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Correspondence of manifestly and implicit Lorentz-covariant treatments of the relativistic Lagrangian mechanics: Euler–Lagrange covariant equations revisited

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The equations of manifestly Lorentz-covariant Lagrangian mechanics are derived from implicitly Lorentz-covariant Euler–Lagrange equations. The resulting equations coincide with the Kalman equations, originally derived by a different way. However, the inverse transformation of the Kalman equations into Euler–Lagrange ones requires some restrictions of the manifestly covariant Lagrangian and the variational procedure. The Kalman equations are modified for one-to-one correspondence with implicitly covariant treatment of the mechanics. It is shown that the use of proper time as a non-variable parameter does not satisfy the correspondence principle of manifestly and implicit treatments.

I. INTRODUCTION

There are implicitly Lorentz-covariant and manifestly Lorentz-covariant treatments of Lagrangian relativistic mechanics [1-5]. They correspond to two different ways of obtaining the Euler–Lagrange equations. In both approaches, the concept of a Lorentz-invariant action S is introduced. In the implicit approach, it is written as a definite integral over the time t of a given inertial frame of reference in the Minkowski space with $(\mathbf{x},t)$-coordinate system:

$$S = \int_{t_a}^{t_b} \Lambda(\mathbf{x}, \mathbf{v}, t)dt\,, \tag{1}$$

where $\Lambda(\mathbf{x}, \mathbf{v}, t)$ is the Lagrange function (Lagrangian) that is not invariant with respect to Lorentz transformations. It depends on the spatial coordinates $\mathbf{x}$, the time t and the three-dimensional velocity of the particle

$$\mathbf{v} = \frac{d\mathbf{x}}{dt}. \tag{2}$$

The limits of integration in Eq. (1) are fixed points in time t_a and t_b corresponding to the initial and final positions of the particle at some fixed points a and b in three-dimensional space: $\mathbf{x}(t_a) = \mathbf{x}_a$ and $\mathbf{x}(t_b) = \mathbf{x}_b$, respectively. The variation of action (1) can be represented as

$$\delta S = \int_{t_a}^{t_b} \left(\frac{\partial \Lambda}{\partial \mathbf{x}} \delta \mathbf{x} + \frac{\partial \Lambda}{\partial \mathbf{v}} \delta \mathbf{v} + \frac{\partial \Lambda}{\partial t} \delta t \right) dt + \int_{t_a}^{t_b} \Lambda \delta dt = 0\,. \tag{3}$$

The variation corresponding to three spatial degrees of freedom and time [6] gives the Euler–Lagrange equations

$$\frac{\partial \Lambda}{\partial \mathbf{x}} - \frac{d}{dt}\frac{\partial \Lambda}{\partial \mathbf{v}} = 0 \tag{4}$$

and

$$\frac{\partial \Lambda}{\partial t} + \frac{d}{dt}\left\{ \mathbf{v}\frac{\partial \Lambda}{\partial \mathbf{v}} - \Lambda \right\} = 0\,, \tag{5}$$

respectively. Using Noether's theorem for a closed system of particles [6], it can be shown that the spatial components of the generalized momentum of a particle, which is an integral of motion, are determined by

$$\mathbf{P} = \frac{\partial \Lambda}{\partial \mathbf{v}}, \tag{6}$$

and the energy satisfying the conservation law has the form

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$$E = \mathbf{v}\frac{\partial \Lambda}{\partial \mathbf{v}} - \Lambda = \mathbf{vP} - \Lambda . \tag{7}$$

Then equations (4)-(5) can be rewritten in the form

$$\frac{\partial \Lambda}{\partial \mathbf{x}} - \frac{d\mathbf{P}}{dt} = 0 , \tag{8}$$

$$\frac{\partial \Lambda}{\partial t} + \frac{dE}{dt} = 0 . \tag{9}$$

Eqs. (8) and (9) are invariant with respect to the following gauge transformation of the Lagrangian

$$\Lambda \to \Lambda + \frac{d}{dt} f(\mathbf{x},t) \equiv \Lambda + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} + \frac{\partial f}{\partial t} , \tag{10}$$

where $f(\mathbf{x},t)$ is an arbitrary differentiable scalar function of three-dimensional coordinates and time.

Thus, within the framework of the implicit treatment, the action is varied along a three-dimensional trajectory between two points in three-dimensional space that the particle passes at fixed points in time of a given inertial frame of reference. This is contrasted with a manifestly treatment, which operates with the Lorentz-invariant Lagrangian L. In this case, the four-coordinates of Minkowski space, $x^{\mu} \equiv \left(x^0, x^1, x^2, x^3\right)$ including three spatial coordinates $\mathbf{x}$, and time coordinate $x^0 = ct$ in a given inertial frame of reference, should appear in the equations of motion of the particle in the same way.

In this work, Greek tensor indices run from 0 to 3, and Latin indices vary from 1 to 3. The two repeating Greek symbols, i.e., covariant and contravariant (lower and upper) indices, imply summation. Latin indices always correspond to purely spatial contravariant components in Minkowski space. The signature is (+ - - -). Summation is also implied by two repeating Latin tensor indices: $A^i B^i \equiv \mathbf{AB}$ (scalar product of the vectors).

Within the framework of manifestly formalism, the concept of the so-called invariant parameter of evolution has become the most widespread. As indicated in the historical review [7], this concept was first declared to the work of V.A. Fock [8]. Further, in various modifications, it is included in almost all authoritative monographs and textbooks outlining relativistic Lagrangian mechanics, for example, in the fundamental monograph [1] that has the large number of citations on this subject. The action is written as an integral with respect to the parameter of evolution θ along the four-dimensional trajectory of the particle motion between two fixed space-time points (events) in the Minkowski four-dimensional space

$$S = \int_a^b L(x^{\mu}(\theta), \dot{x}^{\mu}(\theta)) d\theta , \tag{11}$$

where
$$\dot{x}^{\mu} = \frac{dx^{\mu}}{d\theta} \text{ with } \mu = 0,1,2,3. \tag{12}$$

The action integral (11) is calculated between two fixed events $a \equiv x_a^{\mu} \propto (\mathbf{x}_a, ct_a)$ and $b \equiv x_b^{\mu} \propto (\mathbf{x}_b, ct_b)$, which are two given (fixed) points in Minkowski space limiting the four-dimensional trajectory of the particle. When varying the action (11), it is assumed that the evolution parameter is a coordinate-independent value, so that its variation satisfies the condition

$$\delta d\theta = 0 , \tag{13}$$

and

$$\delta S = \int_a^b d\theta \delta L(x^{\mu}, \dot{x}^{\mu}) = 0 , \tag{14}$$

where

$$\delta L(x^{\mu}, \dot{x}^{\mu}) = \frac{\partial L}{\partial x^{\mu}} \delta x^{\mu} + \frac{\partial L}{\partial \dot{x}^{\mu}} \delta \dot{x}^{\mu} \tag{15}$$

with

$$\delta\dot{x}^{\mu}=\frac{\delta dx^{\mu}}{d\theta}. \tag{16}$$

Lorentz-covariant equations corresponding to this variational principle is

$$\frac{\partial L}{\partial x^{\mu}}-\frac{d}{d\theta}\frac{\partial L}{\partial\dot{x}^{\mu}}=0. \tag{17}$$

After obtaining equations (17), a posteriori statement is introduced that the proper time of a moving particle τ should be taken as the evolution parameter:

$$\theta\equiv\tau. \tag{18}$$

When substituting Eqs. (18) and (12) in Eq. (17), it takes the form

$$\frac{\partial L}{\partial x^{\mu}}-\frac{d}{d\tau}\frac{\partial L}{\partial u^{\mu}}=0, \tag{19}$$

where

$$u^{\mu}=\frac{dx^{\mu}}{d\tau}, \tag{20}$$

$d\tau$ is the differential of the particle's proper time in the Minkowski space.

A four-dimensional interval is determined by

$$ds=cd\tau=\sqrt{g_{\mu\nu}dx^{\mu}dx^{\nu}}, \tag{21}$$

where $g_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$ is the metric tensor of the Minkowski space, a factor c is the speed of light. Then Eq. (3) can be rewritten as

$$ds=cd\tau=\sqrt{dx_{\mu}dx^{\mu}}. \tag{22}$$

Also, it follows directly from

$$u_{\mu}u^{\mu}=c^{2}. \tag{23}$$

Here and further in this work, the speed of light is $c = 1$. Taking it to account, we have $ds=d\tau$, $u_{\mu}u^{\mu}=1$, and

$$\mathbf{u}=\frac{\mathbf{v}}{\sqrt{1-\mathbf{v}^{2}}},\qquad u^{0}=\frac{dt}{d\tau}=\frac{1}{\sqrt{1-\mathbf{v}^{2}}}. \tag{24}$$

Note that the derivation of equations (19) from the principle of least action (14) is performed in violation of logic when a posteriori substitution (18) is made. Then, taking into account (13), one can obtain the expression

$$\delta d\tau=0, \tag{25}$$

which, generally speaking, contradicts the expression (22).

Of course, we have the right a priori to introduce an arbitrary abstract "evolution parameter", the variation of which does not depend on the variations of the four-coordinates. In this case, we obtain the equations of motion (17). But when we identify this parameter with a well–defined proper time of the particle, which, taking into account Eq. (22), does not satisfy condition (13), then we cannot use condition (13) when deriving equations of motion of a particle from the principle of least action. Thus, the Euler–Lagrange equations (19) should be recognized as false, obtained in violation of logic: in their derivation, condition (22) for proper time is not used. Condition (25) is used instead.

Correcting the logic, we identify the evolution parameter θ with proper time τ in the variation procedure. Thus, substituting Eq. (18) into (11), one can obtain the integral with the Lorentz-covariant Lagrangian $L(x^{\mu},u^{\mu})$:

$$S=\int_{a}^{b}L(x^{\mu},u^{\mu})d\tau. \tag{26}$$

The variation of the action (26) can be represented as

$$\delta S=\delta\int_a^b L(x^\mu,u^\mu)d\tau=\int_a^b\left(\left\{\delta L(x^\mu,u^\mu)\right\}d\tau+L(x^\mu,u^\mu)\delta d\tau\right)=0,\tag{27}$$

where the variation of the Lagrangian is given by

$$\delta L(x^\mu,u^\mu)=\frac{\partial L}{\partial x^\mu}\delta x^\mu+\frac{\partial L}{\partial u^\mu}\delta u^\mu,\tag{28}$$

and the variation of the four-dimensional velocity is

$$\delta u^\mu=\delta\left(\frac{dx^\mu}{d\tau}\right)=\frac{\delta dx^\mu}{d\tau}-\frac{dx^\mu}{d\tau^2}\delta d\tau.\tag{29}$$

The variation of the proper time differential (or four-dimensional interval) in accordance with (22) is given by [9]

$$\delta d\tau=\delta\sqrt{dx_\mu dx^\mu}=\frac{dx_\mu\delta dx^\mu}{\sqrt{dx_\mu dx^\mu}}=u_\mu\delta dx^\mu.\tag{30}$$

Substituting (30) into (29), one can obtain

$$\delta u^\mu=\delta\left(\frac{dx^\mu}{d\tau}\right)=\frac{\delta dx^\mu}{d\tau}-u^\mu u_\nu\frac{\delta dx^\nu}{d\tau}.\tag{31}$$

Then, taking into account expression (30), the variation of the Lagrangian (28) takes the form

$$\delta L(x^\mu,u^\mu)=\frac{\partial L}{\partial x^\mu}\delta x^\mu+\frac{\partial L}{\partial u^\mu}\left(\frac{\delta dx^\mu}{d\tau}-u^\mu u_\nu\frac{\delta dx^\nu}{d\tau}\right)\tag{32}$$

or

$$\delta L(x^\mu,u^\mu)=\frac{\partial L}{\partial x^\mu}\delta x^\mu+\left(\frac{\partial L}{\partial u^\mu}-u_\mu u^\nu\frac{\partial L}{\partial u^\nu}\right)\frac{\delta dx^\mu}{d\tau},\tag{33}$$

where, in the second term in parentheses, the designation of the summation index μ has been changed to ν. Substituting expressions (30) and (33) into Eq. (27), we have

$$\delta S=\int_a^b\frac{\partial L}{\partial x^\mu}d\tau\delta x^\mu+\int_a^b\left(\frac{\partial L}{\partial u^\mu}-u_\mu u^\nu\frac{\partial L}{\partial u^\nu}+u_\mu L\right)\delta dx^\mu=0.\tag{34}$$

The second integral of expression (34), being calculated in parts, is equal to

$$\left(\frac{\partial L}{\partial u^\mu}-u^\mu u^\nu\frac{\partial L}{\partial u^\nu}+u^\mu L\right)\delta x^\mu\Bigg|_a^b-\int_a^b\delta x^\mu d\left(\frac{\partial L}{\partial u^\mu}-u_\mu u^\nu\frac{\partial L}{\partial u^\nu}+u_\mu L\right).\tag{35}$$

Since the variations of the four-coordinates at the ends of the four-dimensional trajectory vanish: thus, the first term in expression (35) is zero. With this in mind, substituting Eq. (35) into Eq. (33), we find the final expression for the variation of the action:

$$\delta S=\int_a^b\left\{\frac{\partial L}{\partial x^\mu}-\frac{d}{d\tau}\left(\frac{\partial L}{\partial u^\mu}-u_\mu u^\nu\frac{\partial L}{\partial u^\nu}+u_\mu L\right)\right\}d\tau\delta x^\mu=0.\tag{36}$$

Taking into account the independence of variations of four-dimensional coordinates δx^μ, we obtain a general view of the equations of motion of a particle

$$\frac{\partial L}{\partial x^\mu}-\frac{d}{d\tau}\left(\frac{\partial L}{\partial u^\mu}-u_\mu u^\nu\frac{\partial L}{\partial u^\nu}+u_\mu L\right)=0,\tag{37}$$

or

$$\frac{\partial L}{\partial x^\mu}-\frac{d}{d\tau}\left((g_\mu^\nu-u_\mu u^\nu)\frac{\partial L}{\partial u^\nu}+u_\mu L\right)=0,\tag{38}$$

or

$$\frac{dP^\mu}{d\tau}=-\frac{\partial L}{\partial x_\mu},\tag{39}$$

where the generalized four-momentum is related to the Lagrangian by the equation

$$P^{\mu}=(u^{\mu}u^{\nu}-g^{\mu\nu})\frac{\partial L}{\partial u^{\nu}}-u^{\mu}L. \tag{40}$$

Rewriting by the components, taking into account the signature of the contravariant gradient on the right side (39), we obtain

$$\frac{dE}{d\tau}=-\frac{\partial L}{\partial t}\ \text{ and }\ \frac{d\mathbf{P}}{d\tau}=\frac{\partial L}{\partial \mathbf{x}}, \tag{41}$$

where we have the expression for the spatial components of the momentum

$$\mathbf{P}=\frac{\partial L}{\partial \mathbf{u}}+\mathbf{u}u^{\mu}\frac{\partial L}{\partial u^{\mu}}-\mathbf{u}L, \tag{42}$$

and a formula for energy takes the form

$$E=u^{0}u^{\mu}\frac{\partial L}{\partial u^{\mu}}-u^{0}L-\frac{\partial L}{\partial u^{0}}. \tag{43}$$

Equations (37)-(39) are invariant with respect to the gauge transformation

$$L\to L+\frac{d}{d\tau}F(x^{\mu})\equiv L+u^{\mu}\frac{\partial F}{\partial x^{\mu}}, \tag{44}$$

where $F(x^{\mu})$ is an arbitrary Lorentz-invariant differentiable function of four-dimensional coordinates. This can be seen by directly substituting Eq. (44) into Eq. (37). This transformation is equivalent to the expression (10). Expression (40) means that the four-momentum allows conversion

$$P^{\mu}\to P^{\mu}+\frac{\partial F}{\partial x^{\mu}}. \tag{45}$$

that does not change the equations of motion (37).

In addition, the identity is valid

$$L=-P^{\mu}u_{\mu}\equiv \mathbf{P}\mathbf{u}-Eu_{0}, \tag{46}$$

which can be verified by directly substituting Eq. (40) into (46). We emphasize that the Lagrangian is primary here, and the momentum is determined from it a posteriori using the formula (40). Relation (46) is obviously identical to relation (7) of the implicitly covariant treatment.

The expression (37) is known as the Kalman equations. It was first obtained from the variational principle (27), Ref. [10]. However, the connection of these equations with the implicit Euler–Lagrange equations (4)-(5) has not yet been clarified.

Indeed, the condition of correspondence between manifestly and implicit treatments of Lorentz-covariant mechanics has to be satisfied, since they describe the same physical reality. Later in this work, the equations of manifestly covariant treatment will be obtained in this way. The inverse transformation of the treatments is also considered and the condition of one-to-one correspondence is established.

II. MANIFESTLY COVARIANT EQUATIONS FROM AN IMPLICIT TREATMENT AND VICE VERSA

The Euler–Lagrange equations (4)-(5) obtained within the framework of the implicit treatment of relativistic mechanics should be identical to the corresponding equations obtained within the framework of the manifestly approach. This means that it is possible to obtain the Eqs. (37) corresponding to the manifestly covariant treatment from Eqs. (4)-(5).

Action (1) can be rewritten in Lorentz-covariant form taking into account Eq. (24)

$$S=\int_{t_a}^{t_b}\Lambda(\mathbf{x},\mathbf{v},t)dt=\int_{t_a}^{t_b}\Lambda(\mathbf{x},\mathbf{v},t)\frac{dt}{d\tau}d\tau=\int_{a}^{b}L(x^{\mu},u^{\mu})d\tau. \tag{47}$$

The integral (47) is calculated between two fixed events, $a\equiv x_a^{\mu}\propto(\mathbf{x}_a,ct_a)$ and $b\equiv x_b^{\mu}\propto(\mathbf{x}_b,ct_b)$, which represent two given (fixed) points bounding the four-dimensional trajectory of the particle. The Lagrangian $L(x^\mu, u^\mu)$ depends on the four-dimensional coordinates and four-velocities of the particles. The equality of Eqs. (1) and (47), taking into account Eq. (24), means that there is the following connection between manifestly and implicitly covariant Lagrangians:

$$\Lambda(x^i, t, \mathrm{v}^i) = L(x^\nu, u^\nu(\mathrm{v}^i))\sqrt{1-\mathbf{v}^2}\,. \tag{48}$$

The differential of the Lorentz-covariant Lagrangian can be represented as

$$dL(x^\mu, u^\mu) = \frac{\partial L}{\partial x^\mu}dx^\mu + \frac{\bar{\partial} L}{\partial u^\mu}\bar{d}u^\mu, \tag{49}$$

where the upper underscore in the covariant derivative with respect to four-velocity indicates that we need to take into account the connection of the four-velocity components in accordance with the Eq. (23). To make the differentials of all four-velocity components, du^ν, formally independent, one can use the expression in parentheses in the Eq. (33)

$$dL(x^\mu, u^\mu) = \frac{\partial L}{\partial x^\mu}dx^\mu + \left(g_\mu^\nu - u_\mu u^\nu\right)\frac{\partial L}{\partial u^\nu}du^\nu, \tag{50}$$

where the differentials of the four-dimensional components on the right side, du^ν, are considered now as mutually independent (the same is true for the components of the derivative $\partial L/\partial u^\nu$). In this case, one can formally write

$$\bar{d}u^\mu = \left(g_\mu^\nu - u_\mu u^\nu\right)du^\nu. \tag{51}$$

Eq. (51) satisfies the condition (27):

$$u_\mu \bar{d}u^\mu = u_\mu\left(g^{\mu\nu} - u^\mu u^\nu\right)du_\nu \equiv 0. \tag{52}$$

Thus, the covariant derivative of any differentiable function $f(u^\nu)$ with respect to the velocity, which explicitly takes into account the connection (23), has the form:

$$\frac{\bar{\partial} f(u^\nu)}{\partial u^\mu} = \left(g_\mu^\nu - u_\mu u^\nu\right)\frac{\partial f(u^\nu)}{\partial u^\nu}. \tag{53}$$

Obviously, Eq. (53) satisfied the identity

$$u^\mu \frac{\bar{\partial} f(u^\nu)}{\partial u^\mu} \equiv 0. \tag{54}$$

Eq. (53) has to be used when differentiating any functions by components of a four-dimensional velocity. For example, taking into account Eqs. (53), expressions (38) and (40) can be rewritten in the more concise forms:

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau}\left(\frac{\bar{\partial} L}{\partial u^\mu} + u_\mu L\right) = 0, \tag{55}$$

$$P^\mu = -\frac{\bar{\partial} L}{\partial u^\mu} - u^\mu L. \tag{56}$$

Substituting Eqs. (39) and (56) into Eq. (49), we obtain

$$dL(x^\mu, u^\mu) = -\frac{dP^\mu}{d\tau}dx^\mu - (P_\mu + Lu_\mu)du^\mu, \tag{57}$$

but since $u_\mu du^\mu \equiv 0$the last expression can be rewritten in a more compact form

$$dL = -\frac{dP^\mu}{d\tau}dx^\mu - P_\mu du^\mu \equiv -u_\mu dP^\mu - P^\mu du_\mu. \tag{58}$$

The last expression can be obtained directly from formula (46), which proves the self-consistency of all calculations made here.

Expression (50) is an example of how the manifestly covariant derivative of an arbitrary differentiable function *f* of four-coordinates and four-velocities of a particle with respect to the proper time looks correct, namely:

$$\frac{d}{d\tau}f(x^\mu,u^\mu)=\frac{\partial f}{\partial x^\mu}u^\mu+\left(g_\mu^\nu-u_\mu u^\nu\right)\frac{\partial f}{\partial u^\nu}\frac{du^\nu}{d\tau}. \tag{59}$$

The multiplier $\left(g_\mu^\nu-u_\mu u^\nu\right)$ takes into account the connection of the four-velocity components, so that the derivative $\frac{\partial f}{\partial u^\nu}$ is calculated in the usual way, i.e. under the assumption that all four-velocity components are independent of each other.

Substituting the Lorentz-invariant Lagrangian (48) into the equation for momentum (6), passing to four-dimensional velocities u^μ with respect to Eqs. (24) and (53), we derive

$$\mathbf{P}=\frac{\partial}{\partial\mathbf{v}}\left(L(x^\nu,u^\nu(\mathrm{v}^i))\sqrt{1-\mathbf{v}^2}\right)=\sqrt{1-\mathbf{v}^2}\,\frac{\partial L}{\partial u^\mu}\frac{\partial u^\mu}{\partial\mathbf{v}}-\frac{\mathbf{v}L}{\sqrt{1-\mathbf{v}^2}}. \tag{60}$$

Considering (24) and (53), we obtain

$$\mathbf{P}=\sqrt{1-\mathbf{v}^2}\,(g^{\mu\nu}-u^\mu u^\nu)\frac{\partial u_\nu}{\partial\mathbf{v}}\frac{\partial L}{\partial u^\mu}-\mathbf{u}L, \tag{61}$$

where, taking into account Eq. (24), we have

$$\frac{\partial u_\nu}{\partial\mathrm{v}^i}={u_0}^2u_\nu\mathrm{v}^i-g_\nu^iu_0,\ i=1,2,3. \tag{62}$$

Substituting Eq. (62) into Eq. (61) and considering Eq. (54), after simple transformations we obtain the previously found expression (42). In the same way, Eq. (43) of the manifestly covariant treatment can be obtained from the equation for energy (7) that corresponds to the implicit formulation.

Substituting Eqs. (42) and (43) into the corresponding implicit Euler–Lagrange Eqs. (8)-(9) and replacing there the derivative with respect to t with the derivative with respect to τ, taking into account Eqs. (24) and (48), leads to the Kalman Eqs. (41) or, equivalently, Eqs. (37)-(38).

This confirms that a manifestly covariant formulation in the form of the Kalman equations can be obtained from the implicit formulation of the Euler–Lagrange equations.

To reverse the transition from the Kalman equations to the Euler–Lagrange expressions of implicit treatment (8)-(9), it is necessary to replace the original manifestly covariant Lagrangian L with an implicitly covariant form according to Eq. (48) by the following connection

$$L(x^\nu,u^j)=\Lambda(x^i,t,\mathrm{v}^i(u^j))(1-\mathbf{v}(u^j)^2)^{-\frac{1}{2}}. \tag{63}$$

To pass from the Kalman equations to the corresponding implicitly covariant equations, it is necessary to substitute the manifestly covariant Lagrangian (63), expressed in terms of its implicitly covariant analogue. This means that the derivatives of the Lagrangian (63) with respect to the four-velocity components have to be calculated through the three-dimensional velocity components according to the rules of differentiation of a complex function:

$$\frac{\partial L}{\partial u^k}=\frac{\partial(\Lambda(1-\mathbf{v}^2)^{-\frac{1}{2}})}{\partial\mathrm{v}^i}\frac{\partial\mathrm{v}^i}{\partial u^k}=\left\{(1-\mathbf{v}^2)^{-\frac{1}{2}}\frac{\partial\Lambda}{\partial\mathrm{v}^i}+\Lambda(1-\mathbf{v}^2)^{-\frac{3}{2}}\mathrm{v}^i\right\}\frac{\partial\mathrm{v}^i}{\partial u^k}, \tag{64}$$

where, as follows from Eq. (24),

$$\mathrm{v}^i=\frac{u^i}{\sqrt{1+\mathbf{u}^2}},\ \text{hence}\ \frac{\partial\mathrm{v}^i}{\partial u^k}=\frac{\delta^{ik}-\mathrm{v}^i\mathrm{v}^k}{\sqrt{1+\mathbf{u}^2}}=(\delta^{ik}-\mathrm{v}^i\mathrm{v}^k)(1-\mathbf{v}^2)^{\frac{1}{2}}, \tag{65}$$

where $\delta^{ik}=\mathrm{diag}(1,1,1)$ is the Kronecker's symbol. Then

$$\frac{\partial L}{\partial u^k}=\left\{\frac{\partial\Lambda}{\partial\mathrm{v}^i}+\frac{\Lambda\mathrm{v}^i}{1-\mathbf{v}^2}\right\}(\delta^{ik}-\mathrm{v}^i\mathrm{v}^k). \tag{66}$$

It follows from Eq. (63) that the explicit dependence of the Lagrange function on u^0 has to be excluded

$$\frac{\partial L}{\partial u^0}=0. \tag{67}$$

From here

$$u^{\mu}\frac{\partial L}{\partial u^{\mu}}=u^{k}\left\{\frac{\partial \Lambda}{\partial \mathrm{v}^{i}}+\frac{\Lambda \mathrm{v}^{i}}{1-\mathbf{v}^{2}}\right\}(\delta^{ik}-\mathrm{v}^{i}\mathrm{v}^{k}). \tag{68}$$

Substituting Eqs. (66) and (68) into the expression for momentum (42), after simple transformations we obtain Eq. (6) that corresponds to the implicit treatment. Similarly, converting the expression for energy (43) obtained from the manifestly covariant treatment, one can obtain the Eq. (7) of the implicit treatment.

Thus, it is shown that while the transition from the Euler–Lagrange equation to the Kalman equations does not require the introduction of additional conditions imposed on the original function (48), the reverse transformation from a manifestly covariant treatment to an implicitly covariant one is possible only with a special representation of the Lagrangian (63) that corresponds to an implicitly covariant Lagrangian with three independent speed components. To do this, it is necessary to change the four-dimensional Lorentz-covariant representation of velocity in a manifestly covariant Lagrangian to the three-dimensional velocity that is used in the implicit formulation in accordance with the Eq. (65). The derivatives of the Lagrangian with respect to the components of the four-dimensional velocity in the Kalman equations should be taken in accordance with rules (66) and (67). These transformation guarantees an unambiguous correspondence of the Euler–Lagrange equations of the implicit treatment to the original Kalman equations.

III. SHORTENED KALMAN EQUATIONS

Which approach has a one-to-one correspondence with an implicitly Lorentz covariant treatment? Assume that the Lagrangian explicitly depends only on three spatial components of the four-velocity, excluding u^0 from the number of arguments of this function, i.e. consider $L(x^{\mu},u^{i})$ at i=1,2,3. Then the action integral takes the form

$$S=\int_a^b L(x^{\mu},u^{i})d\tau. \tag{69}$$

That corresponds to a compromise, partially manifestly covariant theory. Note that the number of arguments in the function $L(x^{\mu},u^{i})$, as in the implicit Lagrangian $\Lambda(\mathbf{x},\mathbf{v},t)$, is seven, and they are all independent of each other.

Variation (69) is carried out similarly to Eq. (26), but the variation of the "shortened" manifestly covariant Lagrange function includes only three terms with derivatives with respect to the four-velocity components

$$\delta L(x^{\mu},u^{i})=\frac{\partial L}{\partial x^{\mu}}\delta x^{\mu}+\frac{\partial L}{\partial u^{i}}\delta u^{i} \tag{70}$$

or

$$\delta L(x^{\mu},u^{i})d\tau=\frac{\partial L}{\partial x^{i}}\delta x^{i}d\tau+\frac{\partial L}{\partial x^{0}}\delta x^{0}d\tau+\left(\frac{\partial L}{\partial u^{i}}-u^{k}u_{i}\frac{\partial L}{\partial u^{k}}\right)\delta dx^{i}-u^{k}u_{0}\frac{\partial L}{\partial u^{k}}\delta dx^{0}.$$

Converting the action integral in parts, similarly with Eqs. (34)-(36), we obtain equations of motion of the form

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial u^{i}}-u^{k}u_{i}\frac{\partial L}{\partial u^{k}}+Lu_{i}\right)=\frac{\partial L}{\partial x^{i}}. \tag{71}$$

$$\frac{d}{d\tau}\left(\sqrt{1+\mathbf{u}^{2}}\left(u^{k}\frac{\partial L}{\partial u^{k}}-L\right)\right)=-\frac{\partial L}{\partial x^{0}}. \tag{72}$$

The relations (71)-(72) have the form of Eq. (39). Here the spatial components of the four-momentum are determined by the expression in parentheses in Eq. (71), taken with the opposite sign.

Taking into account the difference between covariant and contravariant components of the 4-vector, one can obtain

$$\mathbf{P} = \frac{\partial L}{\partial \mathbf{u}} + \mathbf{u}u^k \frac{\partial L}{\partial u^k} - \mathbf{u}L, \tag{73}$$

and for energy there is an expression

$$E = \sqrt{1+\mathbf{u}^2}\left(u^k \frac{\partial L}{\partial u^k} - L\right). \tag{74}$$

A comparison of Eqs. (73) and (74) with the previously obtained expressions of the manifestly covariant theory (42) and (43) with $u^0 = \sqrt{1+\mathbf{u}^2}$ demonstrates their similarity, except that formulas (73)-(74) lack a number of terms related to the 0-component of the four-velocity or its derivative. Momentum (73) and energy (74) also satisfy the formula (46) that takes the form $L = \mathbf{Pu} - E\sqrt{1+\mathbf{u}^2}$.

It is easy to show that there is a one-to-one correspondence between the Euler–Lagrange equations (4)-(5) of the implicitly covariant treatment and the shortened Kalman Eqs. (72)-(73): taking into attention the ratio between **v** and **u** (24), these are the same equations.

Consequently, the mathematical model corresponding to the complete Kalman equations (37) describes a potentially wider range of phenomena than the implicitly covariant treatment or shortened Kalman equations.

IV. THE PARTICLE SYSTEM

Consider a closed system consisting of N interacting particles. The interaction between particles occurs through the fields they create. The action for such a closed system, taking into account Eq. (26), is written in the form

$$S = \sum_{k=1}^{N} \int_{a_k}^{b_k} L_{(k)}(x^\mu_{(k)}, u^\mu_{(k)})\, d\tau_{(k)}, \qquad k=1,2,\ldots N. \tag{75}$$

The Lagrange function included in each of the terms of the sum (75) clearly depends only on the coordinates and velocities of a given particle. Summation is carried out for all particles of the system. The limits of integration in the terms for different particles should be chosen between two given moments of coordinate time, the same for all particles:

$$t_{a(k)} = t_a, \quad t_{b(k)} = t_b, \quad k=1,2,\ldots N. \tag{76}$$

It should be noted that the Lagrangian $L_{(k)}(x^\mu_{(k)}, u^\mu_{(k)})$ is a function of the coordinates and velocities not only of the particle with the number k, but also the coordinates and velocities of all other particles with which it interacts, if the coordinates and velocities determine the fields created by these particles. However, due to the delay, i.e. the finiteness of the velocity of propagation of field disturbances when the coordinates and velocities of the source particle change, the Lagrangian of this k-particle explicitly and directly depends only on the coordinates and velocities of the same particle.

The principle of least action $\delta S = 0$ leads to the Kalman Eqs. (37) for each of the N particles

$$\frac{\partial L_{(k)}}{\partial x^\mu_{(k)}} - \frac{d}{ds}\left(\frac{\partial L_{(k)}}{\partial u^\mu_{(k)}} - u_{\mu(k)}u^\nu_{(k)}\frac{\partial L_{(k)}}{\partial u^\nu_{(k)}} + u_{\mu(k)}L_{(k)}\right) = 0, \quad k=1,2,\ldots N. \tag{77}$$

At the same time, the consideration of the effect of other particles on this particle is described by the interaction of this particle with the total field of all other particles.

Let's find which quantities for a closed system of particles are integrals of motion due to the homogeneity of space and time (Noether's theorem). To find the four-momentum, consider a closed system of particles, the action for which is S, and perform an infinitesimal four-dimensional (space-time) parallel transfer, requiring that the action of S remain unchanged.

Parallel transfer means a transformation in which all points of the system are shifted by the same four-dimensional vector ε^μ, i.e. the four-dimensional coordinates of each of the N particles of the system undergo the following change

$$x^\mu_{(k)} \to x^\mu_{(k)} + \varepsilon^\mu, \quad k=1,2,\ldots N. \tag{78}$$

The change in action with such an infinitesimal transformation of coordinates with constant particle velocities has the form

$$\delta S = \delta \sum_{k=1}^{N} \int_{a_k}^{b_k} L_{(k)}(x^{\mu}_{(k)}, u^{\mu}_{(k)}) d\tau_{(k)} = \varepsilon^{\mu} \sum_{k=1}^{N} \int_{a_k}^{b_k} \frac{\partial L_{(k)}}{\partial x^{\mu}_{(k)}} d\tau_{(k)} = 0. \tag{79}$$

Due to its arbitrariness, the requirement is equivalent to the condition

$$\sum_{k=1}^{N} \int_{a_k}^{b_k} \frac{\partial L_{(k)}}{\partial x^{\mu}_{(k)}} d\tau_{(k)} = 0. \tag{80}$$

Taking into account Eqs. (37), the last expression can be rewritten as

$$\sum_{k=1}^{N} \int_{a_k}^{b_k} d\left(\frac{\partial L_{(k)}}{\partial u^{\mu}_{(k)}} - u_{\mu(k)} u^{\nu}_{(k)} \frac{\partial L_{(k)}}{\partial u^{\nu}_{(k)}} + u_{\mu(k)} L_{(k)} \right) = 0. \tag{81}$$

This means that for a closed system, a four-dimensional vector of the form has the conservation property

$$P^{\mu}_{N} = \sum_{k=1}^{N} \left(u^{\mu}_{(k)} u^{\nu}_{(k)} \frac{\partial L_{(k)}}{\partial u^{\nu}_{(k)}} - \frac{\partial L_{(k)}}{\partial u_{\mu(k)}} - u^{\mu}_{(k)} L_{(k)} \right), \tag{82}$$

which, therefore, is an integral of motion and, according to Noether's theorem, is a generalized impulse of the system. Obviously, for a single particle, the generalized four-dimensional momentum is still determined by the formula (40).

V. DISCUSSION

It is shown that the Euler–Lagrange Eqs. (4)-(5) by replacing the Lagrangian of the implicit treatment $\Lambda(\mathbf{x}, \mathbf{v}, t)$ with the Lagrangian of a manifestly covariant theory $L(x^{\mu}, u^{\mu})$ and the corresponding substitution of variables (48) lead directly to the Kalman Eqs. (37), which, therefore, are a manifestly covariant image of the implicitly covariant Euler–Lagrange equations. However, the reverse transition from the Kalman equations to the Euler–Lagrange equations requires the introduction of restrictions on the Lagrangian, so that the Kalman equations correspond to a potentially broader mathematical model in relativistic mechanics. At the same time, the shortened Kalman Eqs. (71)-(72) have a one-to-one correspondence with the Euler–Lagrange equations, in which the time projection of the 4-velocity of the particle is excluded from the arguments of the Lagrangian, so that all seven remaining arguments of the Lagrangian $L(x^{\mu}, u^{i}) \equiv L(\mathbf{x}, \mathbf{u}, t)$ are completely independent.

Let's discuss another variant of the "shortened" Lagrange function, which, like the function included in the action integral (69), has seven completely independent arguments, but instead of the time *t*, it uses the proper time *τ* as an independent variable: $\tilde{L}(\mathbf{x}, \mathbf{u}, \tau)$. Then coordinate time *t* is a dependent variable. Indeed, as follows from Eq. (22), the coordinate time differential of the inertial reporting system is calculated from the given differentials of independent variables according to the formula $dt = \sqrt{d\tau^2 + d\mathbf{x}^2}$.

Using proper time as an independent parameter excludes coordinate time from the list of independent arguments of the Lagrangian. Generally speaking, that is not convenient, since the fields external to the particle depend directly on coordinate time. For example, the four-dimensional vector potential of the electromagnetic field has the form $A^{\mu}(\mathbf{x}, t)$.

As required by the Lorentz covariance condition, variation in coordinate time *t* in all the procedures discussed above is carried out formally in the same way as variation in three spatial coordinates **x**. In other words, there are two fixed points of the trajectory, its beginning and its end; the projections of the four-dimensional interval connecting the beginning and end of the trajectory are predetermined in this problem. That corresponds to the physical formulation and typical measurement procedure. *The length of the trajectory* in four-dimensional Minkowski space that set by the change in the proper time of the particle as it moves along the trajectory, *is not fixed*.

If we replace the coordinate time t with the proper time τ and require, no longer for t, but for τ, the setting of not only the initial but also the final moment of the proper time (i.e., in terms of τ), then this changes the physical formulation of the problem: by that we fix the four-dimensional length of the trajectory in connection with the identity (22). That contradicts the fundamentals of particle dynamics.

Therefore, it is always reasonable to choose coordinate time t as an independent parameter. If the Lagrangian is given in the form $\tilde{L}(\mathbf{x},\mathbf{u},\tau)$, then, taking into account Eq. (24), enter a parametric dependence of the proper time with respect to coordinate time, i.e. $\tau=\tau(t)$. Then the function $\tilde{L}(\mathbf{x},\mathbf{u},\tau(t))$ reduces to the previously considered function $L(x^\mu,u^i)$ with the action integral (69), and the equations of motion have the form (71)-(72).

Let us discuss the construction of Hamiltonian mechanics in Minkowski space, corresponding to a manifestly covariant scheme, i.e. an equal consideration of coordinates **x** and time t. In this case, the velocities will be the components of the four-velocity **u** and u^0 , respectively, and the expression for the Lorentz-covariant Hamiltonian through the Lagrangian should formally have the form

$$H=\mathbf{Pu}-Eu_0-L\equiv-P^\mu u_\mu-L\,. \tag{83}$$

But, taking into account (46), this value is identically zero: $H = 0$. Therefore, such a Lorentz covariant generalization of the Hamiltonian is meaningless [10]. It remains us to the well-known implicitly covariant approach to the construction of Hamilton's equations [11], in which the usual three-component velocity **v** appears, and the Hamiltonian is the 0-component of the four-vector, i.e. energy.

Theoretically, the possibility of parameterizing a four-dimensional trajectory by some independent internal parameter of evolution (which is not subject to variation) is beyond doubt, if such a parameter has a physical meaning. Such an independent parameter cannot be a four-dimensional interval in Minkowski space (proper time), since it depends on four space-time coordinates. Consequently, the condition $\delta d\tau=0$ assumed for the evolution parameter with varying action, assuming its independence from four-coordinates, leading to incorrect Eqs. (19), violates the physical meaning of the relativistic problem.

This is already evident when obtaining the equations of motion of a free particle. The principle of correspondence between manifestly and implicit treatments requires that the Lagrange functions be connected by the formula (48). Since, in the implicit treatment, the Lagrange function of a free particle is $\Lambda=-m\sqrt{1-\mathbf{v}^2}$, then in the manifestly treatment we have $L=-m=\text{const}$. The application of Eqs. (19) to such a Lagrange function in accordance with the usual rules of differentiation leads to a meaningless trivial result

$$\frac{d}{d\tau}m=0\,. \tag{84}$$

How do they get out of this situation within the framework of the concept of an independent parameter of evolution, identified with its own time? It is proposed to formally multiply the Lagrange function by a constant $\sqrt{u_\mu u^\mu}=1$, so that $L=-m\sqrt{u_\mu u^\mu}$. Next, it is differentiated by the components of four-velocity as independent parameters according to Eqs. (19), as if there is a real functional dependence on velocity, and, contrary to the rules of mathematics, a non-zero result is obtained [1-5]:

$$\frac{\partial L}{\partial u^\mu}=\frac{\partial(-m\sqrt{u_\mu u^\mu})}{\partial u^\mu}=-mu_\mu\,!!! \tag{85}$$

In fact, a constant is differentiated here, and we should get strictly zero on the right side (85). The authors of the monograph [1] note that the function $L=-\frac{1}{2}mu_\mu u^\mu$ could be used with the same success in the Eq. (85). The article [12] discusses a whole family of such fake Lagrangians. R.P. Feynman also mentions this ambiguity [13]. Such a situation violates the principle of unambiguous correspondence between manifestly and implicit treatments of relativistic mechanics.

This situation cannot be corrected by introducing an abstract parameter of evolution that does not depend on four-coordinates, and then, a posteriori, identifying it with its proper time. Obviously, this procedure is a trick to hide the choice of a physically unacceptable condition $\delta d\tau = 0$, so that the problem with formula (84) remains the same.

On the contrary, the Lorentz-covariant Eqs. (37) are obtained without violations of the proper time variation. They lead to the correct equation of motion for a free particle, and generally do not cause doubt. Indeed, the Lorentz-invariant Lagrangian of a free particle is determined by its rest mass [9]:

$$L = -m = \text{const}. \tag{86}$$

Substituting (86) into (37), we get

$$m\frac{d}{d\tau}u_{\mu} = 0, \tag{87}$$

as it should be so. Substituting the Lagrangian (86) in the shortened Kalman Eq. (71), as it is easy to see, leads to the correct equation of motion

$$m\frac{d}{d\tau}\mathbf{u} = 0. \tag{88}$$

It is also easy to show that the correct equations of motion of a point charge q in an external electromagnetic field described by an external four-dimensional vector potential A^{μ} can be obtained by substituting in (37) or (71) the corresponding covariant Lagrangian $L = -m - qu_{\mu}A^{\mu}$ [10].

VI. CONCLUSIONS

The erroneous condition is contained in the fundamental work of V.A. Fock [8], in which, apparently, for the first time, incorrect equations (19) containing the particle's proper time as an independent parameter of evolution were obtained [7]. Since then, having received the status of a generally accepted treatment, this has been reproduced in authoritative expositions of the manifestly Lorentz-covariant treatment of Lagrangian mechanics [1-5], in many scientific publications, for example [14], and some academic chats. On the contrary, works that criticize this approach, for example, [15], are relatively little discussed.

Here, the correct equations of motion corresponding to the manifestly Lorentz-covariant treatment are obtained from the Euler-Lagrange equations of the implicit covariant formulation of the mechanics, and it is proved that they coincide with the Kalman equations obtained earlier through the variational principle, in which the proper time of the particle varies. The inverse transformation of the Kalman equations to equations of implicit treatment is possible only if certain special conditions imposed on the Lagrangian are met.

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