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Research Article

The Vertex-Edge Locating Roman Domination of Some Graphs

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In this paper, we introduce the concept of vertex-edge locating Roman dominating functions in graphs. A vertex-edge locating Roman dominating ($ve-LRD$) function of a graph $G=(V,E)$ is a function $f:V(G)\rightarrow\{0,1,2\}$ such that the following conditions are satisfied: (i) for every adjacent vertices u,v with $f(u)=0$ or $f(v)=0$, there exists a vertex w at distance 1 or 2 from u or v with $f(w)=2$, (ii) for every edge $uv\in E$, $max[f(u),f(v)]\neq 0$ and (iii) any pair of distinct vertices u,v with $f(u)=f(v)=0$ does not have a common neighbour w with $f(w)=2$. The weight of ve-LRD function is the sum of its function values over all the vertices. The vertex-edge locating Roman domination number of G denoted by $\gamma^P_{ve-LR}(G)$ is the minimum weight of a ve-LRD function in G. We proved that the vertex-edge locating Roman domination problem is NP complete for bipartite graphs. Also, we present the upper and lower bonds of ve-LRD function for trees. Lastly, we give the upper bounds of ve-LRD function for some connected graphs.

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1. Introduction and Preliminaries

In this paper, we introduce the concept of vertex-edge Locating Roman dominating function. Let $G=(V,E)$ be an undirected graph with vertex set V and edge set E. The number of vertices in G is the order of G and the number of the edges in G is the size of the graph G. The set of all neighbors of vertex u in G is the open neighborhood of u; that is $N_G(u)=\{v\in V|uv\in E(G)\}$. The closed neighborhood of u in G is $G[u]=\{u\}\cup N_G(u)$. The number of vertices at distance 2 from vertex v in G is denoted as $N_2(v)$. The degree of vertex u in G is $d(u)=|N_G(u)|$. The path of order n is written as P_n, the size of P_n is $n-1$. The graphs C_n, K_n denote the cycle and complete graphs of order n respecticely. The

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diameter of G, denoted by $diam(G)$ is define as the shortest maximum distance between two vertices in G, that is $diam(G) = max\{dist(x, y) : x, y \in V(G)\}$.

A rooted tree is a tree whereby the vertex called the root is distinguished from the other vertices of the tree. Let T denotes the rooted tree. Vertex of degree one is the leaf of a tree and the support vertex is a vertex adjacent to a leaf. A star and Bistar are trees with one and two non-leaf vertices respectively.. Let $S(T)$ and $L(T)$ denotes the set of all support vertices and the set of leaves in T respectively. Let $|L(T)| = l(T)$ and $s(T) = |S(T)|$, we denote $L(u)$ as the set of all leaves adjacent to a support vertex u and $l(u) = |L(u)|$. Let $I(T)$ denotes the set of vertices in T that are neither root, support nor leaf vertices. Also, let $i = |I(T)|$.

A subset $D \subset V$ is known as a *dominating set* of G if every vertex u in $V \setminus D$ has a neighbor in D. The dominating set with minimum cardinality is known as the *domination number* $\gamma(G)$ *of* G*. Let* $\beta \in \{0, 1, 2\}$ *and for any vertex* $u \in G$*, we denote the set of vertices with* $f(u) = \beta$ *by* V_β*.*

Slater [1][2] introduced the study of locating dominating sets in graphs whereby he studied many graph related problems with various types of protection. His objective in the work is to locate the intruder. A locating dominating set $D \subset V(G)$ is a dominating set with the property that the set $N(u) \cap D$ is unique for each vertex $u \in V(G) \setminus D$. The locating dominating set of G with minimum cardinality is known as locating domination number of G. Several domination parameters in the concept of locating domination has been considered, for more result, see [3][4][5][6].

A Roman dominating function (RDF) on G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex $v \in V(G)$ with $f(v) = 0$ is adjacent to at leaast one vertex u with $f(u) = 2$. The weight of RDF is the value $f(V(G)) = \sum_{v \in V(G)} f(v)$, denoted by $w(f)$. Roman domination number denoted by $\gamma_R(G)$ is the RDF on G with minimum weight. Cockayne et al. [7] introduced Roman domination which was motivated by the work of Re Velle and Rosing [8] and Stewart [9]. See [10][11] for more results on Roman domination.

A RD-function is called a locating Roman dominating function (LRD-function) if for any pair of vertices u, v with $f(u) = f(v) = 0, N(u) \cap V_2 \neq N(v) \cap V_2$. The minimum weight of LRD-function is known as the locating Roman domination number denoted as $\gamma_R^L(G)$. See [12][13] for more result on LRD -function.

In this paper, we consider the case whereby there will be optimal location of intruder, that is, all the intruder in the whole empire will be located easily. This lead to the study of vertex-edge locating Roman dominating function. Nares Kumar and Venkatakrishnan [14][15] studied the vertex-edge Roman domination. A vertex-edge Roman dominating (ve-LRD) function on a graph G is a function $f: V(G) \rightarrow \{0,1,2\}$ with the property that for every edge $uv \in E$, either $max\{f(u), f(v)\} \neq 0$, or there exists $w \in N(u) \cup N(v)$ such that $f(w) = 2$. The vertex-edge Roman domination number of a graph G denoted by $\gamma_{veR}(G)$ is the minimum weight of a ve-RDF, i.e., $\gamma_{veR}(G) = min\{w(f) : f\ is\ a\ ve - RDF\ on\ G\}$. More result on vertex-edge domination number can be found in [16][17][18]

Our aim in this work, is to apply the analogue of vertex-edge on locating Roman domination and establish the variation vertex-edge locating Roman domination as follows.

A vertex-edge locating Roman dominating function of a graph G, abbrevated ve-LRD function is a function $f: V(G) \rightarrow \{0,1,2\}$ satisfying the conditions that (i) every adjacent vertices u, v with $u \in V_0$ or $v \in V_0$, there exists a vertex $w \in V_2$ such that $w \in N(u) \cup N(v)$; (ii) $max\{f(u), f(v)\} \neq 0$ for every edge $uv \in E$ and (iii) for any pair of distinct vertices u, v of V_0, $N(u) \cap V_2 \neq N(v) \cap V_2$.

In Section 2, we show that the vertex-edge locating Roman domination is NP complete for bipartite graphs and in Section 3, we present the upper and lower bonds of ve-LRD function for trees. In section 4, we presented the ve-LRD function of complete graphs and upper bounds of ve-LRD function for some connected graphs.

2. Complexity

In this section, we presents the complexity result for the vertex-edge locating Roman domination problem in bipartite graphs.

VERTEX-EDGE LOCATING ROMAN DOMINATION (ve-LRD)

Instance: Graph $G = (V, E)$, positive integer $k \leq |V|$.

Question: Does G have a vertex-edge Locating Roman dominating function of weight at most k?

Exact 3-cover (X3C)

Instance: A set X with $|X| = 3q$, a family C of 3-element subsets of X.

Question: Does G have a subcollection $C' \subset C$ such that any member of X appears in only one element of C'?

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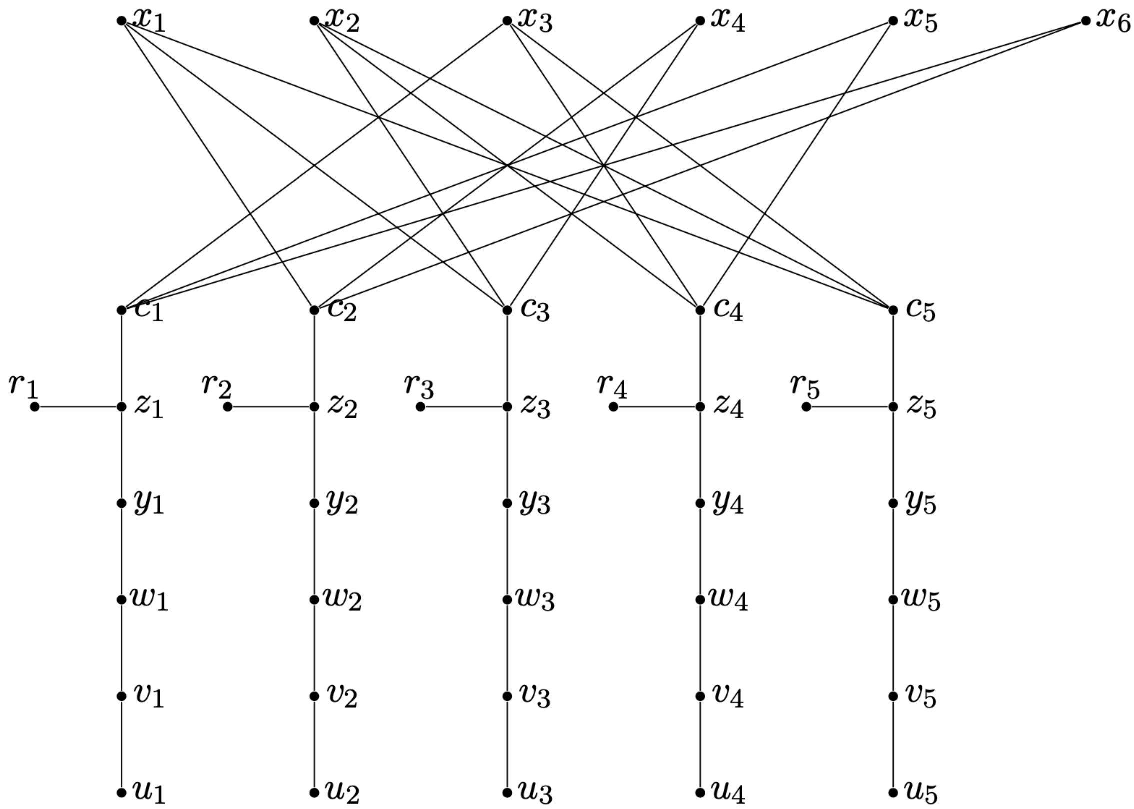

Figure 1. NP-completeness of vertex-edge locating Roman domination for bipartite graphs

Theorem 2.1. *$ve - LRD$ is NP-complete for bipartite graphs.*

Proof. ve-LRD is NP since it can be check in polynomial time that the function $f: V \to \{0,1,2\}$ is an ve-LRD and has weight at most k. Given an instance (X, C) of $X3C$ with $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$.

Bipartite graph G can be constructed as follows: for any $x_i \in X$, create a single vertex x_i. A tree T_j can be built for any $C_j \in C$ which comprises of paths $P_6 = \{u_j, v_j, w_j, y_j, z_j, c_j\}$ and $Q = \{r_1, \ldots r_t\}$ such that edges $r_j z_j$ are added to P_{j6}. To achieve the construction of G, we add edges $c_j x_i$ when $x_i \in C_j$ (see Figure 1). Set $k = 5t + q$. Observe that for every ve-LRD, each P_6 has weight at least 5. The leaf neighbor r_j of z_j, w_j and c_j has weight 0, while vertices u_j, v_j and z_j are assign 1 and y_j must be assigned 2.

Suppose C' is a solution of the instance (X, C) of $X3C$. Then ve-LRD function f on G of weight k can be constructed as follows: Assign value 0 to x_i for each i, then for each j, if $C_j \in C'$, assign value 2 to vertex z_j, value 0 to vertex w_j and 1 to the remaining vertices of P_{j6}. Also, assign 0 to the vertices of Q_j.

If $C_j \notin C'$, assign 2 to vertex y_j in each P_{j6}, assign 0 to vertices w_j, c_j and the set Q_j. Assign 1 to the

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remaining vertices of P_{j6}.

Note that since C' exists, the order of C' is q and so the number of c_j with value 1 is q and every vertex in X is at distance two to vertex z_j with value 2. Therefore, f is ve-LRD with weight $f(V) = 5t + q = k$.

Conversely, suppose that G has ve-LRD function with weight at most k. Let $\alpha = (V_0, V_1, V_2)$, observe from above, each P_{j6} has weight at least 5. We may assume that $\alpha(z_j) = 2$ if $C_j \in C'$ and $\alpha(y_j) = 2$ if $C_j \notin C'$. It is clear that vertices of P_{j6} with value 0 is at distance two or one from the vertex assign 2 such that any pair of vertices with value 0 does not have a common neighbor assign 2 under α. Now since $w(\alpha) \leq 5t + q$, we can see that $X \cap V_0 \neq \phi$. If $\alpha(x_i) > 0$ for some i, then this provides an ve-LRD function of weight at most k with weight greater than α. Therefore $X \subset V_0$. Now, since each vertex of X is at distance two from a vertex in V_2 and the sum of end points of each edge must be greater than 0, each α has exactly three neighbors in $\{x_1, x_2, \ldots, x_{3q}\}$. This will be possible only if there are q vertices z_j of T_j that are assign 2 and q vertices c_j of T_j that are assign 1. We conclude that $C' = \{C_j : \alpha(c_j) = 1\}$ is an exact cover for C. □

3. Vertex-edge locating Roman domination of trees

In this section, we gave the value of vertex-edge domination number of paths. We also gave the upper bound of ve-LRD function for bistar. Lastly, we establish the lower and upper bounds of ve-LRD function for tree T in terms of l leaves, s support vertices and i internal vertices. We begin with the following result.

Proposition 3.1. *For* $n \geq 3$, $\gamma_{veLR}(P_n) = \begin{cases} \frac{4n+k}{5}, & \text{if } n \equiv k \mod 5 \text{ and } k \neq 4 \\ \frac{4n+k}{5} - 1, & \text{if } n \equiv k \mod 5 \text{ and } k = 4 \end{cases}$

Proof. Let $P_n = u_1, u_2, \ldots, u_n$ be a path of order $n > 2$. Let f be a function defined on the $V(P_n)$ as $f : V(P_n) \to \{0, 1, 2\}$. The problem can be split into the following two cases.

Case 1: If $n \equiv k \mod 4, 0 \leq k \leq 3$. The function f is define as

$$f(u_j) = \begin{cases} 0, & \text{if } j \equiv 1 \text{ or } 4 \mod 5 \text{ and } j < n-k \\ 1, & \text{if } j \equiv 0 \text{ or } 2 \mod 5, j < n-k \text{ and } n-(k+1) \leq j \leq n \\ 2, & \text{if } j \equiv 3 \mod 5 \text{ and } j < n-k \end{cases}.$$

Case 2: If $n \equiv k \mod 5$ and $k = 4$.

Define f on $V(P_n)$ as follows:

$$f(u_j) = \begin{cases} 0, & \text{if } j = n-3, n \\ 1, & \text{if } j = n-2 \\ 2, & \text{if } j = n-1 \\ f(u_j) \ in \ case \ 1 \ above, & \text{otherwise.} \end{cases}.$$

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Clearly, f is a ve-LRD of P_n and thus

$$\gamma_{veLR}(P_n) \leq \begin{cases} \frac{4n+k}{5}, & \text{if } n \equiv k \mod 5 \text{ and } k \neq 4 \\ \frac{4n+k}{5} - 1, & \text{if } n \equiv k \mod 5 \text{ and } k = 4 \end{cases}$$

To proof the inverse inequality, we establish it by induction on n. Let P' be the a path obtained from path P_n by removing one vertex (say u_n) from the path P_n. Then P' is a path of order $n' = n - 1$. If $f(u_n) \geq 1$, then the retriction of f on P' will give ve-LRD on P' , that is $w(f) \geq \gamma_{veLR}(P') + 1$. Thus, if $k \neq 4$,

$$\begin{aligned} w(f) &\geq \gamma_{veLR}(P') + 1 \\ &= \frac{4n' + k'}{5} + 1 \\ &= \frac{4(n-1) + k - 1}{5} + 1 \\ &= \frac{4n + k}{5}. \end{aligned}$$

If $k = 4$, we have

$$\begin{aligned} w(f) &\geq \gamma_{veLR}(P') + 1 \\ &= \frac{4n' + k'}{5} - 1 + 1 \\ &= \frac{4(n-1) + k - 1}{5} - 1 + 1 \\ &= \frac{4n + k}{5} - 1. \end{aligned}$$

Thus,

$$\gamma_{veLR}(P_n) \geq \begin{cases} \frac{4n+k}{5}, & \text{if } n \equiv k \mod 5 \text{ and } k \neq 4 \\ \frac{4n+k}{5} - 1, & \text{if } n \equiv k \mod 5 \text{ and } k = 4 \end{cases}$$

Using the induction hypothesis, we get the desired lower bound. Hence, the equality holds. □

Observation: For any star graph S_n, $\gamma_{veLR}(S_n) = 3$.

Proposition 3.2. *For any bistar BS_n of order $n \geq 6$, $\gamma_{veLR}(BS_n) \leq 6$.*

Proof. Let u, v be the support vertices in BS_n and define a function $f : V(BS_n) \to \{0, 1, 2\}$ as follows: If $l_u \leq 2$ and $l_v \geq 2$, then set $f(u) = 2$ and $f(v) = 1$. Also, assign 1 to only one leaf neighbor of support vertex u and assign 0 to the remaining leaves in BS_n. The above assignment will give a ve-LRD function with weight 4.

If $l_u = 3$ and $l_v \geq 3$, set $f(u) = 2, f(v) = 1$. Assign 1 to only two leaves neighbors of u and 0 to the remaining leaves in BS_n. The labeling gives a ve-LRD function with weight 5.

If $l_u, l_v \geq 3$, set $f(u) = f(v) = 1$ and $f(x) = f(y) = 2$, where $x \in l_u$ and $y \in l_v$. Assign 0 to the

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remaining leaves in BS_n. This gives a ve-LRD function with weight 6. In all cases, $\gamma_{veLR}(BS_n) \leq 6$.

Assume that BS_n admits a ve-LRD function h with $w(h) > 6$. Assume that h is of minimum weight. If $\{h(s), h(r)\} \geq 1$, where $s \in l_u \setminus x$ and $r \in l_v \setminus y$, then the restriction of h on $BS_n - \{s, r\}$ is a ve-LRD function on $BS_n - \{s, r\}$ with weight less than 6, a contradiction. Thus $h(s) = h(r) = 0$. Hence, $\gamma_{veLR}(BS_n) \leq 6$. □

Theorem 3.2. *If T is a tree of order $n \geq 6$ with l leaves, s support vertices and i internal vertices and T is not a path, then $\gamma_{veLR}(T) \leq n - l + 2s - i$.*

Proof. We establish the proof by induction on n. Assume that $diam(T) \geq 4$, let $u_0, \ldots, u_t$ be a diametral path . Then u_0 and u_t are the root and leaf respectively and u_{t-1} is a support vertex. We split the proof into the following cases:

Case 1: $d(u_{t-1}) \geq 3$. Then u_{t-1} is adjacent to atleast two leaves. Let T' be the tree obtained from T by deleting u_t. Then T' has order $n' = n - 1$ with $l' = l - 1$, $s' = s$ and $i' = i$. By induction hypothesis, T' admits ve-LRD function f' such that $w(f') \leq n' - l' + 2s - i$. Define a function $f : V(T) \to \{0, 1, 2\}$ as follows:

If $f'(u_{t-2}) = 2$ or $f'(u_{t-1}) \geq 1$, set $f(u_t) = 0$ and $f(x) = f'(x)$ for all $x \in T - u_t$, if $f'(u_{t-2}) < 2$ and $f'(u_{t-1}) \leq 1$, then u_{t-1} is adjacent to a leaf y in T' with $f'(y) \geq 1$, set $f(u_{t-2}) = 2, f(u_{t-1}) = 1$ and $f(y) = f(u_t) = 0$. Also, $f(v) = f'(v)$ for all $v \in T - \{u_{t-2}, u_{t-1}, u_t, y\}$. Then f is a ve-LRD function on T of weight

$$\begin{aligned} w(f) &\leq w(f') \\ &\leq n' - l' + 2s' - i' \\ &= n - 1 - l + 1 + 2s - i \\ &= n - l + 2s - i. \end{aligned}$$

Thus the statement is true.

Case 2: If $d(u_{t-1}) = 2$.

Subcase I: If $d(u_{t-2}) = 2$. Let T' be the tree obtained from T by deleting u_{t-1} and u_t. Then $n' = n - 2, l' = l, s' \leq s$ and $i' \leq i$. T' admits ve-LRD function f' by induction hypothesis such that $w(f') \leq n' - l' + 2s' - i'$. Define $f : V(T) \to \{0, 1, 2\}$ by $f(u_{t-1}) = f(u_t) = 1$ and $f(x) = f'(x)$ for all $x \in T - \{u_{t-1}, u_t\}$. The assignment gives a ve-LRD function f on T with weight

$$\begin{aligned} w(f) &= w(f') + 2 \\ &\leq n' - l' + 2s' - i' + 2 \\ &\leq n - 2 - l + 2s - i + 2 \\ &= n - l + 2s - i. \end{aligned}$$

Therefore, the statement holds.

Subcase II: $d(t_{t-2}) \geq 3$

Let T' be the tree obtained from T by deleting u_{t-1} and u_t. Then $n' = n-2, l' = l-1, s' \leq s$ and $i' = i$. By induction hypothesis, T' admits a ve-LRD function f' with $w(f') \leq n' - l' + 2s' - i'$. Define function f in T as follows: If $f'(u_{t-2}) = 1$ and $f'(u_{t-3}) = 2$, set $f(u_{t-1}) = 0$ and $f(u_t) = 1$. Also, if $f'(u_{t-3}) = 2$ and $f'(u_{t-2}) = 0$, set $f(u_{t-1}) = 1$ and $f(u_t) = 0$. If $f'(u_{t-3}) = 1$ and $f'(t-2) = 2$, set $f(u_{t-1}) = 1$ and $f(u_t) = 0$. Therefore, the labeling gives a ve-LRD function f on T with weight

$$\begin{aligned} w(f) &= w(f') + 1 \\ &\leq n' - l' + 2s' - i' + 1 \\ &\leq n - 2 - l + 1 + 2s - i + 1 \\ &= n - l + 2s - i. \end{aligned}$$

□

Theorem 3.4. *If T is a treee with $diam(T) \geq 4$, l leaves, s support vertices and i internal vertices, then $\gamma_{veLR}(T) \geq \frac{n-l+s-i}{2}$.*

Proof. We use induction on n to establish the proof. Assume that $|T| \geq 5$, let T' be an arbitrary tree of order n' such that $|T'| < |T|$ with $diam(T') \geq 3$. Assume that the statement is true for any tree T'. Also, let l', s', i' be the order of leaves, support vertices and internal vertices in T' respectively. Assume that $diam(T) \geq 4$. Let $u_0, \ldots, u_t$ be a diametral path and f a ve-LRD function on T with minimum weight, that is $w(f) = \gamma_{veLR}(T)$

Claim 1: If $d(u_1) > 2$, then the statement is true.

Proof: The vertex u_1 is adjacent to atleast two vertices u_0 and say leaf y. Let T' be the tree obtained from T by deleting y. Then $n' = n-1, l' = l-1, s' = s$ and $i' \geq i$. Define a function $f : V(T) \to \{0,1,2\}$ on T and f' is a function define on T'. If $f'(u_1) = 1$ and $f'(u_0) = 2$ or $f'(u_2) = 2$, set $f(y) = 0$. Then the restriction of f on T' is a ve-LRD function on T', that is $w(f) \geq \gamma_{veLR}(T')$. If $f'(u_1) = 0$ and any of the vertices $u_j, j = 0,2,3$ is assign 2, set $f(y) = 1$. If $f'(u_1) = 2$ and $f(u_0) = 0$ or $f'(u_2) = 0$ or $f'(u_3) = 0$, set $f(y) = 1$. The restriction of f on T' is a ve-LRD function on T', so $w(f') \geq \gamma_{veLR}(T')$. Therefore in all cases, we have

$$\begin{aligned} w(f) &\geq w(f') \\ &\geq \frac{n' - l' + s' - i'}{2} \\ &\geq \frac{n - 1 - l + 1 + s - i}{2} \\ &= \frac{n - l + s - i}{2}. \end{aligned}$$

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Let assume that $d(u_1) = 2$.

Claim 2: If there exist $j \in \{2, .., t-2\}$ such that u_j is a support vertex in T, then the statement is true.

Proof. Let denote the leaf adjacent to u_i in T by z. Let T' be the tree obtained from T by deleting z. Then $n' = n-1, l' = l-1, s' \leq s$ and $i' \geq i$. By induction hypothesis, $\gamma_{veLR}(T') \geq \frac{n'-l'+s'-i'}{2}$. If $f'(u_i) = 1$ and $f'(u_{i-1})$ or $f'(u_{i+1}) = 2$, then set $f(z) = 0$. The restiction of f on T' is a ve-LRD function on T'; so $w(f) \geq \gamma_{veLR}(T')$. If $f'(u_i) = 0$, set $f(z) = 1$. If $f'(u_i) = 2$ and either $f'(u_{i-1})$ or $f'(u_{i+1})$ or $f'(u_{i+2}) = 0$, set $f(z) = 1$ and $f' = f$ otherwise. If neither $f'(u_{i-1})$ nor $f'(u_{i+1})$ nor $f'(u_{i+2}) = 0$ and $f'(u_i) = 2$, set $f(z) = 0$ and $f' = f$ otherwise. Thus $w(f) \geq w(f') \geq \gamma_{veLR}(T')$. If there exist $v \in N(u_j) \setminus \{z\}$ with $f'(v) = 2$, set $f(z) = 0$, the restriction of f on T' is a ve-LRD function on T', so $w(f) \geq \gamma_{veLR}(T')$. Therefore, in all cases we have

$$\begin{aligned} w(f) &\geq \gamma_{veLR}(T') \\ &\geq \frac{n'-l'+s'-i'}{2} \\ &\geq \frac{n-1-l+1+s-i}{2} \\ &= \frac{n-l+s-i}{2}. \end{aligned}$$

Thus, the statement holds.

Assume that the set $\{u_0, \dots, u_t\}$ does not have a support vertex in T. Then we have the following two cases:

Case 1: $d(u_2) > 2$. Vertex u_2 is adjacent to a support vertex say y since u_2 is not adjacent to any leaf and the path $\{u_0, \dots, u_t\}$ is the diametral path. Note that $y \notin \{u_1, u_3\}$ and y is adjacent to a leaf z. Let T' be a tree obtained from T by deleting vertices y and z. Then $diam(T') = diam(T)$, $n' = n-2, l' = l-1, s' = s-1$ and $i' = i$. If $f(u_2) \geq 1$, then the restriction of f on T' will give a ve-LRD function on T', i.e. $w(f) \geq \gamma_{veLR}(T')$. If $f'(u_2) = 0$, then $f(y) + f(z) > 1$. Define a ve-LRD function f on T as follows: If $f'(u_2) = 1$ and either $f'(u_1)$ or **———————$f'(u_3) = 2$, set $f(y) = 0$, $f(z) = 1$ and $f' = f$ otherwise. Also, if $f'(u_2) = 2$, set $f(y) = 1$ and $f(z) = 0$. If $f'(u_2) = 0$, set $f(y) = f(z) = 1$ and $f' = f$ otherwise. Thus in all cases, we have

$$\begin{aligned} w(f) &\geq \gamma_{veLR}(T') + 1 \\ &\geq \frac{n'-l'+s'-i'}{2} + 1 \\ &\geq \frac{n-2-l+1+s-1-i}{2} + 1 \\ &= \frac{n-l+s-i}{2}. \end{aligned}$$

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Thus the statement holds.

Case 2: $d(u_2) = 2$. If $diam(T) = 4$, then $T = P_5$ and by Proposition 2, $\gamma_{veLR}(P_5) = 4 > \frac{n-l+s-i}{2}$. Let assume that $diam(T) \geq 5$. Let T' be the tree obtained from T by deleting vertices u_0 and u_1. So $diam(T') \geq 3$. Also, $n' = n-2, l' = l, s' = s$ and $i' \leq i$. Assume that $f(u_0) + f(u_1) \geq 1$ and the restriction of f on f' is a ve-LRD function on T' with $w(f') \geq \frac{n-l+s-i}{2}$. Define f on T as follows: If $f'(u_2) = 2$, set $f(u_1) = 1$, $f(u_0) = 0$ and $f = f'$ otherwise. If $f'(u_2) = 1$ and $f'(u_3) = 2$, set $f(u_1) = 0$ and $f(u_0) = 1$, $f' = f$ otherwise. Also, If $f'(u_2) = 1$ and $f'(u_3) \neq 2$, set $f(u_0) = f(u_1) = 1$ and $f = f'$ otherwise. If $f'(u_2) = 0$, set $f(u_0) = f(u_1) = 1$ and $f = f'$ otherwise. Therefore, in all cases above, we have

$$\begin{aligned} w(f) &\geq \gamma_{veLR}(T') + 1 \\ &\geq \frac{n' - l' + s' - i'}{2} + 1 \\ &\geq \frac{n-2-l+s-i}{2} + 1 \\ &= \frac{n-l+s-i}{2}. \end{aligned}$$

Thus, the statement holds. □

4. Vertex-edge locating Roman domination of connected graphs

In this section, we gave the vertex-edge domination number of complete graphs and upper bound for the vertex-edge domination number of connected graphs. We begin with the following result on ve-LRD function of connected graphs.

Lemma 4.1. *Let G be a connected graph of order $n > 3$ and $G \neq K_n$. If $v \in V(G)$ with $d(v) \geq 2$, then $\gamma_{veLR}(G) \leq n-1$.*

Proof. Let $u_1, u_2 \in N_2(v)$ and let $v_1 \in N(v) \cap N(u_1)$ and $v_2 \in N(v) \cap N(u_2)$ such that $\{u_1, v_1, v, v_2, u_2\}$ is a path in G. If v has a leaf neighbor say x, the function $f : V(G) \to \{0,1,2\}$ defined by $f(v_2) = 2$, $f(v_1) = f(x) = f(u_2) = 0$ and $f(y) = 1$ for $y \in V(G) \setminus \{v_1, v_2, x, u_2\}$ is a ve-LRD function on G with weight $n-1$. Therefore, $\gamma_{veLR}(G) \leq n-1$.

If only v_1 has leaf neighbor say $x \in l_{v_1}$, then define $f : V(G) \to \{0,1,2\}$ by $f(v) = 2$, $f(x) = f(u_1) = f(v_2) = 0$ and $f(y) = 1$ for $y \in V(G) \setminus \{v, x, u_1, v_2\}$. The function f define above is a ve-LRD function on G with $w(f) \leq n-1$.

If only v_2 has a leaf neighbor, say $x \in l_{v_2}$, then define $f : V(G) \to \{0,1,2\}$ by $f(v) = 2$, $f(x) = f(v_1) = f(u_2) = 0$ and $f(z) = 1$ for $z \in V(G) \setminus \{v, x, v_1, u_2\}$. The function f gives a ve-LRD function with $w(f) \leq n-1$. If u_1 and u_2 has leaves neighbors, say $x \in l_{u_1} \cup l_{u_2}$, define function $f: V(G) \rightarrow \{0,1,2\}$ by $f(u_1) = f(u_2) = 2$, $f(x) = f(v) = 0$ and $f(t) = 1$ for $t \in V(G) \setminus \{u_1, u_2, x, v\}$. The function gives ve-LRD function with $w(f) \leq n-1$. Thus $\gamma_{veLR}(G) \leq n-1$. □

Corollary 4.2. *If T is a tree of order $n > 3$, then $\gamma_{veLR}(T) \leq n-1$.*

Theorem 4.3. *Let G be a connected graph of order $n \geq 2$, then $\gamma_{veLR}(G) = n$ if and only if $G = P_3, K_n$.*

Proof. Obviously, if $G = P_3$, $\gamma_{veLR}(P_3) = 3$ by proposition 2. Now let $G = K_n$. Suppose $\gamma_{veLR}(G) = n$, then this implies that all vertices in G are adjacent , that is, $G = K_n$. Suppose all vertices in G are not adjacent . Let $u, v \in V(G)$ such that $uv \notin E(G)$. Then $d(v) \leq n-2$ and u, v are at distance 2 from each other. Let vertex $x \in N(u) \cap N(v)$ in G. Since G is connected with $n \geq 3$, then uxv is a path of length 2 and the function f define on $V(G) \setminus \{v\}$ is a ve-LRD function in G which implies that $\gamma_{ve-LR} G \leq n-1$, a contradiction.

Assume that $G = K_n$, then all the vertices are adjacent. For $u, v \in V(G)$, define the function $f: V(G) \rightarrow \{0,1,2\}$ by $f(u) = 2, f(v) = 0$ and $f(y) = 1$ for $y \in V(G) \setminus \{u, v\}$. The above function f gives ve-LRD function of G with weight n. Therefore, $\gamma_{veLR}(G) = n$. □

Corollary 4.4. *Let G be a connected graph of order n such that $\gamma_{veLR}(G) = n$, then $diam(G) \leq 2$.*

Proof. We establish the proof by contradiction. Assume that $diam(G) \geq 3$ and let $P = u_1, u_2, \ldots, u_d$ be a diametral path in G. The vertices $\{u_2, u_6\} \in N_2(u_4)$ which implies that $d(u_2) \geq 2$ and by Lemma 4.1, $\gamma_{veLR}(G) \leq n-1$. This is a contradiction. □

Theorem 4.5. *Let G be a cycle of order $n \geq 3$, then $\gamma_{veLR}(G) = \frac{4n+k}{5}, n \equiv (k \mod 5)$.*

Proof. Applying Proposition 3.1 (case 1) for all values of *k* gives the desired result. □

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