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Fractional Dynamics and Physics Beyond Effective Field Theory (Part 1)

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Abstract

Recent research points out that physics far beyond the Standard Model (SM) scale may substantially deviate from the principles of traditional field theory. The goal of this report is to briefly elaborate on the motivation for *fractal spacetime* and *fractional dynamics* in the deep ultraviolet sector of field theory.

Key words: complex dynamics, fractal spacetime, fractional dynamics, effective field theory, far-from-equilibrium phenomena, primordial cosmology.

1. Introduction and Motivation

The last three decades have given us clues that the evolution of large systems of nonlinearly interacting components is prone to become *non-integrable* in

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the long run. Reliable modeling of such systems requires new concepts and methods inspired by chaos theory, multifractal geometry, critical phenomena, self-organized criticality, and fractional calculus. These non-conventional tools form the mathematical basis of *complex dynamics* [see e.g. 15].

It is well known that variational principles underlie both Lagrangian and Hamiltonian formulations of dynamics and are the key paradigm of classical and quantum physics. What has been only recently realized is that Hamilton's equations of motion cannot properly describe *complex phenomena* [see e.g. 12]. The reason is that standard variational principles fail when applied to processes having *long-range memory effects* and/or *nonlocal spatial interactions* [1-2, 8, 9, 12]. These observations go hand in hand with the known limitations of effective field theory (EFT) in explaining the many open questions of the SM and of far-from-equilibrium gravitational physics. Despite decades of advanced research, the challenges of EFT and field unification are still outstanding. What aggravates the situation is the steady avalanche of unmotivated models, claiming to resolve some unsatisfactory aspects of the theory at the expense of postulating ad-hoc variables, constraints, or exotic symmetry groups.

To make progress, one clearly needs to take a step back and revisit the fundamental principles which have guided theory development for the last couple of centuries. Given the proven benefits of complex dynamics as both framework and modeling tool, several foundational questions need to be scrutinized "from scratch". For example,

1) Is Nature universally described by Lagrangian field theory, specifically, by local effective Lagrangians compatible with the consistency requirements set by Quantum Field Theory (QFT) [16 – 17, 20]?

2) Are symmetries, dualities, and mathematical concepts derived from them universal and applicable to the dynamics of non-integrable phenomena?

3) Do abstract mathematical objects (like spinors, quaternions, octonions, twistors, knots, strings) or exotic algebras offer robust explanations for the challenges of field unification and/or of the SM?

4) Is Nature governed by deterministic laws, or is it fundamentally sensitive to initial conditions at *all levels of observation*?

5) How justified is the belief that the behavior of few-body systems follows a local, unitary, integrable, time-reversible, and analytic evolution *outside* the boundaries of EFT?

6) How strong is the cosmological principle, and is it valid over an arbitrarily large range of cosmological scales? Do unexplained ultra-massive galactic structures provide evidence against the cosmological principle and in favor of a locally non-homogeneous spacetime topology? [in the spirit of 10 - 11, 13 - 14].

7) If the Universe has a large-scale multifractal topology, where does this topology come from [10 – 11, 13]? How likely is it that the root cause is the

onset of a spacetime endowed with continuous dimensions well above the SM scale?

8) What are the implications of *criticality in continuous dimensions* in the ultraviolet sector of physics and primordial cosmology?

The goal of this report is to briefly elaborate on the motivation for *fractal spacetime* and *fractional dynamics* outside the boundaries of EFT. The report is divided into two parts: First part traces the route from Dimensional Regularization of QFT to the fractal topology of spacetime and fractional dynamics above the SM scale. The second part covers the dimensional reduction conjecture, multifractal clustering in cosmology and the SM, and the emergence of Dark Matter in primordial cosmology.

We caution the reader that this report is entirely provisional and far from meeting the requirements of a full-fledged academic paper. Independent research and data are required to validate, develop, or debunk the ideas presented below.

2. From Dimensional Regularization to Fractal Spacetime

Euclidean formulation of the Path Integral in QFT enables a meaningful analogy between QFT and critical phenomena [5]. To highlight this analogy, consider the two-point function of massive scalar field theory. The Euclidean propagator in momentum space reads

$$D_E(p) = \frac{1}{p^2 + m^2} \tag{1}$$

and the correlation function of the corresponding statistical system is given by,

$$\langle \varphi(x)\varphi(0) \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{\exp(ipx)}{p^2 + m^2} \tag{2}$$

in which $|p^2| = p_\mu p_\mu$ and $px = p_\mu x_\mu$. In the limit $m|x| >> 1$, (2) is well approximated by,

$$\langle \varphi(x)\varphi(0) \rangle \approx \frac{1}{|x|^2} \exp(-m|x|) \tag{3}$$

Let us assume that the field is placed on a four-dimensional lattice of points separated by a fixed spacing $a = \Lambda_{UV}^{-1}$, in which Λ_{UV} is the cutoff scale. The spatial coordinate is then given by,

$$|x| = Na = N\Lambda_{UV}^{-1}, \quad N >> 1 \tag{4}$$

and (3) can be written as,

$$\langle \varphi_N \varphi_0 \rangle \propto \exp[-N(m/\Lambda_{UV})] \tag{5}$$

By analogy with statistical mechanics, (5) defines the dimensionless correlation length according to,

$$\langle \varphi_N \varphi_0 \rangle \propto \exp(-N/\xi) \tag{6}$$

where,

$$\xi = \frac{\Lambda_{UV}}{m} \tag{7}$$

Dimensional Regularization in momentum space sets up a relationship between the cutoff scale Λ_{UV} and the dimensional deviation from four-space dimensions $\varepsilon(\mu) = 4 - d(\mu) << 1$ as in [3 - 4]

$$\Lambda_{UV}^2 = \mu^2 \exp(1/\varepsilon) >> \mu^2 \tag{8}$$

where μ is the running scale. The asymptotic limit $m = O(\mu) << \Lambda_{UV}$ leads to the raw estimate,

$$\boxed{\varepsilon(\mu) = 4 - d(\mu) = O[\, m^2(\mu)/\Lambda_{UV}^2\,) << 1} \tag{9}$$

By (7) and (9), we arrive at the effective approximation,

$$\boxed{\xi(\mu) \propto [\varepsilon(\mu)]^{-1/2}} \tag{10}$$

As a diverging correlation length is a characteristic feature of critical phenomena, (10) indicates that removing the dimensional regulator in QFT (that is, taking the classical continuum limit $\varepsilon \to 0$) is *analogous to tuning the corresponding statistical system towards the critical point*. In this sense, (10) underlies the idea of *criticality in continuous dimension* $d(\mu)$, conjectured to play a key role in the ultraviolet regime of field theory and primordial cosmology [18 - 19]. In particular, (10) implies the following:

1) Since, by definition, fractal structures are characterized by continuous dimensions and are the underlying geometry of both critical phenomena and chaotic behavior, (10) leads to the conclusion that taking the limit $\varepsilon \to 0$ in dimensional regularization turns the classical spacetime into a *minimal fractal manifold*.

2) It is known that the diverging correlation length of critical phenomena corresponds to the reciprocal of a mass parameter ($\xi \propto m^{-1}$). It follows from (10) that the *concept of mass is closely tied to the continuous dimensionality of fractal spacetime*, as argued in [3 – 4].

3. From Dimensional Regularization to Fractional Dynamics

According to [6–7], Dimensional Regularization of scalar field theory amounts to changing the momentum-space differential volume $d^4 v = d^4 p$ in Feynman diagrams to,

$$d^4\nu_\varepsilon = \left|p/\Delta\right|^{-\varepsilon} d^4 p \tag{11}$$

in which Δ is a fixed momentum scale and the dimensional deviation is set to $\varepsilon = 4 - d > 0$. Since (11) replicates the concept of measure in fractional calculus, the natural question is: Does (11) provide a natural bridge between Dimensional Regularization and fractional dynamics?

To answer this question and following [21], consider the probability distribution function in momentum space $P(p,\tau)$ with τ playing the role of an independent parameter. The normalization condition requires,

$$\int_{-\infty}^{+\infty} P(p,\tau) = \int_{-\infty}^{s} P(p,\tau) + \int_{s}^{+\infty} P(p,\tau) = 1 \tag{12}$$

Fractional integration of arbitrary order α on $(-\infty, s)$ and $(s, +\infty)$ is defined through the equations,

$$(I_{+}^{\alpha} P)(s,\tau) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{s} \frac{P(p,\tau)dp}{(s-p)^{1-\alpha}} \tag{13a}$$

$$(I_{-}^{\alpha} P)(s,\tau) = \frac{1}{\Gamma(\alpha)} \int_{s}^{+\infty} \frac{P(p,\tau)dp}{(p-s)^{1-\alpha}} \tag{13b}$$

Fractional generalization of the normalization condition (12) reads,

$$(I_{+}^{\alpha}P)(s,\tau)+(I_{-}^{\alpha}P)(s,\tau)=1 \tag{14}$$

which can be shown to assume the form,

$$\int_{-\infty}^{+\infty}\overline{P}(p,\tau)\,d\nu_{\alpha}(p)=1 \tag{15}$$

where,

$$\overline{P}(p,\tau)=\frac{1}{2}[P(s-p,\tau)+P(s+p,\tau)] \tag{16}$$

and,

$$d\nu_{\alpha}(p)=\frac{1}{\Gamma(\alpha)}\left|p\right|^{\alpha-1}dp \tag{17}$$

Relation (17) may be generalized to a four-dimensional momentum space, namely,

$$d^{4}\nu_{\alpha}(p)\propto\left|p\right|^{4(\alpha-1)}d^{4}p \tag{18}$$

Comparative inspection of (11) and (18) sets up an analogy between the order of fractional integration α to dimensional deviation ε as in,

$$\boxed{\alpha \Leftrightarrow 1-\frac{\varepsilon}{4}=\frac{1}{4}d} \tag{19}$$

Let $x_t = x_t(x_0)$ denote a time-dependent one-dimensional variable of a generic dynamical system. It can be shown that the generalization of the Liouville equation (LE) on a fractal spacetime (9) can be presented as [21]

$$\frac{d\overline{P}(x_t,t)}{dt}+\Omega_\varepsilon(x_t,t)\overline{P}(x_t,t)=0 \tag{20}$$

in which the omega function is

$$\Omega_\varepsilon(x_t,t)=\frac{d}{dt}\ln(\left|x_t\right|^{-\varepsilon}\frac{\partial x_t}{\partial x_0}) \tag{21}$$

To give an elementary example of (20)-(21), consider a non-relativistic particle of unit mass $(m=1)$ acted upon by Newtonian forces. The equations of motion are,

$$p = \frac{dx}{dt}, \quad \frac{dp}{dt} = F(x) \tag{22}$$

and the omega function (21) reduces to,

$$\Omega_{\varepsilon}(x,p) = -\varepsilon\,[\frac{p}{x} + \frac{F(x)}{p}] \tag{23}$$

Since, by (9), dimensional deviation is scale-dependent, equations (10), (20), (21) and (23) show that the probability density $\overline{P}(x_t, t)$ is also scale-dependent, with the exception of classical spacetime ($\varepsilon = 0$) where Liouville equation defaults to its standard form.

4. Fractional Dynamics and Physics Above the SM Scale

As embodiment of flow continuity in physics, the relevance of LE can hardly be overstated. LE provides the starting point in the derivation of the *Fokker-Planck equation (FPE)*, which models the diffusion of a Brownian particle in nonlinear potentials. As it is known, Brownian motion is a Markovian process, whereby each step of particle diffusion is independent of the previous steps. By extension, *fractional Fokker-Planck equation* (FPPE) applies to non-Markovian processes exhibiting long-range memory and/or long-range spatial interactions.

To appreciate the relevance of FFPE above the SM scale ($v = 246\,\text{GeV}$), consider an ensemble of N classical fields $z(x)$ operating in a spacetime having nearly vanishing fractality (9). As argued in [1 - 4], this spacetime background is conjectured to surface somewhere between the electroweak and Planck scales ($M_{Pl} > \mu >> v$). In line with [8], fractional dynamics of $z(x)$ can be presented as

$$z(x) = F_x z; \;\; z(x_0) = z \tag{24}$$

where F_x denotes the evolution operator, $z = \left(z_1, z_2, ..., z_N\right)$ and $x = \left(x_\mu\right)$, $\mu = 0,1,2,3$. Let's assume for simplicity that the behavior of z is time-dependent only, that is, $z(x) = z(x_0) = z(t)$. If z's are acted upon by the nonlinear potential $U(z) = -\int_z f(z')dz'$, the single-time probability density function (pdf) of $z(x)$ obeys the FPPE equation written as,

$$\frac{\partial \overline{P}(z,t)}{\partial t} = K_{\varepsilon} D_t^{\varepsilon} L_{FP} \overline{P}(z,t) \tag{25}$$

Here, the Fokker-Planck operator consists of a drift and a diffusion term, respectively,

$$L_{FP} = -\frac{\partial}{\partial z}[\frac{f(z)}{k_B T}] + \frac{\partial^2}{\partial z^2} \tag{26}$$

in which K_{ε} is the diffusion coefficient and $k_B T$ the thermal energy. Both quantities are classically related through the Einstein equation at $\varepsilon = 0$,

$$K = \frac{k_B T}{\gamma} \tag{27}$$

in which γ stands for the dissipation coefficient. The fractional operator D_t^{ε} reflects the contribution of long-range memory effects and has the form,

$$D_t^{\varepsilon} \varphi(t) = \frac{\partial}{\partial t} \frac{1}{\Gamma(1-\varepsilon)} \int_0^t \frac{\varphi(t')}{(t-t')^{\varepsilon}} dt' \tag{28}$$

To enable a comparison with QFT, it is instructive to appeal to the generic *two-time correlation function* of the field, expressed in closed form as [8 - 9],

$$C_z(t_2,t_1) \propto (\langle z^2 \rangle_B - \langle z \rangle_B^2) \frac{B(t_1/t_2, 1-\varepsilon, \varepsilon)}{\Gamma(1-\varepsilon)\Gamma(\varepsilon)} + \langle z^2 \rangle_B \tag{29}$$

for $t_2 \geq t_1$, where the Beta function is given by,

$$B(s,a,b) = \int_0^s y^{a-1}(1-y)^{b-1} dy \tag{30}$$

and

$$\langle z^2 \rangle_B = \int_{-\infty}^{\infty} z^2 \exp(-U(z)/k_B T) dz \Big/ \int_{-\infty}^{\infty} \exp(-U(z)/k_B T) dz \tag{31}$$

We close this section with a couple of observations concerning (29) - (31), namely,

a) (29) – (31) *must default* to the QFT two-point correlation function in the classical limit $\varepsilon = 0$.

b) (29) – (31) may be viewed as generalization of *cosmological correlation functions* and it may naturally account for the scaling relationships of standard cosmology [see e.g. 8, 10-11, 13].

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