i \wedge dx^j$$

so that in component form,

$$F_{ij} = A_{j,i} - A_{i,j} = \partial_i A_j - \partial_j A_i$$

In the context of the above flat space-time metric, the Lorentz group is $O(p,q)$, namely, the group of all $n \times n$ real matrices g which preserve the metric η, ie,

$$g^T \eta g = \eta$$

or equivalently,

$$G(x,x) = G(gx, gx) \forall x \in \mathbb{R}^n$$

where G is the (p,q)-Lorentz metric:

$$G(x,x) = x^T \eta x = \sum_{i=1}^{p}(x^i)^2 - \sum_{i=p+1}^{n}(dx^i)^2$$

2.The Clifford algebra of Dirac Gamma matrices in n=(p,q)-dimensional space-time and spin representations of SO(p,q)

We introduce matrices $\gamma_1, ...\gamma_n$ or appropriate size so that they generate a Clifford algebra for the symmetric bilinear form G on $\mathbb{R}^n$: $G(x,y) = x^T \eta y$,

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\eta_{ij}I$$

or equivalently, writing

$$\gamma(v) = \sum_{i=1}^{n} \gamma_i v_i,$$

$$\gamma(v)\gamma(u) + \gamma(u)\gamma(v) = 2G(v,u)I = (2v^T\eta u)I, v, u \in \mathbb{R}^n$$

Let T be an element of $O(p,q)$. Then

$$\gamma(Tv)\gamma(Tu) + \gamma(Tu)\gamma(Tv) = 2G(Tv,Tu)I = 2G(v,u)$$

because by definition, $O(p,q)$ consists of all $n \times n$ matrices that preserve the quadratic form $x \to Q(x) = G(x,x) = x^T\eta x$. Now, choose a basis $B = \{e_1, ..., e_n\}$ for $\mathbb{R}^n$ and let $T \in O(p,q)$. Let $[T]_B = ((T_{ij}))$, the matrix of T in the basis B. Then, we have with $[v]_B = ((v_i))$, that

$$\gamma(Tv) = \sum_{i,j=1}^{n} \gamma_i T_{ij} v_j = \sum_j \gamma_j^T v_j$$

where

$$\gamma_j^T = \sum_i T_{ij}\gamma_i$$

Thus,

$$\gamma_i^T\gamma_j^T + \gamma_j^T\gamma_i^T = 2\eta_{ij}I$$

In other words, every $T \in O(p,q)$ induces by duality, a linear transformation on the vector space $V = span\{\gamma_i : i = 1, 2, ..., n\}$ that preserves the anticommutation relations of the Clifford algebra. Now suppose that we are able to find for $T \in O(p,q)$, a matrix $S(T)$ of the same size as the $\gamma_i's$ such that

$$S(T)\gamma_j S(T)^{-1} = \gamma_j^T = \sum_i T_{ij}\gamma_i$$

Note that this equation is consistent because

$$S(T)\gamma_i S(T)^{-1}.S(T)\gamma_j S(T)^{-1} + S(T)\gamma_j S(T)^{-1}.S(T)\gamma_i S(T)^{-1}$$

$$= S(T)(\gamma_i\gamma_j + \gamma_j\gamma_i)S(T)^{-1} = S(T)(2\eta_{ij}S(T)^{-1} = 2\eta_{ij}I$$

It follows immediately from the definition of $S(T)$

$$S(T_2T_1)\gamma_j S(T_2T_1)^{-1} = S(T_2)S(T_1)\gamma_j(S(T_2)S(T_1))^{-1}$$

for $T_1, T_2 \in O(p,q)$. Thus, it is reasonable to expect that $T \to S(T)$ will be a representation of $O(p,q)$:

$$S(T_2T_1) = S(T_2)S(T_1), T_1, T_2 \in O(p,q)$$

In particular, we can restrict S to $SO(p,q)$ and the resultant S is then called the spinor representation of the Lorentz group $SO(p,q)$. This representation acts in the same vector space as that on which which $\gamma_i's$ act. Such a representation can be constructed easily by first defining the Gamma matrices γ_i as $a_i + a_i^*$ and $-i(a_i - a_i^*)$ where the a_i and a_i^* are annihilation and creation operators acting in the Fermion Fock space $\wedge\mathbb{R}^n$ corresponding to the vectors e_i and then constructing $S(T)$ as $exp(\sum_{i,j}\theta(i,j)[\gamma_i,\gamma_j]/4)$ where $T = exp(\sum_{i,j}\theta(i,j)\epsilon(i,j))$ with $\epsilon(i,j)(k,m) = \eta_{ik}\eta_{jm} - \eta_{im}\eta_{jk}$. In other words, if $theta$ is an element of the Lie algebra $\mathfrak{so}(p,q)$, then $dS(\theta) = \sum_{i,j}\theta(i,j)[\gamma_i,\gamma_j]/4$ is the corresponding element of the Lie algebra of $S(SO(p,q))$. $S(SO(p,q))$ is called the spin group of $SO(p,q)$ and its action on the Fermionic Fock space or equivalently on the space on which the Gamma matrices act defines the spin representation of the Lorentz group $SO(p,q)$.

3.(p,q)-dimensional Maxwell equations and their SO(p,q)-invariance with applications to (p,q)-dimensional image processing for electromagnetic fields

Returning back to the n-dimensional Maxwell equations, we construct the action functional as

$$S[A] = C\int F_{ij}F^{ij}d^nx, F^{ij} = \eta_{ik}\eta_{jm}F_{km}$$

This action is $SO(p,q)$ invariant and consequently, the corresponding field equations will also be $SO(p,q)$ invariant. The action principle $\delta_A S = 0$ give us the field equations

$$\partial_j F^{ij} = 0$$

To verify $SO(p,q)$ invariance, let $T \in SO(p,q)$. Then under T, F^{ij} will transform to

$$(TF)^{ij} = T_k^i T_m^j F^{km}$$

and ∂_j will transform to

$$\partial_j^T = T_j^i \partial_i$$

Then

$$\partial_j^T (TF)^{ij} = T_j^r T_k^i T_m^j \partial_r F^{km} = T_k^i \delta_m^r \partial_r F^{km} =$$

$$T_k^i \partial_r F^{kr} = 0$$

Remark: Indices are raised and lowered using the metric η_{ij}. If $((T_{ij})) = T \in SO(p,q)$, then we have the equation

$$T^T \eta T = \eta$$

which can be expressed as

$$\eta_{rs}T_{ri}T_{sj} = \eta_{ij}$$

or equivalently, using the raising and lowering of indices,

$$T_i^s T_{sj} = \eta_{ij}$$

or

$$T_i^s T_s^j = \delta_i^j$$

with the Einstein summation convention over the repeated indices being implied. W e can choose alter the one form A to $A' = A + d\Lambda$ where Λ is an arbitrary scalar field, without affecting the field $F = dA = dA' = F'$ since $d^2 = 0$. We choose in particular, Λ so that $\Box\Lambda = -divA$, ie

$$\Box\Lambda = \partial^i\partial_i\Lambda = -\partial_i A^i$$

This gauge condition ensures that $divA' = 0$ and hence the field equation

$$\partial_j F^{'ij} = 0$$

gives

$$\Box A' = 0$$

Henceforth, we remove the prime from A thereby implicitly assuming that A satisfies the gauge condition $divA = 0$ and hence the $SO(p,q)$ wave equation

$$\Box A = \partial^i\partial_i A = 0$$

or equivalently,

$$(\sum_{i=1}^{p} \partial_i^2 - \sum_{i=p+1}^{n} \partial_i^2) A_k(x) = 0$$

For $T \in SO(p,q)$, we write $T = ((T_j^i))$ rather than $((T_{ij}))$. Thus, the $SO(p,q)$ property is expressed rather as

$$T_i^j \eta_{jk} T_m^k = \eta_{im}$$

Under a (generalized) Lorentz transformation $T \in SO(p,q)$, the coordinate system changes to

$$y^i = T_j^i x^j$$

so that

$$\eta_{ij} y^i y^j = \eta_{ij} T_m^i T_k^j x^m x^k = \eta_{km} x^m x^k$$

as it should be. It should be noted that under such a transformation, the potential $A^i(x)$ changes to

$$B^i(y) = T_j^i A^j(x) = T_j^i A^j(T^{-1}y)$$

or in matrix notation,

$$B(y) = TA(T^{-1}y)$$

or

$$B = TAT^{-1}, T \in SO(p,q)$$

We write

$$B = \pi(T)A$$

as a transformation from one n-vector field A on $\mathbb{R}^n$ to another vector field B on $\mathbb{R}^n$. It is clear then that

$$\pi(T_1T_2) = \pi(T_1)\pi(T_2), T_1, T_2 \in SO(p,q)$$

so that π is a representation of $SO(p,q)$ in the space of n-vector fields. It is usual to assume some sort of integrability condition on the vector fields. For example, in conventional electromagnetic field theory the energy of the field may be taken a finite so that the partial derivatives of the vector potential are square integrable. If we decompose the representation π into the irreducible representations of $SO(p,q)$, then since this is a real group, its irreducible components will have principal, supplementary and discrete series. It should be noted that if $A(x)$ satisfies the $SO(p,q)$ wave equation as well as the Lorentz gauge condition, ie,

$$\partial^i \partial_i A^k(x) = 0, \partial^k A_k(x) = 0$$

then $B(y) = \pi(T)A(y)$ will also satisfy these two equations. To see this, we observe that for $y = Tx$ or $y^i = T^i_j x^j$,

$$\partial/\partial x^j = (\partial y^i/\partial x^j)(\partial/\partial y^i) = T^i_j . \partial/\partial y^i$$

so that

$$\Box_x = \eta_{jk}(\partial/\partial x^j).(\partial/\partial x^k) =$$

$$\eta_{jk} T^i_j T^m_k (\partial/\partial y^i).(\partial/\partial y^m) =$$

$$\eta_{im}(\partial/\partial y^i).(\partial/\partial y^m) = \Box_y$$

Remark: $T \in SO(p,q)$ implies $T^T \eta T = \eta$ implies on taking inverse and noting that $\eta^{-1} = \eta$ that $T^{-1}\eta T^{-T} = \eta$ which implies $T\eta T^T = \eta$ which implies that $T^T \in SO(p,q)$. We have thus shown that the generalized wave operator in $\mathbb{R}^n = \mathbb{R}^{p,q}$ is invariant under generalized Lorentz transformations. It is also easy to see that the gauge condition is also invariant:

$$\partial B^j(y)/\partial y^j = (\partial x^i/\partial y^j)\partial T^j_k A^k(x)/\partial x^i$$

$$= (T^{-1})^i_j T^j_k \partial A^k(x)/\partial x^i = (T^{-1}T)^i_k \partial A^k(x)/\partial x^i =$$

$$\delta^i_k \partial A^k(x)/\partial x^i = \partial A^k(x)/\partial x^k$$

In particular, $divA = 0$ implies $divB = 0$, ie, $div(TAT^{-1}) = 0$. When we make a change of the frame by a generalized Lorentz transformation, the transformed field will also therefore satisfy the Maxwell equations and the resulting vector potential transforms to TAT^{-1} while the resulting field tensor F transforms to $(T \otimes T)F.T^{-1}$. If we take as our representation space, the set of antisymmetric field tensors F with a square integrability condition, then we get a unitary representation $T \to U(T)$ of the locally compact non-Abelian group $SO(p,q)$:

$$(U(T)F)(x) = (T \otimes T)F(T^{-1}x)$$

It is usual to denote the restriction of $T \otimes T$ to $\Lambda^2(\mathbb{R}^{p,q})$ by $T \wedge T$ and regarding an antisymmetric field F as an element of the vector bundle $\Lambda^2 T^*(\mathbb{R}^{p,q})$ namely as a differential form of degree two over $M = \mathbb{R}^{p,q}$ endowed with the flat Riemannian metric η. Specifically, $F(x) = F_{ij}(x)dx^i \wedge dx^j$. Then U becomes a reprsentation of $SO(p,q)$ in the vector bundle $\Lambda^2 T^*(\mathbb{R}^{p,q})$ and we can study the irreducible representations of U. Note that we can write

$$U(T)F = (T \wedge T)FoT^{-1}$$

We can using this representation theoretic formalism of generalized Lorentz transformations acting on an n-dimensional electromagnetic field answer questions such as estimate the transformation $T \in SO(p,q)$ from noisy measurements of the original field F and the transformed field $H = U(T)F + W$ where W is noise, or construct invariants $I(F)$ for $SO(p,q)$ ie, $I(U(T)F) = I(F)$ for all $T \in SO(p,q)$ and all fields F. The latter will give us for example given two fields F_1, F_2 and their transformed versions $F_1' = U(T_1)F_1$, $F_2' = U(T_2)F_2$ for some $T_1, T_2 \in SO(p,q)$, the information that F_1' came by transforming F_1 and not from F_2 and likewise F_2' came by transforming F_2 and not from F_1. In other words, we can solve the feature extraction problem or equivalently the pattern classification problem for n-dimensional electromagnetic fields in $\mathbb{R}^{p,q}$.

4.Analogy of (p,q)-dimensional Maxwell equations with the four dimensional case

In analogy with the four dimensional case, we define the electric field components by

$$E_{i,j-p} = A_{j,i} - A_{i,j}, 1 \le i \le p, p+1 \le j \le n$$

and the magnetic field components by

$$B_{i-p,j-p} = A_{j,i} - A_{i,j}, p+1 \le i, j \le n$$

Recall that there are p time coordinates $x^j, j = 1, 2, ..., p$ and q spatial coordinates $x^j, j = p+1, ..., n$. For $p > 1$, we have additional fields, namely

$$H_{ij}, 1 \le i, j \le p$$

which have no analogy in the conventional four dimensional or more precisely $\mathbb{R}^{1,3}$-dimensional space-time. The definitions imply the following "Homogeneous Maxwell equations" or the Maxwell curl equations

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0, i, j, k = 1, 2, ..., n$$

as can be verified by substituting .$F_{ij} = A_{j,i} - A_{i,j}$. These imply

$$\partial_k E_{i,j-p} - \partial_j E_{i,k-p} + \partial_i B_{j-p,k-p} = 0, i = 1, 2, ..., p, j, k = p+1, ..., n$$

or equivalently,

$$\partial_{p+k}E_{ij} - \partial_{p+j}E_{ik} + \partial_i B_{jk} = 0, j, k = 1, 2, ..., q, i = 1, 2, ..., p$$

This is the n-dimensional generalization of the four dimensional homogeneous Maxwell equation

$$curlE + \partial_t B = 0$$

Note that for each temporal index $i = 1, 2, ..., p$, we have an "electric field vector" $(E_{i,j} : j = 1, 2, ..., q)$ with q components and we have a "magnetic field tensor" $(B_{ij} : i, j = 1, 2, ..., q)$ with $q(q-1)/2$ independent components. This magnetic field tensor cannot be replace in general by a magnetic field vector since $q(q-1)/2$ equals q only when $q = 3$. The n-dimensional generalization of the four dimensional Maxwell equation

$$divB = 0$$

is given by

$$B_{ij,k} + B_{jk,i} + B_{ki,j} = 0, i, j, k = 1, 2, ..., q$$

However in the n-dimensional situation, we have another set of homogeneous Maxwell equations when $p > 1$ which has no analogy in four dimensional physics, namely

$$H_{ij,k} + H_{jk,i} + H_{ki,j} = 0, i, j, k = 1, 2, ..., p$$

and

$$\partial_k H_{ij} + \partial_i E_{jk-p} - \partial_j E_{ik-p} = 0, i, j = 1, 2, ..., p, k = p + 1, ..., n$$

or equivalently,

$$\partial_{p+k}H_{ij} + \partial_i E_{jk} - \partial_j E_{ik} = 0, i, j = 1, 2, ..., p, k = 1, 2, ..., q$$

This last equation is non-void only when $p > 1$, ie, when there is more than one time coordinate.

5. Maxwell's equations in an n-dimensional maximally symmetric space in the presence of moving charges

Consider an $n - 1$ dimensional surface in n dimensional Euclidean space defined by the equation

$$C_{ij}x^i x^j = 1$$

or equivalently,

$$x^T Cx = 1$$

where $C = ((C_{ij}))$ is a positive definite matrix. If we diagonalize C, then this equation becomes the surface equation of an $n - 1$ dimensional ellipsoid immersed in n-dimensional Euclidean space. If further, after diagonalizing, ie, rotating the frame, we also scale coordinates, then this becomes the surface fo an $n - 1$ dimensional sphere in $\mathbb{R}^n$. More generally, if C is a non-singular

Hermitian matrix with p positive eigenvalues and q negative eigenvalues, then diagonalizing C using a rotation of space time followed by an appropriate scaling of the coordinates, this surface assumes the form

$$\sum_{i=}^{p}(x^i)^2 - \sum_{i=p+1}^{q}(x^i)^2 = 1$$

or equivalently,

$$x^T \eta x = 1, \eta = diag[I_p, -I_q]$$

Since the metric $x^T \eta x$ is invariant under $SO(p,q)$, it follows immediately that this this surface is also invariant under $SO(p,q)$. We now introduce an additional coordinate z and define the following n-dimensional surface M immersed in $\mathbb{R}^{n+1}$:

$$x^T Cx + Kz^2 = 1$$

which is an abbreviation for

$$C_{ij}x^i x^j + Kz^2 = 1$$

We wish to determine the linear transformations T of $(x,z) \in \mathbb{R}^{n+1}$ under which this surface M remains invariant. Any linear transformation on $\mathbb{R}^{n+1}$ can be expressed as

$$(x', z') = T(x,z) = (Rx + bz, r^T x + cz)$$

where

$$R \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n, r \in \mathbb{R}^n, x \in \mathbb{R}$$

The condition for M to be invariant under T is that

$$x^{'T}Cx' + K(z')^2 = 1$$

ie

$$(Rx + bz)^T C(Rx + bz) + K(r^T x + cz)^2 = 1$$

whenever

$$x^T Cx + Kz^2 = 1$$

Comparing coefficients, this gives the following necessary and sufficient conditions:

$$R^T CR = C, R^T Cb + Kcr = 0, b^T Cb + Kc^2 = K$$

The number of independent equations here for R, b, r, c is clearly $n(n+1)/2 + n + 1 = (n+1)(n+2)/2$ and hence the number of degrees of freedom in T is $(n+1)^2 - (n+1)(n+2)/2 = n(n+1)/2$. The metric induced on M from the Euclidean metric $ds^2 = dx^T dx + dz^2$ on $\mathbb{R}^{n+1}$ is also clearly invariant under T. Such metric on M therefore is invariant under a maximum number of $n(n+1)/2$ independent transformations on M or equivalently under a set of $n(n+1)/2$ linearly independent vector fields. Such vector fields that leave the metric invariant are called Killing vector fields. This means that the Riemannian manifold M having the metric

$$ds^2 = dx^T dx + dz^2 = dx^T dx + (d(K^{-1/2}(1 - x^T Cx)^{1/2})^2$$

induced by the Euclidean metric on $\mathbb{R}^{n+1}$ is a maximally symmetric space.

Remark: More generally, if a p-dimensional surface M is defined by the equations

$$z = f(x), z \in \mathbb{R}^{n-p}, x \in \mathbb{R}^p$$

then the metric on M induced by the Euclidean metric on $\mathbb{R}^n$ is given by

$$ds^2 = dz^T dz + dx^T dx = dx^T (I + f'(x)^T f'(x))dx$$

Then, if T is transformation on $\mathbb{R}^n$ that leaves M invariant in the sense that if $(x', z') = T(x, z)$, then $z' = f(x')$ whenever $z = f(x)$, then the induced transformation on M will leave this induced metric invariant since

$$(dz')^T dz' + (dx')^T dx' = (dx')^T (I + f'(x')^T f'(x'))dx'$$

The maximally symmetric space M can be used to define a comoving metric on an $n+1$ dimensional manifold with coordinates (t, x) as:

$$d\tau^2 = dt^2 - dl^2$$

where

$$dl^2 = dz^2 + dx^T dx, z = K^{-1/2}(1 - x^T Cx)^{1/2}$$

by making $C = C(t), K = K(t)$ be functions of coordinate time t. Denoting this metric by $g_{ij}(t, x), i, j = 0, 1, ..., n, x^0 = t$, we can then introduce the comoving energy-momentum tensor

$$T_{ij} = (\rho(t) + p(t))v_i v_j - p(t), v_0 = 1, v_i = 0, i = 1, 2, ..., n$$

and set up the $n+1$-dimensional Einstein field equations

$$R_{ij} = k(T_{ij} - Tg_{ij}/2), i, j = 0, 1, ..., n, k = -8\pi G$$

This will give us the n+1-dimensional cosmological dynamics for a given K that can be solved for $C(t), K(t), \rho(t), p(t)$ in a consistent way given an equation of state $p(t) = f(\rho(t))$. The total number of independent Einstein field equations are one for R_{00} plus $n + n(n-1)/2 = n(n+1)/2$ for $R_{ij}, 1 \leq i \leq j \leq n$ giving in all $n(n+1)/2 + 1$ equations for the symmetric matrix $C(t)$ totally $n(n+1)/2$ variables in number plus $K(t)$ one in number. There is an additional variable $\rho(t)$ to be solved for and that can be determined from the vanishing of the covariant derivative of the energy momentum tensor leading to the matter conservation equation. Indeed, the matter conservation equation

$$T^{ij}_{:j} = 0$$

gives

$$((\rho+p)v^i v^j)_{:j} - p^{,i} = 0$$

or

$$((\rho+p)v^j)_{:j}v^i + (\rho+p)v^j v^i_{:j} - p^{,i} = 0$$

Contracting with v^i gives

$$((\rho+p)v^j)_{:j} - v^j p_{,j} = 0$$

or equivalently since $v^j = 0, j = 1,2,...,n, v^0 = 0$, we get the following equation of continuity:

$$((\rho+p)\sqrt{-g})_{,0} - p_{,0} = 0$$

or

$$(\rho+p)\sqrt{-g} - p = c_0(x)$$

a function of only the n spatial variables x. Now, the space-time metric is

$$d\tau^2 = dt^2 - dx^T dx - d(K(t)^{-1/2}(1 - x^T C(t)x)^{1/2})$$

$g(t,x) = det(g_{ij}(t,x)))$ can be computed in principle from this equation and can be matched to

$$g(t,x) = -[(c_0(x) + p(t))/(\rho(t) + p(t))]^2$$

to get an additional equation relating $C(t), K(t)$ to $\rho(t)$. Recall that $p(t) = f(\rho(t))$ is the assumed equation of state. Further, substituting the above equation of continuity into the momentum equation, namely conservation of matter, we get

$$(\rho(t) + p(t))v^j v^i_{:j} - p^{,i} + v^i v^j p_{,j} = 0$$

Using the comoving condition, this simplifies to

$$(\rho(t) + p(t))\Gamma^i_{00} - p^{,i} + p_{,0}\delta^i_0 = 0$$

This is an additional fluid dynamical equation. We next look at Maxwell's equations in such a maximally symmetric space taking into account an interaction with the current field coming from the motion of N charged particles in curved space-time. First assume that there is a classical current density J^μ. The action functional for the (n+1)-dimensional Maxwell field as well as the motion of the particles in the background maximally symmetric space is derived from the action

$$S[A] = \int[(-1/4)F^{ij}F_{ij}\sqrt{-g} - J^i A_i\sqrt{-g}]d^{n+1}x - \sum_{k=1}^N m_k \int d\tau_k$$

where

$$d\tau_k = (g_{ij}(x_k)dx^i_k dx^j_k)^{1/2}$$

We are assuming that the current J^i is produced by the motion of N charged particles with charges $e_1, ..., e_N$:

$$J^i(t,x) = \sum_{k=1}^{N} e_k v_k^i(t)\delta^n(x - x_k(t))(-g(t,x))^{-1/2} - - - (1)$$

where

$$v_k^i(t) = dx_k^i(t)/d\tau_k = \frac{dx_k^i(t)}{dt}(\frac{d\tau_k}{dt})^{-1}, i = 0, 1, ..., n,$$

and x_k is the $n+1$ dimensional position vector of the k^{th} charged particle. Note that

$$d\tau_k/dt = (g_{00}(x_k) + 2\sum_{m\geq 1} g_{0m}(x_k)dx_k^m/dt + \sum_{s,m\geq 1} g_{sm}(x_k)(dx_k^s/dt)(dx_k^m/dt))^{1/2}$$

Note that $u_k^0 = 1$ since $x_k^0 = t$. Carrying out the variation w.r.t the x_k^i, we get the n-dimensional geodesic equations

$$\frac{d^2x_k^i}{d\tau_k^2} + \Gamma^i_{rs}(x_k)\frac{dx_k^r}{d\tau_k}.\frac{dx_k^s}{d\tau_k}$$

$$= e_k F^{ij}(x_k)\frac{dx_k^j}{d\tau_k} - - - (2)$$

and on carrying out the variation w.r.t the fields A_i we get the Maxwell equations with the discrete current source:

$$(F^{ij}\sqrt{-g})_{,j} = -\sqrt{-g}J^i - -(3)$$

The equations (1),(2),(3) must be jointly solved to obtain the particle trajectories $x_k^i(t), i = 1, 2, ..., d, k = 1, 2, ..., N$ and the electromagnetic fields $F_{ij}(x)$.

6.Green's functions for the wave operator in (p,q)-dimensions

We now take a look at the generalization of the retarded Green's function for the wave operator in four dimensional space time, ie, in $\mathbb{R}^{1,3}$ to the n-dimensional case, ie, in $\mathbb{R}^{p,q}$. In order to motivate this discussion, we first observe that in flat (p,q) dimensional space-time, the Maxwell action interacting with a n-dimensional current density J^i is given by

$$(1/4)\int F^{ij}F_{ij}d^nx - \int J^iA_id^nx, F_{ij} = A_{j,i} - A_{i,j}$$

and carrying the variation w.r.t the A_i gives

$$F^{ij}_{,j} = J^i$$

Taking into account the generalized (p,q)-Lorentz gauge condition

$$A^i_{,i} = 0$$

gives us the (p,q)-dimensional wave equation with source:

$$\Box A^i(x) = \partial^j \partial_j A^i(x) = \eta_{kj} \partial_k \partial_j A^i = -J^i(x)$$

To solve this, we for an arbitrary source, we must first solve for the Green's function in (p,q)-dimensional space-time:

$$\Box G(x) = \delta^n(x)$$

On taking (p,q)-dimensional Fourier transforms, this gives

$$G(x) = -(2\pi)^{-n} \int exp(ik.x) d^n k / k^2 - - - (3)$$

where

$$k.x = k_i x^i = \eta_{ij} k^i x^j = \sum_{i=1}^{p} k^i x^i - \sum_{i=p+1}^{n} k^i x^i$$

and

$$k^2 = \eta_{ij} k^i k^j = k^i k_i = \sum_{i=1}^{p} (k^i)^2 - \sum_{i=p+1}^{n} (k^i)^2$$

Evaluating the n-dimensional integral in (3) is not easy. Let $d\Omega_p(\hat{k}(1:p))$ denote the solid angle measure in p-dimensional Euclidean space and likewise $d\Omega_q(\hat{k}(p+1:n))$. We write $cos(\theta(\hat{k}(1:p), \hat{x}(1:p)))$ for the cosine of the angle between the p-unit vectors $\hat{k}(1:p)$, and $\hat{x}(1:p)$ and likewise for q. Then, we can write

$$G(x) = -(2\pi)^{-n} \int [exp(i|k(1:p)||x(1:p)|cos(\theta(\hat{k}(1:p), \hat{x}(1:p)))$$

$$-i|k(p+1:n)||x(p+1:n)|cos(\theta(\hat{k}(p+1:n), \hat{x}(p+1:n))))].$$

$$\times [|k(1:p)|^2 - |k(p+1:n)|^2]^{-1} |k(1:p)|^p |k(p+1:n)|^q d\Omega_p(\hat{k}(1:p)).d|k(1:p)| d|k(p+1:n)| d\Omega_q(\hat{k}(p+1:n))$$

Writing for instance

$$\int exp(i|k(1:p)||x(1:p)|cos(\theta(\hat{k}(1:p), \hat{x}(1:p)))) d\Omega_p(\hat{k}(1:p))$$

$$= F_p(|k(1:p)||x(1:p)|)$$

(Note that this integral is independent of $\hat{x}(1:p)$), we get

$$G(x) = -(2\pi)^{-n} \int \frac{F_p(|k(1:p)||x(1:p)|) F_q(|k(p+1:n)||x(p+1:n)|)}{|k(1:p)|^2 - |k(p+1:n)|^2} d|k(1:p)|.d|k(p+1:n)|$$

In the special case when $p=1,q=n-1$, ie, the situation of n-dimensional Maxwell equations with one time coordinate x^0 and $n-1$ spatial coordinates $x^1,...,x^{n-1}$, we have

$$(\partial_0^2-\sum_{i=1}^{n-1}\partial_i^2)G(x)=\delta^n(x)$$

which gives on taking the $n-1$ spatial Fourier transform we get

$$(\partial_0^2+k^2)G(x^0,k)=\delta(x^0)$$

Laplace transforming w.r.t x^0 gives us

$$(s^2+k^2)G(s,k)=1$$

so that

$$G(x^0,k)=k^{-1}sin(kx^0)\theta(x^0),k=\sum_{i=1}^{n-1}(k^i)^2$$

Taking the $n-1$-dimensional inverse spatial Fourier transform w.r.t $(k^i:i=1,2,...,n-1)$ then gives with $t=x^0$,

$$G(x)=G(x^0,x^1,...,x^{n-1})=(2\pi)^{-n+1}\int\frac{sinkt)}{k}exp(ikr.cos(\theta))k^{n-2}dk.d\Omega_{n-2}(\theta))$$

where $d\Omega_{n-2}(\theta)$ is the solid angle measure as a function of the elevation angle measured w.r.t the pole direction $(x^i:i=1,2,...,n-1)$ and

$$r=\sqrt{\sum_{i=1}^{n-1}(x^i)^2}$$

Defining

$$I_{n-2}(r)=(2\pi)^{-n+1}\int exp(ir.cos(\theta))d\Omega_{n-2}(\theta)$$

we get

$$G(x)\int_0^\infty k^{n-3}sin(kt)I_{n-2}(kr)dk,t=x^0$$

Expanding

$$I(r)=\sum_{m\geq 0}c(m)r^m$$

gives us

$$G(x)=\sum_{m\geq 0}c(m)r^m\int_0^\infty k^{m+n-3}.sin(kt)dk$$

Observe that for m odd, $k^m.sin(kt)$ is even in k and hence

$$\int_0^\infty k^m.sin(kt)dk=(1/2)\int_{-\infty}^\infty k^m.sin(kt)dk$$

$$= (1/2) Im \int_{\mathbb{R}} k^m exp(ikt) dk$$

$$= (\pi/2) Im((-i)^m \delta^{(m)}(t)) = (-\pi/2) sin(m\pi/2)\delta^{(m)}(t)$$

This finally gives us the following expansion of the Green's function:

$$G(x) = (-\pi/2) \sum_{m \geq 0, m+n odd} sin(m\pi/2) c(m) r^m \delta^{(m)}(t) = G(t,r)$$

7. Radiation by accelerating charges in $(1, n-1)$ dimensional space-time

Letting $G(x)$ denote the Green's function, we can write down the Maxwell equations source after adopting the Lorentz gauge:

$$A_i(x) = \int G(x-x) J_i(x') d^n x', i = 0, 1, ..., n-1$$

The electric field components are

$$E_i(x) = F_{0i} = A_{i,0} - A_{0,i}, i = 1, 2, ..., n-1$$

and the magnetic field components are

$$F_{ij}, 1 \leq i < j \leq n-1$$

Equivalently, we write

$$A_i(t, \mathbf{r}) = \int G(t-t', |r-r'|) J_i(t', r') dt' d^{n-1} r'$$

To proceed further, we require an expression for the Poynting vector in n dimensions. This is obtained from the energy-momentum tensor:

$$T_j^k = (\partial L / \partial A_{i,j}) A_{i,k} - L \delta_j^k$$

which gives after raising the index j and adding a total divergence term to make the energy momentum tensor symmetric,

$$T^{kj} = (1/4) F_{ab} F^{ab} \eta_{kj} - F^{ak} F_a^j$$

The energy flux is given by

$$T^{b0} = -F^{ab} F_a^0 = -F_{ab} F_{a0} = F_{ab} F_{0a}$$

Note that b is a spatial index and the sum runs over all spatial indices a. This formula is the generalization of the four dimensional formula $E \times B$ once we note that F_{0a} is the electric field and F_{ab} is the magnetic field. We must calculate the energy flux T^{b0} in the far field zone retaining only terms of order $1/r^{n-2}$ so that the integral of $T^{b0} n_b$ over the $n-2$ dimensional sphere gives a finite nonzero quantity. This means that $A_i(t,\mathbf{r})$ and hence the electric and magnetic fields $F_{0a}, F_{ab}, 1 \leq a < b \leq n-1$ must be evaluated upto order $1/r^{n/2-1}$. Note that for $n=4$ this reduces to the standard formula for radiation fields, namely, that the electric and magnetic fields in the radiation zone must be evaluated upto $O(1/r)$ so that the Poynting energy flux vector is evaluated upto $O(1/r^2)$. Now recall the Green's function for the wave operator in $(1,n-1)$ dimensions:

$$G(x)\int_0^\infty k^{n-3}sin(kt)I_{n-2}(kr)dk, t=x^0$$

where

$$I_{n-2}(u) = (2\pi)^{-n+1}\int exp(iu.cos(\theta))d\Omega_{n-2}(\theta)$$

We can write on changing the integration variable from k to $u=kr$,

$$G(x) = r^{2-n}\int_0^\infty u^{n-3}I_{n-2}(u)sin(ut/r)du$$

As a check, taking $n=4$ gives

$$I_2(u) = (2\pi)^{-3}\int_0^\pi exp(iu.cos(\theta))2\pi.sin(\theta)d\theta$$

$$= (2\pi)^{-2}.2sin(u)/u$$

giving

$$G(x) = (2/\pi^2)r^{-2}\int_0^\infty sin(u)sin(ut/r)du$$

the causal part of which is clearly a constant times $\delta(t/r-1)/r^2 = \delta(t-r)/r$ as expected. In the general case of $(1,n-1)$ dimensions, we define the one sided Fourier transform of $I_{n-2}(u)$:

$$\int_0^\infty I_{n-2}(u)exp(iuv)du = J_{n-2}(v)$$

and then get assuming n to be even, that

$$\int_0^\infty u^{n-3}I_{n-2}(u)sin(ut)du = Im(\int_0^\infty u^{n-3}I_{n-2}(u)exp(iut)du)$$

$$= \partial_t^{n-3}Im((-i)^{n-3}J_{n-2}(t))$$

$$= -sin((n-3)\pi/2)\partial_t^{n-3}Re(J_{n-2}(t)) = sin((n-1)\pi/2)\partial_t^{n-3}Re(J_{n-2}(t))$$

Thus, for even n, the Green's function for the $(1,n-1)$ manifold is given by

$$G(x) = G(t,r) = C(n)r^{2-n}.J_{n-2}^{(n-3)}(t/r)$$

where

$$C(n) = sin((n-1)\pi/2)$$

Note that

$$C(4m) = -1, C(4m+2) = 1$$

The vector potential is then

$$A_i(t, \mathbf{r}) = \int G(t-t', |\mathbf{r}-\mathbf{r}'|)J_i(t', \mathbf{r}')dt'd^{n-1}\mathbf{r}'$$

and in the far field zone $r >> r'$, we get approximately

$$A_i(t, \mathbf{r}) = \int G(t-t', r-\hat{r}.\mathbf{r}'))J_i(t', \mathbf{r}')d^{n-1}r'$$

Note that the Fourier transform in the time domain of the exact potential is given by

$$A_i(\omega, \mathbf{r}) = \int A_i(t, \mathbf{r})exp(-i\omega t)dt$$

$$\int G_1(\omega, |r-r'|)J_i(\omega, r')d^{n-1}r'$$

where

$$G_1(\omega, r) = \int G(t, r)exp(-i\omega t)dt$$

with

$$G(t, r) = G(x) = r^{2-n}\int_0^\infty u^{n-3}I_{n-2}(u)sin(ut/r)du$$

Clearly,

$$G_1(\omega, r) = (i)^{-1}\pi r^{3-n}\int_0^\infty u^{n-3}I_{n-2}(u)(\delta(r\omega-u)-\delta(r\omega+u))du$$

Assuming $\omega > 0$, this evaluates to

$$G_1(\omega, r) = -i\pi.r^{3-n}(r\omega)^{n-3}I_{n-2}(\omega r)$$

$$= -i\pi.\omega^{n-3}I_{n-2}(\omega r)$$

8. The coupled (n+1)-dimensional Einstein Maxwell equations in the presence of N charged particles carrying masses

Assuming that the metric has the form

$$d\tau^2 = (1+2\phi)dt^2 - ((1-2\phi)\delta_{rs} + h_{rs})dx^r dx^s + 2h_{0r}dtdx^r$$

in analogy with small perturbations of the Schwarzchild metric in four dimensional space-time, where h_{rs} is two degrees smaller than ϕ and h_{0r} is one degree smaller than ϕ. Specifically, as per the principles of perturbation theory in general relativity, ϕ is $O(1/c^2)$, h_{rs} is $O(1/c^4)$ and h_{0r} is $O(1/c^3)$. Note that the summation indices r, s run over $1, 2, ..., n$. The energy momentum tensor of the matter field of the N charged particles carrying masses $m_1, ..., m_N$ and charges $e_1, ..., e_N$ is given by

$$T^{ij} = \sum_k m_k \delta^n(x - x_k)(-g(x))^{-1/2}(dx_k^i/d\tau_k)(dx_k^j/dt), i, j = 0, 1, ..., n$$

Note that $x^0 = t$. The energy-momentum tensor of the electromagnetic field generated by the charged particles in motion is given by

$$S^{kj} = (1/4)F_{ab}F^{ab}\eta_{kj} - F^{ak}F_a^j$$

The action functional of the gravitational field plus the electromagnetic field plus the motion of the particles plus the interaction between the charged particles and the electromagnetic field is given by

$$S = C_1 \int R\sqrt{-g}d^{n+1}x - (1/4)\int F^{ij}F_{ij}\sqrt{-g}d^{n+1}x - \int J^i A_i \sqrt{-g}d^{n+1}x - \sum_k \int m_k d\tau_k$$

where J^i is the n-current produced by the charged particles and is given by

$$J^i(x) = \sum_k e_k \delta^n(x - x_k)(-g(x))^{-1/2}dx_k^i/dt$$

The curvature scalar R is calculated from the above metric and from the above action, we derive the Einstein-Maxwell equations as well as the geodesic equations for the particles in the electromagnetic field generated by the charged particles themselves (note that the geodesic part of the dynamical equations of motion of the particles correspond to motion under the gravitational field produced by the particles themselves, thus motion of the charged particles is under the mutually generated gravitational and electromagnetic fields, or putting it in another way, the motion of each particle takes place in the gravitational and electromagnetic fields produced by the other particles. Variation of the total action w.r.t the metric gives

$$R_{ij} - (1/2)Rg_{ij} = K(T_{ij} + S_{ij}),$$

Variation of the total action w.r.t the electromagnetic n-potential gives

$$(F^{ij}\sqrt{-g})_{,j} = -J^i\sqrt{-g} = -\sum_k e_k \delta^n(x - x_k)dx_k^i/dt$$

Variation of the total action w.r.t the x_k^i gives the equation of motion of the particles:

$$d^2x_k^i/d\tau_k^2 + \Gamma^i_{jm}(x_k)(dx_k^j/d\tau_k)(dx_k^m/d\tau_k) = e_k g_{jm}F^{ij}(x_k)dx_k^m/d\tau_k$$

To proceed further, we must solve the Einstein field equations approximately for the metric. First observe that for any particle, with x denoting x_k and v denoting v_k,

$$-d\tau/dt = -(g_{00} + 2g_{0r}v^r + g_{rs}v^r v^s)^{1/2} =$$

$$-(1+2\phi+2h_{0r}v^r-(1-2\phi)v^2-h_{rs}v^rv^s)^{1/2}$$

$$\approx -1-\phi+v^2/2$$

upto $O(1/c^2)$. Note that we take v to be $O(1/c)$ because, actually v occurs in the combination v/c in the metric differential. If we wish to also include $O(1/c^4)$ terms, then we get

$$-d\tau/dt \approx -1-\phi+v^2/2-h_{0r}v^r+\phi^2/2-2\phi v^2$$

The energy-momentum tensor of the matter component can thus be approximated using

$$T^{ij}=\sum_k m_k\delta^n(x-x_k)(dx_k^i/dt)(dx^j/dt)(d\tau_j/dt)^{-1}$$

By expressing the Ricci tensor in terms of the functions ϕ, h_{rs}, h_{0k}, we can thus in principle solve for these functions in terms of $x_k^i(t), v_k^i(t)$ and we can also solve for the electromagnetic potentials A_i in terms of $x_k^i(t), v_k^i(t)$ using perturbation theory. Note that if we assume that the metric is a weak perturbation of the flat space-time $SO(1,n-1)$ metric η, ie,

$$\eta_{ij}dx^idx^j=dt^2-\sum_{k=1}^{n-1}(dx^k)^2, t=x^0$$

then we can write upto linear orders in the metric perturbations,

$$g_{ij}=\eta_{ij}+h_{ij}, \sqrt{-g}=1-h/2, h=h_i^i=\eta_{ij}h_{ij}, g^{ij}=\eta_{ij}-h^{ij}, h^{ij}=\eta_{ik}\eta_{jm}h_{km}$$

where

$$h_{00}=2\phi, g_{rs}=-2\phi.\delta_{rs}-h_{rs}, 1\leq r,s\leq n, g_{0r}=h_{0r}, 1\leq r\leq n$$

Then upto linear orders in the metric perturbations,

$$F^{ij}\sqrt{-g}=g^{ia}g^{jb}\sqrt{-g}F_{ab}=(\eta_{ia}-h_{ia})(\eta_{jb}-h_{jb})(1+h/2)F_{ab}$$

$$=\eta_{ia}\eta_{jb}F_{ab}+f_{ijab}F_{ab}$$

where f_{ijab} is a linear function of the metric perturbations:

$$f_{ijab}=-\eta_{ia}h_{jb}-\eta_{jb}h_{ia}+\eta_{ia}\eta_{jb}h/2$$

The n+1-dimensional Maxwell equations in curved space-time in the presence of point charges can then be expressed upto linear terms in the metric perturbations as

$$\eta_{ia}\eta_{jb}F_{ab,j}+(f_{ijab}F_{ab})_{,j}=-\sum_k e_k\delta^n(x-x_k(t))dx_k^i(t)/dt---(a)$$

To proceed further, we assume the the gauge condition (ie, the $n+1$-dimensional Lorentz gauge condition in curved space-time)

$$(A^i\sqrt{-g})_{,i} = 0$$

which upto linear terms in the metric perturbation reads

$$((\eta_{ij} - h^{ij})(1 + h/2)A_j)_{,i} = 0$$

or equivalently,

$$\eta_{ij}A_{j,i} = (k_{ij}A_j)_{,i}$$

where k_{ij} is also a linear function of the metric perturbations:

$$k_{ij} = h_{ij} - \eta_{ij}h/2$$

Taking this gauge condition into account, Maxwell's equations (a) become

$$-\eta_{ia}\eta_{jb}A_{a,jb} + \eta_{ia}(k_{jb}A_b)_{,ja} + (f_{ijab}F_{ab})_{,j} = -\sum_k e_k\delta^n(x - x_k(t))dx^i_k(t)/dt$$

or equivalently,

$$\Box A_a = (k_{jb}A_b)_{,ja} + \eta_{ai}(f_{ijcb}F_{cb})_{,j} + \sum_k e_k\delta^n(x - x_k(t))\eta_{ai}dx^i_k(t)/dt, \Box = \eta_{ij}\partial_i\partial_j$$

Assume that A^0_a is a free wave in flat $(1, n)$ space-time. It satisfies the wave equation

$$\Box A^0_a = 0$$

The perturbation to this electromagnetic wave caused by moving charges and gravitational effects is then obtained by applying the Green's function for $(1, n)$ space time to the above equation:

$$\delta A_a(x) = \int G(x - x')[(k_{jb}A^0_b)_{,ja}(x') + \eta_{ai}(f_{ijcb}F^0_{cb})_{,j}(x') + \sum_k e_k\delta^n(x' - x_k(t'))\eta_{ai}dx^i_k(t')/dt']d^{n+1}x'$$

Note that this formula is also applicable to (p, q)-space time provided that we use the formula for the (p, q)-Green's function derived above. This evaluates to

$$\delta A_a(x) = \int G(x - x')[(k_{jb}A^0_b)_{,ja}(x') + \eta_{ai}(f_{ijcb}F^0_{cb})_{,j}(x')d^{n+1}x'$$

$$+\sum_k e_k \int G(t - t', \mathbf{r} - x_k(t')))\eta_{ai}dx^i_k(t')$$

Sometimes, especially when there is no external plane wave, and we are interested in calculating the motion of the charges in the mutual gravitational and electromagnetic field produced by themselves, it is more accurate to consider the unperturbed electromagnetic field to be that produced by the charges in motion:

$$A_a^0 = \sum_k e_k \int G(t-t', \mathbf{r} - x_k(t')))\eta_{ai} dx_k^i(t')$$

The perturbation to this field caused by gravitational effects is then

$$\delta A_a(x) = \int G(x-x')[(k_{jb}A_b^0)_{,ja}(x') + \eta_{ai}(f_{ijcb}F_{cb}^0)_{,j}(x')d^{n+1}x'$$

Appendix A.1

Some further remarks on the (p,q)-wave equation: The Green's function for the (p,q) wave operator with $p+q=n$ is

$$(\sum_{i=1}^p \partial_i^2 - \sum_{i=p+1}^n \partial_i^2)G(x) = \delta^n(x)$$

and by Fourier transforming, we get

$$G(x) = (2\pi)^{-n} \int exp(ik_1.x_1 - k_2.x_2))d^pk_1.d^qk_2/(k_1^2 - k_2^2)$$

where

$$x_1 = (x^1, ..., x^p), x_2 = (x^{p+1}, ..., x^n), k_1.x_1 = \sum_{i=1}^p k^i x^i, k_2.x_2 = \sum_{i=p+1}^n k^i x^i$$

where

$$k_1 = (k^1, ..., k^p), k_2 = (k^{p+1}, ..., k^n), k_1^2 = \sum_{i=1}^p (k^i)^2, k_2^2 = \sum_{i=p+1}^n (k^i)^2$$

This integral can be expressed as

$$G(x) = G(r_1, r_2) = (2\pi)^{-n} \int_{k_1,k_2>0} exp(ik_1r_1cos(\theta_1) - ik_2r_2cos(\theta_2))k_1^{p-1}k_2^{q-1}d\Omega_{p-1}(\theta_1)d\Omega_{q-1}(\theta_2)dk_1dk_2$$

where

$$r_1 = \sqrt{\sum_{i=1}^p (x^i)^2}, r_2 = \sqrt{\sum_{i=p+1}^n (x^i)^2}$$

Defining

$$F_p(u) = (2\pi)^{-p} \int exp(iu.cos(\theta))d\Omega_p(\theta), u \in \mathbb{R}$$

(Note that $F_p(-u) = F_p(u)$), we can write

$$G(x) = G(r_1, r_2) = \int_{k_1,k_2>0} F_p(k_1r_1)F_q(k_2r_2)k_1^{p-1}k_2^{q-1}dk_1dk_2/(k_1^2 - k_2^2)$$

$$= r_1^{-p} r_2^{-q} \int_{u_1,u_2>0} u_1^{p-1} u_2^{q-1} F_p(u_1)F_q(u_2)du_1du_2/((u_1/r_1)^2 - (u_2/r_2)^2)$$

$$= (1/4)r_1^{-p}r_2^{-q} \int \frac{u_1^{p-1}u_2^{q-1}F_p(u_1)F_q(u_2)}{(u_1/r_1)^2 - (u_2/r_2)^2} du_1du_2$$

One of the solutions is obtained by replacing the denominator in the integrand by

$$(u_1/r_1)^2 - (u_2/r_2)^2 - i\epsilon, \epsilon \to 0+$$

so that the integral w.r.t u_1 has a pole in the upper half u_1-plane at $(r_1u_2/r_2)+i\epsilon$ and another in the lower half u_1-plane at $-(r_1u_2/r_2)^2 - i\epsilon$. Closing the contour in the upper half u_1 plane by a large semicircle (this is justified when $r_1 > 0$ (ie the (p,q)-dimensional generalization of the causal condition on the Green's function), we get from the Cauchy residue theorem that

$$G(x) = G(r_1,r_2) = (i\pi/4)r_1^{1-p}r_2^{1-q} \int_{\mathbb{R}} (r_1u_2/r_2)^{p-1}u_2^{q-1}F_p(r_1u_2/r_2)F_q(u_2)du_2/u_2$$

$$= (i\pi/4)r_2^{2-p-q} \int_{\mathbb{R}} u_2^{p+q-2}F_p(r_1u_2/r_2)F_q(u_2)du_2/u_2$$

For example, in the standard $(1,3)$-dimensional case, we have $p = 1, q = 3$ and $F_1(u)$ is proportional to $exp(iu)$ while $F_3(u)$ is proportional to $sin(u)/u$ so $F_1(r_1u_2/r_2)F_3(u_2)$ is proportional to $exp(-iu_2(1 - r_1/r_2))/u_2$ and hence $u_2^{p+q-2}F_1(r_1u_2/r_2)F_3(u_2)/u_2$ is proportional to $exp(-iu_2(1 - r_1/r_2))$ which integrates w.r.t u_2 to give $\delta(1-r_1/r_2) = r_2\delta(r_1-r_2)$ and hence $G(r_1,r_2)$ becomes proportional to $r_2^{-2}r_2\delta(r_1-r_2) = \delta(r_1-r_2)/r_2$ and recalling that $r_1 = t, r_2 = r$, this becomes the standard four dimensional retarded Green's function formula $\delta(t-r)/r$ of the four dimensional theory.

Appendix A.2

1.**The Maxwell-Dirac equations in** (p,q)**-dimensional flat space-time**

Let η denote the (p,q)-metric: $\eta = diag[I_p, I_q]$. Consider Dirac matrices $\gamma_k, k = 1,2,...,n$ so that

$$\gamma_k\gamma_m + \gamma_m\gamma_k = 2\eta_{km}I$$

Dirac's wave equation is

$$(i\gamma_k\partial_k - m)\psi(x) = 0, x \in \mathbb{R}^n$$

where summation over the repeated index k is implies. This implies

$$(i\gamma_k\partial_k + m)(i\gamma_m\partial_m - m)\psi(x) = 0$$

or equivalently the (p,q)-dimensional Klein-Gordon equation

$$(\eta_{km}\partial_k\partial_m + m^2)\psi(x) = 0$$

ie

$$(\Box + m^2)\psi = 0, \Box = \sum_{k=1}^{p} \partial_k^2 - \sum_{k=p+1}^{n} \partial_k^2$$

This equation corresponds to the (p,q)-dimensional relativistic energy-momentum relation:

$$\sum_{k=1}^{p} P_k^2 = \sum_{k=p+1}^{n} P_k^2 + m^2$$

Consider now the Lie algebra $\mathfrak{so}(p,q)$ of $SO(p,q)$. Any element X in this Lie algebra satisfies

$$X^T \eta + \eta X = 0$$

or equivalently, without implying summation,

$$X_{ji}\eta_{jj} + \eta_{ii}X_{ij} = 0$$

This means that

$$X_{ji} + X_{ij} = 0, 1 \leq i,j \leq p, p+1 \leq i,j \leq n,$$

$$X_{ji} - X_{ij} = 0, 1 \leq i \leq p, p+1 \leq j \leq n,$$

$$X_{ji} - X_{ij} = 0, p+1 \leq i \leq n, 1 \leq j \leq p$$

A basis for $\mathfrak{so}(p,q)$ is therefore given by the set of $n \times n$ matrices $E_{ij} - E_{ji}.1 \leq i,j \leq p, p+1 \leq i,j \leq n$ and $E_{ij} + E_{ji}, 1 \leq i \leq p, p+1 \leq j \leq n$. Using the commutation relations for these matrices, it is easily seen that $[\gamma_i, \gamma_j]/4, 1 \leq i < j \leq n$ form a basis for a representation of $\mathfrak{so}(p,q)$. In fact, we can express the above basis for $\mathfrak{so}(p,q)$ as the set of all matrices J_{ij} where

$$(J_{ij})^k_m = \delta^k_i \eta_{jm} - \delta^k_j \eta_{im}$$

A quick check that these are indeed matrices in $\mathfrak{so}(p,q)$ follows from the readily verifiable identity

$$J^k_m \eta_{kl} + \eta_{mk} J^k_l = 0,$$

for $J = J_{ij}$. We next verify the commutation relations

$$[J_{ij}, J_{kl}] = \eta_{jk} J_{il} + \eta_{il} J_{jk} - \eta_{ik} J_{jl} - \eta_{jl} J_{ik}$$

and complete the story by using the anticommutation relations of the Dirac matrices to show that $S_{ij} = [\gamma_i, \gamma_j]/4$ satisfy the same commutation relations:

$$[S_{ij}, S_{kl}] = \eta_{jk} S_{il} + \eta_{il} S_{jk} - \eta_{ik} S_{jl} - \eta_{kl} S_{ik}$$

This representation of $\mathfrak{so}(p,q)$ is called the spinor representation. Note that we are assuming $n = p+q$ to be even. It then follows that since $\gamma_{n+1} = \gamma_1 ... \gamma_n$ anticommutes with all the $\gamma_j' s$, that γ_{n+1} commutes with all the $S_{ij}' s$ and in fact upto linear combinations, 1 and γ_{n+1} are the only matrices that commute with all the $S'_{ij}s$. Thus, the centre of the spinor representation has dimension two and therefore the spinor representation decomposes into the direct sum of two irreducible representations.

2. Dirac's equation in (p,q) dimensional curved space-time taking into account interaction with the electromagnetic field

Dirac's equation in curved space time [2,8] is

$$[e_a^\mu \gamma^a (i\partial_\mu + i\Gamma_\mu + eA_\mu) - m]\psi = 0$$

where Γ_μ is the SO(p,q) spinor connection of the gravitational field and e_a^μ is the tetrad of the metric $g^{\mu\nu}$:

$$g^{\mu\nu} = \eta^{ab} e_a^\mu e_a^\nu$$

with $\eta = diag[I_p, -I_q]$ being the flat space-time metric in (p,q)-space time. The spinor connection can be derived from the assumption that the covariant derivative of the tetrad e_μ^a must vanish:

$$0 = e^a_{\mu,nu} - \Gamma^\rho_{\mu\nu} e^a_\rho + \omega^{ab}_\nu e_{b\mu} = 0, \Gamma_\mu = \omega^{ab}_\mu [\gamma_a, \gamma_b]/4, \gamma_a = \eta_{ab}\gamma^b$$

Inverting this equation gives us the spinor connection as

$$\omega^{ab}_\nu = -e^{b\mu}(e^a_{\mu,\nu} - \Gamma^\rho_{\mu\nu} e^a_\rho) = -e^{b\mu} e^a_{\mu:\nu}$$

Note that a,b are generalized Lorentz indices, ie, (p,q)-space-time indices while μ,ν are curved space-time indices. It is easily seen that this Dirac equation is invariant under both local $SO(p,q)$ (generalized Lorentz) transformations as well as under (p,q)-space-time diffeomorphisms. Note that if $\Lambda(x)$ is an element of the group $SO(p,q)$ that is a function of the (p,q)-space time coordinates and if U is the spinor representation of $SO(p,q)$, then

$$U(\Lambda(x))\gamma^a U(\Lambda(x))^{-1} = \Lambda^a_b(x)\gamma^b$$

$\Lambda(x)$ is called a local $SO(p,q)$ transformation or a local generalized (p,q) Lorentz transformation, local because it is a function of the space-time coordinates. Under such a local $SO(p,q)$ transformation, we have that the tetrad undergoes the transformation

$$e^a_\mu(x) \to \Lambda^a_b(x) e^b_\mu(x)$$

and the wave function transforms as

$$\psi(x) \to U(\Lambda(x))\psi(x)$$

From these facts, it easily follows that the spinor connection of the gravitational field transforms under a local SO(p,q) transformation as

$$\Gamma_\mu(x) \to U(\Lambda(x))\Gamma_\mu(x)U(\Lambda(x))^{-1} - (\partial_\mu U(\Lambda(x))).U(\Lambda(x))^{-1}$$

and that the Dirac equation remains invariant. In order to prove the above law of transformation of the spinor connection, we must consider an inifnitesimal $SO(p,q)$ transformation:

$$\Lambda(x) = I + \delta.\theta(x)$$

where $\theta(x) = ((\theta^a_b(x)))$ is an element of the Lie algebra $\mathfrak{so}(p,q)$, ie

$$\theta^a_b(x)\eta_{ac} + \eta_{ba}\theta^a_c(x) = 0$$

which is the same as

$$\theta_{cb}(x) + \theta_{bc}(x) = 0$$

where the local SO(p,q) indices a, b, c are raised and lowered using the flat (p,q)-metric η. δ is an infinitesimal real number. It is instructive to derive the curved space-time Klein-Gordon equation for a charged particle in (p,q)-dimensional space-time from the Dirac equation by expanding

$$[\gamma^\mu(x)(i\partial_\mu + i\Gamma_\mu(x) + eA_\mu(x)) + m].[\gamma^\mu(x)(i\partial_\mu + i\Gamma_\mu(x) + eA_\mu(x)) - m]\psi(x) = 0$$

where

$$\gamma^\mu(x) = e^\mu_a(x)\gamma^a$$

are Local Dirac matrices satisfying the anticommutation relations

$$\gamma^\mu(x)\gamma^\nu(x) + \gamma^\nu(x)\gamma^\mu(x) = 2g^{\mu\nu}(x)I$$

in view of the anticommutation relations satisfied by the flat space-time Dirac matrices

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta_{ab}I$$

and the tetrad property:

$$\eta_{ab}e^\mu_a(x)e^\nu_b(x) = g^{\mu\nu}(x)$$

This Klein-Gordon equation can equivalently be expressed as

$$[[\gamma^\mu(x)(i\partial_\mu + i\Gamma_\mu(x) + eA_\mu(x))]^2 - m^2]\psi(x) = 0$$

and the above squared operator can be expressed in terms of the Riemann curvature tensor of the metric $g_{\mu\nu}(x)$. It is known in the literature as Lichnerowicz formula for the square of the Dirac operator. This formula has been obtained in the more general case when A_μ is a non-Abelian Yang-Mills potential. By assuming the metric to be a weak perturbation of (p,q)-dimensional flat space-time, we can solve for the wave function approximately using perturbation theory with our formula for the (p,q)-dimensional flat spacet-time Green's function. We do not discuss the details of this procedure here as although the computations are tedious, they are straightforward.

Conclusions

In this paper, we have explained how to use properties of the $SO(p,q)$ group to formulate Maxwell's equations in n-dimensions with p time coordinates and q space coordinates by starting with an $n = p + q$ vector potential. We derive the (p,q) dimensional wave equation for the electromagnetic potentials and explain how to calculate the associated Green's function in order to solve the (p,q)-dimensional Maxwell equation in the presence of an n-current density. We have discussed properties of this Green's function and also explained how to use to calculate the power radiated out into $n-1$ space when there is one time variable. We have also explained how to derive generalizations of the homogeneous Maxwell field equations in (p,q)-dimensional space-time from the potentials. We then generalize the n-dimensional Maxwell equations to curved space-time and arrive at the Einstein-Maxwell equations in n-dimensional space-time. These equations are used to describe the motion of N points charges carrying masses moving under their mutual gravitational and electromagnetic interactions in the general theory of relativity. For this formulation, we first write down expressions for the energy-momentum tensor of the matter, of the electromagnetic radiation field and the current density produced by N discrete point masses moving in n-dimensional space with one time variable and then derive the Einstein-Maxwell equations from the standard variational principle. In this paper, we also formulate Dirac's relativistic wave equation in (p,q)-dimensional space time and use it to describe the quantum mechanics of a an electron moving in such a space-time manifold. We discuss the $SO(p,q)$ invariance of Dirac's equation by using the technique of spin representations of $SO(p,q)$. Specifically, we show how to construct Dirac matrices in (p,q)-dimensional space-time using creation and annihilation operators on Fermion Fock space. The spin group as the outer cover of $SO(p,q)$ is then constructed using Lie algebraic methods applied to commutators of the Dirac matrices. Spin representations are also constructed using the action of Dirac matrices on the Fermion Fock space. Finally, we construct Dirac's relativistic wave equation in the presence of an electromagnetic field and curvature of the n-dimensional space-time manifold. In order to construct such an equation, we introduce the spinor connection of the gravitational field using the gravitational tetrad as well as the commutators of the Dirac matrices which we have shown to be generators of the Spin representation of $SO(p,q)$. Higher dimensional Maxwell equations as pointed out by Professor Newcomb can be used to model psychic fields. It is therefore natural to consider the Combined Maxwell-Dirac-Einstein field equations in higher dimensions as a method to describe the effects of gravitation and charges on such psychic fields where charges are not classical point charges but rather characterized by the Dirac wave operator field. In other words, the charges that we speak of are to be regarded as higher dimensional generalizations of the sea of electrons with some of the electrons removed to form positrons as first enunciated by Dirac. Finally, we formulate Dirac's equation in (p,q)-dimensional curved space-time in terms of the terad, the spin connection of the gravitational field and (p,q)-space time Dirac matrices. The spin connection is derived from the condition that the covariant derivative of the tetrad basis vanishes and it is shown to lead to the curved space-time Dirac equation having all three symmetries: $U(1)$ gauge invariance along with the electromagnetic field, local $SO(p, q)$-invariance also called local generalized Lorentz invariance, and diffeomorphism invariance.

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