Research Article

The Pauli Spin Algebra, Related Clifford Algebra Representation, and Derivation of the Dirac Hamiltonian Operator

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I present a novel approach to derivation of the Dirac Hamiltonian operator on the secondly quantized Fock space, based on the the Pauli spin $\mathrm{su}(2)$ - algebra and its Clifford algebra representation. The relativistic Lorentz invariance of the related Schrödinger equation is stated as a natural consequence of its operator structure.

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1. Introductory setting: tensor algebra, Clifford algebra and Fock space representation

1.1. Historical retrospective notes

In the study there is undertaken an attempt to clarify from the mathematical point of view the electron spin origin and its Hamiltonian evolution description, as they had been formulated about one hundred years ago by Austrian mathematical physicist Wolfgang Pauli, who invented spin property owing to experimental data (the classical Zeeman effect [1]) that implied half-integer angular momentum for the electron, which he termed as a "*classical non-describable two-valuedness*" and suitably described by means of the so called Pauli spin matrices, representing the Lie algebra of the unitary $\mathrm{SU}(2)$ symmetry group [2] of the related quantum states in a suitably defined Hilbert space. As this spin symmetry property was deeply related with a spin conservation law of the electron space-time evolution, described by means of the corresponding Schrödinger equation, governed by some still unknown Hamiltonian operator, Pauli posed [3][4] a problem to retrieve from existing experimental and the unitary $\mathrm{SU}(2)$ symmetry group properties a physically acceptable Hamiltonian operator, describing both electron evolution and its spin properties. This problem was not realized by W. Pauli

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himself on time, as soon enough the British mathematical physicist P. Dirac had suggested [5] a completely different relativistic theory based approach to this problem, having reduced it to constructing suitably fitting square roots to the so called Klein-Gordon differential operator on the Minkowski space-time M^4, and derived both the corresponding Dirac evolution equation and the related Hamiltonian operator, governing the electron evolution.

As the P. Dirac derivation of the electron Hamiltonian operator had been based on some square root matrix tricks and had been in no way related to experimental data, describing spin symmetry properties, a rigorous derivation of this Hamiltonian operator and a description of the spin Pauli algebra symmetry properties, based on these experimental data, had been remained for decades not analyzed in the literature [6][7] until the present work which presents in detail a mathematically rigorous approach to solving the Pauli electron spin and Hamiltonian problem.

Taking into account the Pauli spin symmetry algebra, its related Clifford algebra dual structure, there is constructed [8][9][10] the corresponding exact Clifford algebra $Cl_3(\sigma)$ representations on the Hilbert-Fock space, generated by the corresponding creation-annihilation operators, naturally describing both physically observable quantum particles and the corresponding operator invariants, commuting with the corresponding Pauli spin operator. As a simple by-product of the devised algebraic commutation property, there is constructed the corresponding Hamiltonian operator, completely coinciding with that, before derived by P. Dirac. Moreover, its relativistic Lorentz invariance is obtained as a simple consequence of the corresponding Schrödinger evolution equation, governed by this Hamiltonian.

As was mentioned by A. Jadczyk in [11], it would be instructive to think of the family of Clifford algebras $\{Cl(F,\, \eta) : \eta \in \mathrm{Quad}(F)\}$ as a $\mathbb{Z}_2$- graded vector bundle $Cl(F)$ over the space $\mathrm{Quad}(F)$ of all quadratic forms on F. Moreover, instead of considering the usual Dirac equation, where the Clifford algebra acts on an irreducible graded "*spin module*", one considers the so called Ivanenko–Landau–Kähler equation [12][13], where spinors are represented by differential forms - elements of the exterior bundle of the cotangent space. The idea of representing spinors as differential forms has its clear physical justification that was stated by P. Lounesto [14], explicitly in the following form : ". . . However the physical justification of the theory of spin manifolds could be questioned on the following basis: why should we need to know the global properties of the universe, if we want to explore the local properties of a single electron?" Being aware of the fact that choosing a global minimal graded ideal of the Clifford algebra bundle over space-time is questionable, he further stated, that ". . . In curved manifolds it is more appropriate to use abstract representation modules as spinor spaces and not minimal left ideals, not even subalgebras, of Clifford algebra". So, one expect that the

only acceptable solution to these problems is by using the whole exterior algebra of differential forms, as in Ivanenko-Landau-Kähler equation, on a curved spacetime manifold M, locally diffemorphic to M^4.

1.2. Tensor algebra

We consider a complex vector space $(F;\eta(\cdot|\cdot))$, rigged with a bilinear form $\eta: F\times F\to\mathbb{C}$, and the related tensor algebra $T(F)$ over it, that is a unital associated algebra, containing F as a linear subspace and satisfying [15][11][16][17][14][18][19] the following property: every linear mapping $\varphi: F\to A$ into a unital associative algebra A extends to an algebra homomorphism $\theta_\varphi: T(F)\to A$. If $\iota: F\to T(F)$ denotes the embedding of F into $T(F)$, then one can write down that $\varphi=\theta_\varphi\circ\iota$. If $\{e_i\in F: i\in I(F)\subset\mathbb{N}\}$ denotes a basis in F, then the unite $1\in F\subset T(F)$ and $\{e_{j_1}\otimes e_{j_2}\otimes\ldots\otimes e_{j_n}\}, n\in\mathbb{N}$, form a natural basis in $T(F)$, where $\otimes$ denotes the "tensor" multiplication in $T(F)$. The latter imposes on it the natural grading:

$$T(F)=\oplus_{n=0}^{\infty}T^n(F), \tag{1.1}$$

where

$$T^n(F):=F^{\otimes n}=\underbrace{F\otimes F\otimes ...\otimes F}_{n\quad times}$$

is a subspace, generated by the basis elements $\{e_{j_1}\otimes e_{j_2}\otimes\ldots\otimes e_{j_n}\}$, mentioned above, in particular, $T^0(F)=\mathbb{C}, T^1(F)=F, T^2(F)=F\otimes F$ and so on. On the tensor algebra $T(F)$ there are defined two natural involutions:

$$\begin{array}{l} v: u_1\otimes u_2\otimes\ldots\otimes u_n=(-1)^n u_1\otimes u_2\otimes\ldots\otimes u_n, \\ \tau: u_1\otimes u_2\otimes\ldots\otimes u_n=\ u_n\otimes u_{n-1}\otimes\ldots\otimes u_1 \end{array} \tag{1.2}$$

for arbitrary $u_j\in F, j=\overline{1,n}, n\in\mathbb{N}$. The main involution $\nu: T(F)\to T(F)$ defines the $\mathbb{Z}_2$ gradation of $T(F)$: an element $u\in T(F)$, for which $v(u)=u$, is called *even* and for which $\nu(u)=-u$, is called *odd*. Moreover, even elements form a subalgebra of the tensor algebra $T(F)$.

On the tensor algebra $T(F)$ there is determined for arbitrary $p\in\mathbb{N}$ the left operator action $a^+(v): T^p(F)\to T^{p+1}(F)$:

$$a^+(v)u_1\otimes u_2\otimes\ldots\otimes u_p:=\ v\otimes u_1\otimes u_2\otimes\ldots\otimes u_j\otimes\ldots\otimes u_p \tag{1.3}$$

for any $v\in F$ and $u_1\otimes u_2\otimes\ldots\otimes u_p\in T^p(F), p\in\mathbb{N}$. Taking into account the existence of the bilinear form $\eta: F\times F\to\mathbb{C}$, it naturally induces on the tensor algebra $T(F)$ the corresponding "adjoint" linear operator mapping $a(v): T^p(F)\to T^{p-1}(F)$

$$a(v)\left(u_1\otimes u_2\otimes\ldots\otimes u_p\right):=\sum_{j=1}^{p}(-1)^{j-1}\eta(v|u_j)u_1\otimes u_2\otimes\ldots\otimes\hat{u}_j\otimes\ldots\otimes u_p, \tag{1.4}$$

where $u_1\otimes u_2\otimes\ldots\otimes u_p\in T^p(F), p\in\mathbb{N}$,and the cap " $\hat{\circ}$ "over the vector means its omission in the product, and satisfying the following conditions:

$$\begin{aligned} a(v)\,a(w)+a(w)a(v)&=0,\\ a(v)a^{+}(w)+a^{+}(w)a(v)&=\eta(v|w)1, \end{aligned} \tag{1.5}$$

for all $v,w\in F$.

Apply now to the tensor algebra $T(F)$ the linear antisymmetrization operator $Alt:T(F)\to\Omega(F)$, acting on uniform elements $u_1\otimes u_2\otimes\ldots\otimes u_p\in T^p(F), p\in\mathbb{N}$, as

$$Alt\left(u_1\otimes u_2\otimes\ldots\otimes u_p\right):=\frac{1}{p!}\sum_{\sigma\in\Sigma_p}(-1)^{\sigma}u_{\sigma^{-1}(1)}\otimes u_{\sigma^{-1}(2)}\otimes\ldots\otimes u_{\sigma^{-1}(p)}:=u_1\times u_2\times\ldots\times u_p, \tag{1.6}$$

we obtain the natural gradation on the obtained [15][16][14][18] Grassmann algebra $\Omega(F)=\sum_{n=0}^{\infty}\Omega^p(F)$, where " $\times$ " denotes the classical external multiplication on $\Omega(F)$. Assuming now that the semilinear form $\eta:F\times F\to\mathbb{C}$ is unitary, that is $\overline{\eta(u,v)}=\eta(v,u)$ for arbitrary $u,v\in F$, owing to the linearity of the mappings (1.3) and (1.4), the corresponding operators $a^{+}(v)$ and $a(v):\Omega^p(F)\to\Omega^{p+1}(F), p\in\mathbb{N}$, where

$$a^{+}(v)u:=v\times u \tag{1.7}$$

with $v\in F, u\in\Omega^p(F)$, and operator action

$$a(v)\left(u_1\times u_2\times\ldots\times u_p\right):=\sum_{j=1}^{p}(-1)^{j-1}\eta(v|u_j)u_1\times u_2\times\ldots\times u_j\times\ldots\times u_p \tag{1.8}$$

with $v,u_j\in F, j=\overline{1,p}$, persists to satisfy the same algebraic relationships (1.5). Moreover, if to take into account that the unitary vector space $(F;\eta)$ naturally induces the unitary structure on the whole tensor product $T(F)$ and, suitably, on the antisymmetrized Grassmann algebra $\Omega(F)$, we can easily check that operators $a^{+}(v)$ and $a(v):\Omega^p(F)\to\Omega^{p+1}(F)$are adjoint to each other, that is $\eta(a^{+}(u)\alpha|\beta)=\eta(\alpha|a(u)\beta)$ for all $\alpha\in\Omega^{p+1}(F)$ and $\beta\in\Omega^p(F)$, $p\in\mathbb{N}$, and are called [20][2][21][5][22] the *creation* and *annihilation* operators, respectively. The constructed above algebraic structure - the Grassmann algebra $\Omega(F)$ and related creation - annihilation operators is called the *algebraic Fock space* [20][21][23][24] over the unitary vector space $(F;\eta)$, being basic object for describing the important in physical applications [25][26][20] algebra of self-adjoint observable operators.

1.3. Clifford algebra

Let F be a complex vector space, $\eta:F\times F\to\mathbb{C}$ be a symmetric bilinear form and $J(\eta)$ be the two sided ideal in the tensor algebra $T(V)$, generated by elements of the form $u\otimes u-\eta(u,u)1$, where $u\in V\subset T(V)$. The

ideal $J(\eta) \subset T(V)$ consists of all finite sums of elements of the form $u_1 \otimes u_2 \otimes \ldots \otimes u_p \otimes (u \otimes u - \eta(u,u)1) \otimes v_1 \otimes v_2 \otimes \ldots \otimes v_q$, where $u, u_j, v_k \in V, j = \overline{1,p}, k = \overline{1,q}$. Then the Clifford algebra is the quotient algebra $Cl(F;\eta) := T(F)/J(\eta)$, related to the vector space $(F;\eta)$.

In case of the space $F = \mathbb{C}^n$, $n \in \mathbb{N}$, the corresponding symmetric bilinear form $\eta: \ \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ allows the canonical diagonal matrix representation

$$\eta = \mathrm{diag}(\underbrace{1,1,...,1}_{n})$$

and the corresponding Clifford algebra is denoted by $Cl_n(\mathbb{C})$. There are two separate cases to consider, according to whether the integer $n \in \mathbb{N}$ is even or odd. When $n \in \mathbb{N}$ is even, the algebra $Cl_n(\mathbb{C})$ is central simple. Owing to the classical Artin–Wedderburn theorem [15][11][16][18], it is isomorphic to a matrix algebra over $\mathbb{C}$. When $n \in \mathbb{N}$ is odd, the center of the algebra $Cl_n(\mathbb{C})$ includes except scalars so called the pseudoscalars. The latter makes it possible to find a normalized pseudoscalar ξ, such that $\xi^2 = 1$. If to define the central projection operators $P_{\pm} := \frac{1}{2}(1 \pm \xi) : Cl_n(\mathbb{C}) \ \to \ Cl_n(\mathbb{C})$, whose images $P_{\pm}Cl_n(\mathbb{C}) \subset Cl_n(\mathbb{C})$ split the Clifford algebra $Cl_n(\mathbb{C})$ as the direct sum

$$Cl_n(\mathbb{C}) = \ Cl_n^{+}(\mathbb{C}) \oplus Cl_n^{-}(\mathbb{C}), \tag{1.9}$$

satisfying the involution relationships: $\nu : Cl_n^{\pm}(\mathbb{C}) = Cl_n^{\mp}(\mathbb{C})$. The latter, in particular, means that these Clifford subalgebras are isomorphic. Since these two isomorphic subalgebras are each central simple and so, again, isomorphic to a matrix algebra over $\mathbb{C}$. The related sizes of the representation matrices can be determined from the fact that the dimension $\dim Cl_n(\mathbb{C}) = 2^n$.

2. Pauli spin algebra, the related Clifford algebra and its representation

We consider, following the approach, devised in [8][9][10][27], the complex Euclidean space $\left(\mathbb{C}^2; \langle \cdot | \cdot \rangle\right)$ and the unitary group $\mathrm{SU}(2)$, whose Lie algebra basis consists of the following Pauli spin matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.1}$$

satisfying the following commutator relationships:

$$[\sigma^1, \sigma^2] = 2i\sigma^3, [\sigma^2, \sigma^3] = 2i\sigma^1, [\sigma^3, \sigma^1] = 2i\sigma^2, \tag{2.2}$$

which can be equivalently rewritten as

$$\sigma^j \sigma^k + \sigma^k \sigma^j = 2\delta^{jk} \tag{2.3}$$

for all $j,k=\overline{1,3}$. The Clifford algebra $Cl_3(\sigma)$of the dimension $\dim Cl_3(\sigma)=2^3=8$, generated by the Pauli matrices (2.1), supplemented with the unit matrix $\sigma^0=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, describes the internal degrees of freedom [20][2][21] of such a "*fermionic*" particle [6] as electron, whose internal structure is, to our regret, of a weakly known up-to-date nature. Yet algebraic properties of the Clifford algebra $Cl_3(\sigma)$ make it possible to propose some useful inferences about some hidden symmetry properties of electron, motivated by studying its representation structure. There is accepted [15][18][24] the following definition of the Clifford algebra representation.

Definition 2.1. The left module of a Clifford algebra $Cl_m(\sigma)$, generated by a set $\{\sigma\}\subset F_{int}$, is a linear finite-dimensional space F_{int} over $\mathbb{C}$, endowed with a linear representation mapping $\rho: Cl_m(\sigma)\otimes F_{int}\to F_{int}$, such that

$$\rho(ab\otimes f)=\rho(a\otimes\rho(b\otimes f)) \tag{2.4}$$

for all $a,b\in Cl_m(\sigma)$ and $f\in F_{int}$. In particular, a representation mapping $\rho: Cl_m(\sigma)\otimes F_{int}\to F_{int}$is a homomorphism of the corresponding algebras.

Construct now a physically motivated representation of the spin σ - matrices above (2.1), generating the eight-dimensional Clifford algebra $Cl_3(\sigma)$, by means of the Clifford algebras, generated by canonical creation-annihilation operators on the Fock type space F_{int}. Since the basis dimension of the Clifford algebra $Cl_3(\sigma)$ is odd, we will take a pseudoscalar central element $\xi=i\sigma^1\sigma^2\sigma^3\in Cl_3(\sigma)$, $\xi^2=1$, and make use of the Peirce decomposition [28][29] by means of the projectors $P_\pm:=\frac{1\pm i\sigma^1\sigma^2\sigma 3}{2}$, giving rise to the splitting like (1.9) into the direct sum of its isomorphic Pierce subalgebras:

$$Cl_3(\sigma)=Cl_3^+(\sigma)\oplus Cl_3^-(\sigma), \tag{2.5}$$

where $Cl_3^\pm(\sigma):=P_\pm Cl_3(\sigma)$, $Cl_3^\pm(\sigma)=Cl_3^\mp(\sigma)$ and $\dim\ Cl_3^\pm(\sigma)=4$, and which representations are completely described by the following proposition.

Proposition 2.2. *The direct sum of the Clifford subalagebras (2.5) allows the representation* $\rho: Cl_3(\sigma)\to \mathrm{End}\left(F_{int}^{(+)}\oplus F_{int}^{(-)}\right)$*,of the Clifford algebra* $Cl_3(\sigma)$*, where*

$$\begin{gathered}\rho(Cl_3(\sigma))\otimes\left(F_{int}^{(+)}\oplus F_{int}^{(-)}\right)\simeq\\ \simeq Cl_2(b_+,b_+^+)\oplus\ Cl_2(b_-,b_-^+): F_{int}^{(+)}\oplus F_{int}^{(-)}\to F_{int}^{(+)}\oplus F_{int}^{(-)}\end{gathered} \tag{2.6}$$

on the space $F_{int}=F_{int}^+\oplus F_{int}^-\simeq\mathbb{C}^4$*, where* $F_{int}^\pm\simeq\mathbb{C}^2$*, in the following form:*

$$\begin{gathered}\rho(\sigma_1)P_\pm=b_\pm^++b_\pm,\rho(\sigma_2)P_\pm=i(b_\pm^+-b_\pm),\rho(\sigma_3)P_\pm=\Gamma_\pm,\\ \rho(Cl_3^\pm(\sigma))=<b_++b_+^+,i(b_+^+-b_+),2b_+^+b_+-P_+>,\\ \rho(Cl_3^\pm(\sigma))=<b_-+b_-^+,i(b_-^+-b_-),-2b_-^+b_-+P_->.\end{gathered} \tag{2.7}$$

Here the operators $\Gamma_{\pm} := \exp{(i\pi b^{+}_{\pm}b_{\pm})} = 1 - 2b^{+}_{\pm}b_{\pm}$ *are the so called central "chirality" operators* [15][18] [24] *and the creation and annihilation operator mappings*

$$b_{\pm} := a(\bar{e}_{\pm}), b^{+}_{\pm} = a^{+}(e_{\pm}), \tag{2.8}$$

defined by the normalization conditions $\eta(\bar{e}_{\pm}|e_{\pm}) = 1, \eta(e_{\pm}|e_{\pm}) = 0 = \eta(e_{\mp}|e_{\pm})$ *on the basis vectors* $\{e_{\pm}, \bar{e}_{\pm}\} \in \mathbb{C}^4$, *satisfy the canonical algebraic relationships:*

$$\begin{array}{c} b_{\pm}b^{+}_{\pm} + b^{+}_{\pm}b_{\pm} = P_{\pm},\ b_{\pm}b^{+}_{\mp} + b^{+}_{\mp}b_{\pm} = 0, \\ b^{+}_{\pm}b^{+}_{\pm} = 0 = b_{\pm}b_{\pm}. \end{array} \tag{2.9}$$

Moreover, there hold the following anticommutation relationships

$$\Gamma_{\pm}\, b_{\pm} + b_{\pm}\Gamma_{\pm} = 0 = \Gamma_{\pm}b^{+}_{\pm} + b^{+}_{\pm}\Gamma_{\pm}, \tag{2.10}$$

and the related important isometry properties

$$(b^{+}_{\pm} + b_{\pm})F^{(\pm)}_{int} = F^{(\pm)}_{int}. \tag{2.11}$$

Proof. Consider the Clifford algebra splitting $Cl_3(\sigma) = Cl^{+}_3(\sigma) \oplus Cl^{-}_3(\sigma)$, where the dimension of each component $\dim Cl^{\pm}_3(\sigma) = 4$, and take into account that $Cl^{\pm}_3(\sigma) \simeq Cl_2(\mathbb{C})$, the corresponding basic creation-annihilation operators $\{b_{\pm} := a(\bar{e}_{\pm}), b^{+}_{\pm} = a^{+}(e_{\pm}) \in Cl_2(b_{\pm}, b^{+}_{\pm}) : e_{\pm}, \bar{e}_{\pm} \in \mathbb{C}^4\}$ are adjoint to each other: $a(\bar{e}_{\pm})^{*} = a^{+}(e_{\pm}) \in Cl_2(b_{\pm}, b^{+}_{\pm})$. The latter makes it possible to construct the so called "*chirality*" operators $\Gamma_{\pm} := \exp{(i\pi b^{+}_{\pm}b_{\pm})}$, satisfying the condition $\Gamma^2_{\pm} = 1$, as the spectrum of the commutating to each other selfadjoint operators $b^{+}_{\pm}b_{\pm} : F^{(\pm)}_{int} \to F^{(\pm)}_{int}$ consists of $\{0, 1\}-$ integers: $\mathrm{Spec}\left(b^{+}_{\pm}b_{\pm}\right)|_{F^{(\pm)}_{int}} = \{0, 1\}$. Taking into account that the operators $P_{\pm} := 1/2(1 \pm \xi)$, $P^2_{\pm} = P_{\pm}$, are central projectors and the dimension of the representation spaces $F^{(\pm)}_{int}$ equals 2, the Clifford algebra $Cl_3(\sigma)$ and its subalgebras $Cl^{\pm}_3(\sigma)$ allow the creation-annihilation representations (2.7), which satisfy additional relationships (2.10) and (2.11), proving the proposition. □

We see that the dimension of the constructed above Clifford algebra $Cl_3(\sigma)$ representation (2.7) satisfies the equality $\dim Cl_3(\sigma) = 8 = \dim \mathrm{End}\ F^{(+)}_{int} + \dim \mathrm{End}\ F^{(-)}_{int}$, giving rise to the dimension equality for the representation space $F_{int} \simeq \mathbb{C}^4$ of the Clifford alagebra $Cl_3(\sigma)$, as $\dim\ F_{int} = \dim{(F^{(+)}_{int} \oplus F^{(-)}_{int})} = 4$. The latter gives rise to the related matrix representation of the $su(2)$ Lie algebra on the internal Fock space $F_{int} = F^{(+)}_{int} \oplus F^{(-)}_{int}$ by means of the following vector matrix operator :

$$\alpha := \{\alpha_j : \alpha_j = \sigma_1 \otimes \sigma_j \in \quad \mathrm{End}(\mathbb{C}^4) \otimes \mathbb{E}^3, \tag{2.12}$$

transposing the components of the direct sum $F_{int} = F^{(+)}_{int} \oplus F^{(-)}_{int}$, called [30] Dirac 2 - spinors, and satisfying [5][30] the algebraic relationships, similar to those of (2.3):

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk} \tag{2.13}$$

for all $j,k=\overline{1,3}$. As another important inference from the above reasonings, regarding the Clifford algebra $Cl_4(\alpha,\beta)$ representation, we can construct a matrix-vector operator for the electron spin:

$$\begin{aligned}\mathbf{S} \ :=\ \hbar/2\sigma^0\otimes\sigma=\{S^j:S^j=\hbar/2\sigma^0\otimes\sigma^j\in\\ \in\hbar/2\rho(Cl_3(b_+,b_+^+)\oplus Cl_3(b_-,b_-^+)),j=\overline{1,3}\}\in \quad \text{End}\ \left(\mathbb{C}^4\right)\otimes\mathbb{E}^3,\end{aligned}$$

where $\hbar>0$denotes [26][21][5] the well known Planck constant, and which takes into account the isomorphism of the subalgebras $Cl_3^{\pm}(\sigma)$, guaranteeing from both geometrical and physical points of view equivalence of three-dimensional space rotations of quantum electron states, regardless of the "chiral" state or fermion parity sector the system is embedded. This independence is what allows electron spin to remain a pure geometric characteristic of a particle even in the extended 4-component space of Clifford representations, playing an important role in the construction of the associated spatially dependent symmetries and their corresponding invariants. In particular, it is easy to check that the the spin operator component $S^3\in\text{End}\ \mathbb{C}^4$ possesses two eigenvalues $s^{\pm}=\pm\ \ \hbar/2$, corresponding to their four eigenfunctions $(f(s^{\pm}),0)^{\intercal},(0,f(s^{\pm}))^{\intercal}\in\mathbb{C}^2\times\mathbb{C}^2$, on which the following actions

$$\begin{aligned}S^1(f(s^{\pm}),0)^{\intercal}\ =\ |s^{\pm}|^2(f(s^{\mp}),0)^{\intercal},S^1(f(s^{\pm}),0)^{\intercal}\ =\ |s^{\pm}|^2(f(s^{\mp}),0)^{\intercal},\\ S^2(f(s^{\pm}),0)^{\intercal}=-is^{\pm}\big|(f(s^{\mp}),0)^{\intercal},S^2(f(s^{\pm}),0)^{\intercal}\ =\ -is^{\pm}(f(s^{\mp}),0)^{\intercal},\\ S^3(f(s^{\pm}),f(s^{\pm}))^{\intercal}=s^{\pm}(f(s^{\pm}),f(s^{\pm}))^{\intercal},S^3(f(s^{\pm}),0)^{\intercal}\ =\ -is^{\pm}(f(s^{\pm}),0)^{\intercal}\end{aligned}\tag{2.15}$$

hold. The related scalar operator $\langle\mathbf{S}|\mathbf{S}\rangle\in\text{End}\ (\mathbb{C}^4)$, called the square of the full electron spin, possesses the unique eigenvalue $2|s^{\pm}|(|s^{\pm}|\ +\hbar)$. Other important properties of the spin operator (2) and inferences from them are discussed in detail in the next section below.

3. Second quantized Fock space, translational invariance and momentum conservation law

Let now the mentioned above so called "*second quantized*" Fock space structure $\mathscr{F}_{ext}:=(\Omega(F);\ (\cdot|\cdot))$, as a suitably constructed cyclic Hilbert space, be generated by the complex vector space $F=L_2(\mathbb{R}^3;\mathbb{C}^2)\otimes\mathbb{Z}_2$ and basic creation-annihilation generalized operators $\psi_j^+(f)$ and $\psi_k(g)$:

$$\begin{aligned}\psi_j(f\ )\psi_k(g)+\psi_k(g)\psi_j(f\ )=0,\\ \psi_k(g)\psi_j^+(f\ )+\psi_j^+(f\ )\psi_k(g)\ =\ \delta_{jk}(g|f),\\ \psi_j^+(f\ )\psi_k^+(g)+\psi_k^+(g)\psi_j^+(f\ )\ =\ 0\end{aligned}\tag{3.1}$$

for arbitrary f and $g\in L_2(\mathbb{R}^3;\mathbb{C}^2)\otimes\mathbb{Z}_2$, where indices $j,k\in\overline{1,4}$, are responsible for the related spin operator (2) spatial invariance property. It is worth to observe here that the set $\{\psi_j(f\ ),\psi_k^+(g):j,k=\overline{1,4},$ $f,g\in F\}$ generate the generalized Clifford algebra $\prod\limits_F Cl_4(\psi,\psi^+)$ of the creation and annihilation operators on $\Omega(F)$. Namely, let us consider a system of $N\in\mathbb{N}$ electron particles, located in the physical three-

dimensional space $\mathbb{R}^3$, which can be described by means of the normalized "*wave*" function $\Phi_N \in \mathscr{F}_{ext,N} := \Omega(L_2(\mathbb{R}^3;\mathbb{C}^2)\otimes\mathbb{Z}_2)$:

$$\Phi_N = \frac{1}{\sqrt{N!}}\int_{\mathbb{R}^{3\times N}} dx_1 dx_2 \dots dx_N \varphi_{(j_1,j_2,\dots,j_N)}(x_1,x_2,\dots,x_N)\psi^+_{j_1}(x_1)\psi^+_{j_2}(x_2)\dots\psi^+_{j_N}(x_N)|0), \quad (3.2)$$

where, by definition, $(\Phi|\Phi) = 1, \psi^+_{j_k}(x) := \psi^+_{j_k}(e_x), j_k = \overline{1,4}, \{e_x \in L_2(\mathbb{R}^3;\mathbb{C}^2), x \in \mathbb{R}^3\}-$ a set of orthonormal functions, $|0) \in \mathscr{F}_{ext}$ is the corresponding Clifford algebra unit, called often the "*vacuum*", and the function $\varphi \in \left(L_2(\mathbb{R}^3;\mathbb{C}^2)\otimes\mathbb{Z}_2\right)^{\times N} \simeq L_2(\mathbb{R}^{3\times N};\mathbb{C}^2)\otimes\mathbb{Z}_2, ||\varphi||_2 = 1,$ allows to calculate the probability density

$$p_{(j_1,j_2,\dots,j_N)}(x_1,x_2,\dots,x_N) := (\varphi_{(j_1,j_2,\dots,j_N)}(x_1,x_2,\dots,x_N)|\varphi_{(j_1,j_2,\dots,j_N)}(x_1,x_2,\dots,x_N)) \quad (3.3)$$

to locate simultaneously the set of these $N \in \mathbb{N}$ electron particles at points $(x_1,x_2,\dots,x_N) \in \mathbb{R}^{3\times N}$, respectively. The quantity of these particles is suitably described by the number particle operator

$$N = \int_{\mathbb{R}^3} \psi^+(x)\psi(x)dx, \quad (3.4)$$

where $\psi(x) := (\psi_1(x),\psi_2(x),\psi_3(x),\psi_4(x))^\intercal, \psi^+(x) := (\psi^+_1(x),\psi^+_2(x),\psi^+_3(x),\psi^+_4(x))$ - sets of basic creation and annihilation operators on the second quantized Fock space $\mathscr{F}_{ext}$ at $x \in \mathbb{R}^3$.

Take now into account that our set of $N \in \mathbb{N}$ electron particles, located at points $(x_1,x_2,\dots,x_N) \in \mathbb{R}^{3\times N}$, satisfy the spatial translation group symmetry invariance, that is

$$\begin{gathered} \int_U p_{(s_1,s_2,\dots,s_N)}(x_1,x_2,\dots,x_N)dx_1dx_2\dots dx_N = \\ = \int_{U+a} p_{(s_1,s_2,\dots,s_N)}(x_1,x_2,\dots,x_N)dx_1dx_2\dots dx_N \end{gathered} \quad (3.5)$$

for arbitrary open domain $U \subset \mathbb{R}^3$, shifted by arbitrary vector $a \in \mathbb{R}^3$. The corresponding translation symmetry group generator is called the "*momentum*" and given by the operator expression $p = \frac{\hbar}{i}\frac{\partial}{\partial x}$, $x \in \mathbb{R}^3$, where $\hbar$, as before, denotes the well known Planck constant.

Turning now back to the constructed above spin operator (2), we take into account the well known experimental fact, discovered by Davisson and Germer [31][32] in 1927, in which the electron beam splitting into exactly two beams showed that the electron spin component along the magnetic field axis can only have two orientations: spin-up $(+1/2)$ or spin-down (-1/2), what, in part, can be reduced to the invariance in time of the Euclidean projection of the electron spin on its momentum, the so called "*helicity*", presenting a physical conservation law of a free quantum electron. The latter can be rewritten as the invariance in time of the following "*second quantized*" [20][21][24] operator quantity

$$S_p = \int_{\mathbb{R}^3} dx \psi^+ \langle \mathbf{S}|\mathbf{p}\rangle \psi, \quad (3.6)$$

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that is the equation

$$dS_p/dt = 0 \tag{3.7}$$

is satisfied for all temporal parameters $t \in \mathbb{R}$. The invariance relationship (3.6) can be effectively used to constructing Hamiltonian dynamics of the electron particle in time, if to make use of the fundamental Heisenberg [25][26][20][21][24] principle: the temporal evolution of any operator $A \in \mathrm{End}\ \mathscr{F}_{ext}$ on the second quantized Fock space $\mathscr{F}_{ext}$ is given by the commutator type equation

$$dA/dt = i/\hbar[H, A] \tag{3.8}$$

for all $t \in \mathbb{R}$. This construction is in detail presented in the section below.

4. Dirac Hamiltonian operator structure and its derivation

The matrix-vector of spin (2) and spin projection (3.6) operators on the basic spin representation space $\mathscr{F}_{int} = \mathscr{F}^{+}_{int} \oplus \mathscr{F}^{-}_{int}$ can be equivalently represented both as

$$\mathbf{S} = \quad \hbar/2\ \sigma^0 \otimes \sigma|_{\Phi_{int}} = \hbar/2\ \sigma^0 \otimes \sigma|_{(\sigma^1 \otimes 1)F_{int}} \tag{4.1}$$

and

$$\begin{gathered} S_p = \ \int_{\mathbb{R}^3} dx \psi^{+}(x) \langle \mathbf{S}|\mathbf{p}\rangle \psi|_{F_{int}} \quad = \tfrac{\hbar}{2} \int_{\mathbb{R}^3} dx \psi^{+}(x) \langle \sigma^0 \otimes \sigma|\mathbf{p}\rangle \times \\ \times \left(\sigma^1 \otimes I\right) \psi|_{(\sigma^1 \otimes 1)F_{int}} = \tfrac{\hbar}{2} \int_{\mathbb{R}^3} dx \psi^{+}(x) \langle \alpha|\mathbf{p}\rangle \psi|_{(\sigma^1 \otimes 1)F_{int}}, \end{gathered} \tag{4.2}$$

respectively, where the matrix $\alpha \in \mathrm{End}\ \mathscr{F}_{int}$ coincides with that of (LABEL:P11). Combining the operator relationships (3.6) and (3.7) together, we obtain following commutator relationships

$$\begin{gathered} \left[\int_{\mathbb{R}^3} dx \psi^{+}(x) \langle \alpha|\mathbf{p}\rangle \psi, S_p\right]\big|_{(\sigma^1 \otimes 1)F_{int}} = 0 = \\ = \left[\int_{\mathbb{R}^3} dx \psi^{+}(x) \langle \alpha|p\rangle \psi + M, S_p\right]\big|_{F_{int}} = [H_f, S_p]|_{F_{int}} = dS_p/dt|_{F_{int}} = 0, \end{gathered} \tag{4.3}$$

satisfied for all temporal parameters $t \in \mathbb{R}$. The obtained above commutator relationship in (4.3) makes it possible to derive the unique first order selfadjoint second quantized Hamiltonian operator, generated by the basis $< \beta, \alpha >$ of the complete Clifford algebra $Cl_4(\alpha, \beta)$, as

$$H_e = \ \int_{\mathbb{R}^3} dx \psi^{+}(x) \langle \alpha|\mathbf{p}\rangle) \psi \quad + M, \tag{4.4}$$

defined on the whole second quantized Fock space $\mathscr{F}_{el} := \mathscr{F}_{ext} \otimes \mathscr{F}_{int}$, The obtained above result can be formulated as the following proposition.

Proposition 4.1. *The first order selfadjoint operator $H_e : \mathscr{F}_{el} \to \mathscr{F}_{el}$ on the Hilbert space $\mathscr{F}_{el} = \mathscr{F}_{ext} \otimes \mathscr{F}_{int}$, providing the evolution in the operator space $\mathrm{End}\ \mathscr{F}_{el}$ via the Heisenberg temporal*

commutator flow (3.8) and taking into account the isometry properties (2.11) is given by the unique operator expression (4.4), endowed with the mass operator $M = \int_{\mathbb{R}^3} \psi^+(x) m\beta\psi(x) \in \mathrm{End}\ \mathscr{F}_{el}$, *generated by the element* $\beta = \sigma^3 \otimes I \in Cl_4(\alpha, \beta)$, *satisfying the commutator relationship* $[M, S_p] = 0$. *The so called "bare" mass parameter* $m \in \mathbb{R}_+$ *is interpreted as the electron "rest" mass.*

The corresponding one-particle representation of the secondly quantized operator (4.4) easily gives rise to the classical self-adjoint Dirac Hamiltonian operator expression

$$\mathscr{H}_e = \langle \alpha | \mathbf{p} \rangle + m\beta \tag{4.5}$$

on the Hilbert space $L_2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathbb{Z}_2$, first suggested [5] by P. Dirac in 1928, and whose consequence like the existence of antiparticles was soon confirmed by Carl Anderson in 1932 [33] by the experimental discovery of a new elementary particle - positron dual to electron. As a by product, one also obtains from the expression (4.4) and relationship (3.8) the trivial condition $[H_f, N] = 0$, meaning that the whole number of particles is conserved, in particular, it means that all finite-particle Fock subspaces $\mathscr{F}_{el,n}, n \in \mathbb{N}$, where, by definition, $\mathscr{F}_{el} := \oplus_{n=0}^{\infty} \mathscr{F}_{el,n}$, are invariant in time.

An additional and surprising inference, following from the representation (4.5), is that the corresponding one-electron Schrödinger type equation

$$\partial\varphi/\partial t = -i/\hbar \mathscr{H}_e \varphi \tag{4.6}$$

for $\varphi \in L_2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathbb{Z}_2$, rewritten in the Dirac form

$$i\hbar\beta\partial\varphi/\partial t - \langle \gamma | p \rangle - m\varphi = 0, \tag{4.7}$$

where $\gamma := \beta\alpha$, is *a priori* relativistic, being covariant [2][34][35] with respect to the Lorentz symmetry group $\mathscr{L}^+(1,3)$,where the light velocity $c > 0$ is assumed to be scaled to unity.

Really, as any element $\Lambda \in \mathscr{L}^+(1,3)$ transforms [30][36][37] the $(\gamma^0 := \beta, \gamma)-$ spinors by means of the expressions $g(\Lambda)^{-1}\gamma^j g(\Lambda) = \sum_{k=0}^3 \Lambda_k^j \gamma^k, j = \overline{1,3}$, where $g(\Lambda) \in \mathrm{Spin}_{\mathbb{C}}(1,3)$, it is easy to check that the Dirac equation (4.6) persists to be covariant with respect to the Lorentz symmetry group $\mathscr{L}^+(1,3)$. Really, consider the infinitesimal version of the above covariance expression, which is easily obtained, if to put, by definition, $\Lambda_k^j = \delta_k^j + e_k^j$, where $e_k^j \to 0$, $j = \overline{0,3}$, and $g(\Lambda) = 1 + \frac{1}{2}\sum_{j,k,l=0}^3 \Sigma^{jk}\delta_k^l e_j^k$ for some matrices $\Sigma^{jk} \in \mathrm{End}\ \mathbb{C}^4, j, k = \overline{0,3}$: it looks then as $\left[\gamma^j, \Sigma^{kl}\right] = \delta^{jk}\gamma^l - \delta^{jl}\gamma^k$ for all $j, k, l = \overline{0,3}$. The latter relationship allows the following matrix solution: $\Sigma^{jk} = \frac{1}{8}(\gamma^j\gamma^k + \gamma^k\gamma^j) \in Cl_4(\alpha, \beta)$for $j, k = \overline{0,3}$, thus confirming the Lorentz covariance of the Dirac equation. Moreover, the vector components of the electron vector - spin $\mathbf{S} \in \mathrm{End}\ \mathbb{C}^4 \otimes \mathbb{E}^3$ are, by definition, represented as $S^j = \frac{\hbar}{8i}\sum_{j,k=1}^3 \varepsilon_{kl}^j \Sigma^{kl} = \frac{\hbar}{4i}\sum_{j,k=1}^3 \varepsilon_{kl}^j \gamma^k\gamma^l = \frac{\hbar}{2}\sigma^0 \otimes \sigma$, where $\varepsilon_{kl}^j, j, k, l = \overline{1,3}$, is the Levi-Civita alternating tensor, or equivalently, as $\mathbf{S} = \frac{\hbar}{2}\sigma^0 \otimes \sigma = \frac{1}{2}\alpha$, where $\sigma = (\sigma^1, \sigma^2, \sigma^3)^\intercal$ - the usual "vector" of Pauli matrices.

So, the relativistic covariance of the Dirac equation is stated. Furthermore, this can be verified also by constructing the operator expression:

$$(\partial/\partial t + i/\hbar\mathscr{H}_e)^*(\partial/\partial t + i/\hbar\mathscr{H}_e)\varphi = \ (\partial^2/\partial t^2 + \hbar^{-2}m^2 - \Delta)\varphi = 0, \tag{4.8}$$

which coincides with the relativistic invariant classical Klein-Gordon [2][21][34][35] operator in the Minkowski space M^4, also implying the relativistic Lorentz covariance [38][34] of the evolution Dirac equation (4.7). The latter can be formulated as the final proposition.

Proposition 4.2. *The Dirac Hamiltonian operator (4.5) is, by construction, selfadjoint on the Hilbert space* $L_2(\mathbb{R}^3;\mathbb{C}^2)\otimes\mathbb{Z}_2$ *and generates the relativistic Lorentz covariant Schrödinger equation (4.6).*

Additionally, it was observed still by Dirac [5] that the formal algebraic sum of the spin $I\otimes\mathbf{S}:\mathscr{F}_{ext}\otimes\mathscr{F}_{int}\to\mathscr{F}_{ext}\otimes\mathscr{F}_{int}$ and the angular momentum $\mathbf{J}\otimes I:\mathscr{F}_{ext}\otimes\mathscr{F}_{int}\to\mathscr{F}_{ext}\otimes\mathscr{F}_{int}$ operators also remains to be invariant in time, that is the commutator

$$[H_f, \mathbf{J}\otimes I + I\otimes\mathbf{S}] = 0, \tag{4.9}$$

easily following from the Hamiltonian operator expression (4.4). Yet another interesting commutator relationships

$$[\mathscr{H}_f, \mathbf{S}] = -2i\hbar\,\alpha\times\mathbf{p}\Rightarrow\ [\mathscr{H}_f, \langle\mathbf{S}|\mathbf{p}\rangle] = 0 \tag{4.10}$$

holds, supporting the determining from the very beginning constraint that the spin operator projection $\langle\mathbf{S}|\mathbf{p}\rangle:\mathscr{F}_{ext}\to\mathscr{F}_{ext}$ is conserved. In particular, in case of electron, interacting with a constant magnetic field $\mathbf{B}\in\mathbb{E}^3$ and directed along the electron momentum $\mathbf{p}:\mathscr{F}_{ext}\to\mathscr{F}_{ext}$, the corresponding interaction energy $V_B=\langle\mathbf{S}|\mathbf{B}\rangle$ then persists to be constant:

$$dV_B/dt = 1/\hbar[\mathscr{H}_f, V_B] = -2i/\hbar\langle\alpha\times\mathbf{p}|\mathbf{B}\rangle = -2i/\hbar\langle\alpha|\mathbf{p}\times\mathbf{B}\rangle = 0 \tag{4.11}$$

for all $t\in\mathbb{R}$.

Remark 4.3. The electron spin operator $\mathbf{S}\in\mathrm{End}\,(\mathscr{F}_{ext}\otimes\mathscr{F}_{int})$ is, in some sense, modelled by means of a virtual space of internal parameters $Z^3_{int}\simeq\mathbb{R}^3$ and its internal "*momentum*" operator $\mathbf{p}_{int}:=\frac{\hbar}{i}\frac{\partial}{\partial z}, z\in Z^3_{int}$, in the internal microworld Z^3_{int}as follows: $\mathbf{S}\simeq z\times\mathbf{p}_{int}$, whose components satisfy the classical spin algebra relationships

$$[S^1,S^2] = \ i\hbar S^3, [S^2,S^3] = \ i\hbar S^1, [S^3,S^1] = \quad i\hbar S^2 \tag{4.12}$$

and imitate those

$$[J^1,J^2] = i\hbar J^3, [J^2,J^3] = i\hbar J^1, [J^3,J^1] = i\hbar J^2 \tag{4.13}$$

for the classical angular momentum of electron in the real space $\mathbb{R}^3$, yet nothing more.

Remark 4.4. As was noted in [8], the appearance of the "mass" parameter in the Dirac Hamiltonian expression (4.5) in no way is physically motivated by the internal $\mathrm{SU}(2)$ symmetry group of the electron, discussed in detail above, but may rather be related to some virtual mechanism of the $\mathrm{SU}(2)$ - Higgs type symmetry breaking, as it was developed within the standard [17][39] Weinberg-Salam model.

5. Conclusions

Taking into account analytical properties of discussed above quantized Fock states, associated to electron particle and describing its evolution in the ambient Minkowski space-time, one can draw an important conclusions. As first one, the electron spin operator $\mathbf{S}$ is a physical quantity, responsible for its *hidden internal symmetry structure*, mathematically related [34][35] with the symmetry group $\mathrm{SU}(2)$ of the electron quantum states and their representations. The second one says that the compatibility of quantum states with the related many-particle representations of the algebra of observable operators is realized by the basic Clifford algebra $Cl_4(\psi,\psi^+)\otimes Cl_4(\sigma,\beta)$ on the Fock space $\mathscr{F}_{el}=\mathscr{F}_{ext}\otimes\mathscr{F}_{int}$. Moreover, there was demonstrated that the Clifford algebra based description of quantum electron states proves [25][9][27] to be also effective for studying such an important problem as the energy spectrum of electron, naturally interacting with the ambient quantum electromagnetic field.

Notes

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