New Taylor theorem for non-differentiable functions and a simplification of the original Taylor theorem

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Abstract

We introduce new Taylor expansions when the function is not differentiable. Moreover, we simplify the classical Taylor theorem.

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We introduce a pioneering approach that overcomes an obstacle in mathematical sciences. In doing so, we introduce Taylor-like expansions even if the original function is not differentiable. Needless to say, this result is very useful in many applications. We introduce new Taylor expansions when the function is not differentiable. Moreover, we simplify the classical Taylor theorem (see Taylor (1715)).

Consider this function $f(x)$ on R that is not differentiable with respect to x. Also, consider this finite difference quotient for a non-differentiable function

$$\frac{f(x) - f\left(c_1\right)}{x - c_1} \qquad (1)$$

where c is an arbitrary constant. There is a subderivative $\bar{f}'$ (or a value $\hat{f}$ such that

$$\frac{f(x) - f\left(c_1\right)}{x - c_1} = \bar{f}'\left(x_i\right) \qquad (2)$$

where $x_i \in \left(c_1, x\right)$. Therefore,

$$f(x) = f\left(c_1\right) + \bar{f}'\left(x_i\right)\left(x - c_1\right) \qquad (3)$$

We note that, practically, it doesn't make a difference whether the expansion includes a derivative or a subderivative. Also, if f is locally non-differentiable (i.e., the cusp is at a small interval), we can possibly find a local derivative such that $f'\left(x_j\right) = \bar{f}'\left(x_i\right)$. Thus

$$f(x) = f\left(c_1\right) + f'\left(x_j\right)\left(x - c_1\right) \qquad (4)$$

Alternatively, there is a value $\hat{x}$ that satisfies $\hat{x} = \frac{f(x) - c_2}{x - c_1}$, where $\hat{x}$ depends on x and c. Thus, the expansion can be presented in a simpler form as follows

$$f(x) = c_2 + \hat{x}\left(x - c_1\right) \qquad (5)$$

This can be extended to a higher order. To do so, consider this expansion

$$\hat{x} = c_3 + \check{x}\left(x - c_1\right) \qquad (6)$$

Substituting (5) into (6) yields

$$f(x) = c_2 + c_3\left(x - c_1\right) + \check{x}\left(x - c_1\right)^2 \qquad (7)$$

This recursive substitution can be used to obtain any order as follows%

$$f(x) = c_0 + \sum_{i=1}^{n} c_i\left(x - c_1\right)^i + \check{x}\left(x - c_1\right)^{n+1} \qquad (8)$$

Therefore, we don't have to deal with derivatives or subderivatives and we don't need to know the functional form. This form of the expansion is more convenient and useful, since it is simpler (we deal with arbitrary constants); and it enables us to avoid the differentiability problem. Furthermore, it can have more applications (for example, the solutions of differential equations). Consequently, this form can be used even if f is differentiable. Thus, this simplifies the classical Taylor's expansions.

The extension to a multiple-variable function is straightforward. For example, for a two-variable function, consider this expansions

$$f(x, y) = c_0 + \hat{x}\left(x - c_1\right) + \hat{y}\left(y - c_2\right)$$

and

$$\hat{x} = c_3 + \ddot{x}\left(x - c_1\right) + \ddot{y}\left(y - c_2\right)$$

$$\hat{y} = c_6 + x\left(x - c_1\right) + y\left(y - c_2\right)$$

Recursive substitutions yields

$$f(x,y) = c_0 + \sum_j \sum_i c_{i,j} \left(x - c_1\right)^i \left(y - c_2\right)^j + R$$

where R is the remainder.

References

- Taylor, Brook (1715). Methodus Incrementorum Directa et Inversa
  [Direct and Reverse Methods of Incrementation] (in Latin). London. p. 21--23
  (Prop. VII, Thm. 3, Cor. 2).