On relativistic kinetic energy and the mass of photons.

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Abstract

The formula for kinetic energy in the special relativistic context (with mass dilation) is shown to differ from what might be expected. As a consequence, photons are allowed to have a mass and may be quite different from what has been assumed until now. Moreover, it is shown that massive objects can reach the velocity of light under the action of a constant force in a finite time quite comparable to what would happen without the mass dilation property.

Key words: energy, mass, photon, mass dilation, Lorentz factor.

1 Introduction

In special relativity, it is assumed that the mass of a moving massive object evaluated in a given spatial reference frame $\mathcal{R}$ is given by the formula

$$m = m(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1}$$

where m_0 is the mass of the same object at rest in $\mathcal{R}$ and v, c are respectively the scalar velocity of the object measured in $\mathcal{R}$ and the scalar velocity of light. This formula refers to the so-called mass dilation property. If we evaluate the kinetic energy of that object by the formula

$$KE = \frac{1}{2}m(v)v^2 = \frac{1}{2}\frac{m_0 v^2}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{2}$$

we can see that it becomes infinite as v approaches c. In some texts the kinetic energy is evaluated alternatively as

$$KE_{rel} = (\gamma - 1)m_0 c^2 \tag{3}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the Lorentz factor, which produces the same difficulty and is about twice as large as v approaches c. This has been used as an argument in special relativity (SR) to assert that photons, carrying a finite energy, do not have a mass at rest. It is also stated in (SR) that photons always travel in empty space with the velocity c, precluding definitely the existence of any meaning for the mass of a photon "at rest", because the photon is at rest with respect to nothing material, any material object moving with respect to any photon with the velocity of light c. However, these considerations become strange when we realize that light propagates in liquids and solids at a velocity smaller than c. So, in empty space, the velocity of the photon equals that of the wave, and in the glass for instance, the wave is slower than the corpuscles, a circumstance which never happens for other wave propagation phenomena such as sound or electricity, where the velocity of corpuscles is much smaller than the wave velocity.

In this paper, we derive in Section 2 the relativistic kinetic energy of any massive object from the total work of forces used to pass from rest to velocity v. The result is very different from both (2) and (3), in particular it remains bounded by m_0c^2 when v approaches c. If our approach is correct, it implies, as shown in Section 3, that photons do have a mass, computable in terms of their frequency. Then arises the question of the nature of isolated photons: real corpuscules or large groups of tiny interacting subcorpuscles ? Next, in Section 4 we show that a massive object in empty space on which a constant force is applied reaches the velocity of light in a finite computable time comparable to what would happen without the mass dilation property. The final section 4 is devoted to a few remarks.

2 A relativistic kinetic energy formula

Let us consider a massive object initially at rest (at time $t = 0$ in $\mathcal{R}$ with mass m_0, and with final velocity $v = v(t)$ obtained by the action of a force

$$F = F(s), \quad 0 \le s \le t.$$

We start from Newton's second law in the form

$$F = m(v)v' \tag{4}$$

and, by the conservation of energy principle, we infer that the kinetic energy $\mathcal{E}$ acquired by the massive object is equal to the total mechanical work of the force F, which provides the formula

$$\mathcal{E} = \mathcal{E}(t) = \int_0^t (F(s), dx(s)) = \int_0^t m(v(s))(v(s), v'(s))ds \tag{5}$$

Replacing $m(v(s))$ by its value given by (1) and observing that

$$(v(s), v'(s) = \frac{1}{2}\frac{d}{ds}(||v(s)||^2)$$

we find

$$\mathcal{E}(t) = m_0 \int_0^t \frac{\frac{d}{ds}(||v(s)||^2)}{2\sqrt{1 - \frac{||v(s)||^2}{c^2}}} ds$$

whence

$$\mathcal{E}(t) = -m_0c^2 \int_0^t \frac{d}{ds}\left[\sqrt{1 - \frac{||v(s)||^2}{c^2}}\right] ds = m_0c^2 \left(1 - \sqrt{1 - \frac{||v(t)||^2}{c^2}}\right) \tag{6}$$

Since

$$1 - \sqrt{1 - \frac{||v(t)||^2}{c^2}} = \frac{||v(t)||^2}{c^2\left(1 + \sqrt{1 - \frac{||v(t)||^2}{c^2}}\right)},$$

from (6) we conclude that

$$\mathcal{E} = \mathcal{E}(t) = \frac{m_0v^2}{1 + \sqrt{1 - \frac{v^2}{c^2}}} \tag{7}$$

where $v := v(t) = ||v(t)||$ stands for the *scalar velocity* at time t. We rewrite (7) in the more suggestive following form:

Theorem 2.1. *The (relativistic) kinetic energy of a punctual body with mass m_0 and velocity v is given by the formula*

$$\mathcal{E}(m_0, v) = \frac{m_0v^2}{1 + \sqrt{1 - \frac{v^2}{c^2}}} \tag{8}$$

We now state a few observations about this result.

Remark 2.2. *As expected, the result depends neither on the path followed by the massive object, nor on the evolution of $v(s)$ on the time interval $[0, t]$, it only depends on the mass m_0 at rest and the final scalar velocity v.*

Remark 2.3. *If $v << c$, we have $\mathcal{E}(m_0, v) \sim \dfrac{m_0v^2}{2}$ as in the case of classical Newtonian mechanics. In addition, whenever $v < c$, we have $\mathcal{E}(m_0, v) < m_0v^2 < m_0c^2$.*

Remark 2.4. *If $v = c$, we find $\mathcal{E}(m_0, c) = m_0c^2$. This may be the most surprising conclusion of this section.*

3 About photons.

The previous result has very important consequences on photons, assuming that the energy of the electromagnetic wave corresponds to that of the photons. Indeed in the absence of additional fields (gravitational for instance) which might bring a form of potential energy, all the photon's energy is in kinetic form (displacement and vibration). Assume now an electromagnetic wave with frequency ν, associated with a single photon of mass m_0, by Planck-Einstein' s formula we have, even without having to assume that photons travel at speed c, the inequality

$$h\nu = \mathcal{E}(m_0, v) \leq m_0 c^2$$

hence

$$m_0 \geq \frac{h\nu}{c^2}.$$

In particular we have

- Photons have positive masses at rest!
- Their mass when moving at velocity c becomes infinite.
- However their kinetic energy is finite, and at most equal to $m_0 c^2$ with m_0 their mass at rest.
- High frequency waves are carried by photons with high masses at rest. In particular *different photons have different masses*, unless usual elementary particles.

Remark 3.1. *The last property suggests that what we usually consider an isolated "photon" could in fact be a combination of more elementary corpuscules ("sub-photons") interacting together as a product of slow sinusoidal waves with suitably shifted phases giving the illusion of fast oscillation. We can even imagine that the actual quantum of energy is the sub-photon.*

Remark 3.2. *To fix the ideas, let us compute the mass of the "photon " associated with a red light. For a red light , $\nu \sim 4.10^{14} hz$ since $h \sim 6.6.10^{-34}$, we find $h\nu \sim 2.6.10^{-19}$ and finally ,*

$$m_0 \geq \frac{h\nu}{c^2} \sim (2.6/9).10^{-19-16} \sim 3.10^{-36} kg.$$

One might object that the energy of transversal vibration is also involved in the total energy of the photon, but the corresponding displacement of the mass is at sub-luminic speed, so if photons really travel at the velocity of light, in the worst case we overestimated the mass by a factor 2.

Remark 3.3. *The order of magnitude of the photon's mass given by our calculation is one billionth of a proton's mass or one millionth of an electron's mass, and seems to contradict the upper bound of [1]. In addition, according to the paper [6], the limit of detectability for the photon's mass is around $10^{-69} kg$, so if our estimate turns out to be valid, it shoud be possible to detect the photon's mass for light and even much less energetic electromagnetic waves.*

4 Reaching the velocity of light in finite time.

Here we consider equation (4) in the special case of a constant force F. Starting with an initial velocity equal to 0 (which is always feasible by a Galilean change of referential), we rewrite (4) in the equivalent form

$$v' = \frac{F}{m(v)} = \frac{F}{m_0}\sqrt{1-\frac{v^2}{c^2}} \tag{9}$$

where F, v, v' can be taken as positive scalars, the trajectory lying on a half-line. Starting with $v(0) = 0$, we can see that $v = v(t)$ is increasing and the equation indicates that the value $v = c$ cannot be crossed, the velocity remaining equal to c later if this value is reached in finite time. It turns out to be the case: (10) can be written as

$$\frac{d}{dt}\left[\arcsin\left(\frac{v(t)}{c}\right)\right] = \frac{F}{cm_0} \tag{10}$$

which implies

$$v(t) = c\sin\frac{tF}{cm_0}$$

whenever

$$0 < t \leq T := \frac{\pi}{2}\frac{cm_0}{F}.$$

and

$$v(t) = c, \quad \forall t \geq T.$$

Theorem 4.1. *Under a constant force F, starting from rest in the reference frame $\mathcal{R}$, any punctual mass m_0 reaches the velocity of light at the exact time*

$$T := \frac{\pi}{2}\frac{cm_0}{F}.$$

Remark 4.2. *For $t > T$, as a consequence of* (10)*, no action parallel to the motion can change the velocity of the massive object which will go on moving at speed c in the same direction for ever, except if it collides with some material object or encounters a transversal perturbation.*

Remark 4.3. *The time T is just $\frac{\pi}{2}$ times larger than the time required to reach the speed of light in the Newtonian framework.*

Remark 4.4. *As a consequence of time dilation, it is easy to see that if τ is the proper time measured in the moving referential of the massive object, the equation becomes*

$$v'(\tau) = v'(t)t'(\tau) = \frac{F}{m_0}\sqrt{1-\frac{v^2}{c^2}}t'(\tau) = \frac{F}{m_0}, \tag{11}$$

Since $t'(\tau) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ (the Lorentz factor). For the moving mass itself, everything happens as if the mass dilation had no influence on the time needed to achieve the velocity of light.

5 Final remarks

- If the photons do have a mass, the action of gravity on light becomes understandable in the Newtonian framework, and we can study general black holes such as in [2]. Note that the mass dilation only takes place in the direction of the ray, therefore a gravitational field transversal to the ray is perfectly able to deviate the photon.

- The author was induced to compute the relativistic kinetic energy when trying to understand the origin of inertia. That problem seems to be one of the most important of fundamental physics, together with the origin of the gravitational force, cf. about this the references [3, 4, 5] .

References

[1] A. S. Goldhaber and M. M. Nieto, The mass of the photon, *Scientific American*, **234** (1976), 86-97.

[2] A. Haraux, A Newtonian approach to general black holes, *Evolution Equations and Control Theory*, **12**, issue 6 (2023), 1447–1455.

[3] E. Mach, Die Mechanik in ihrer Entwicklung : historisch-kritisch dargestellt " La mecanique dans son developpement : expose historique et critique ", *Leipzig, F. A. Brockhaus, coll. " Internationale wissenschaftliche Bibliothek "*, (no LIX), 1883, 1ere ed., 1 vol., X-483, ill.

[4] (MR59668) [10.1093/mnras/113.1.34] D. W. Sciama, On the origin of inertia, *Royal Astronomical Society, Provided by the NASA Astronomical Data Systems*, **1** (1953), 34-42.

[5] T. Van Flandern, Dark Matter, Missing Planets and New Comets: Paradoxes Resolved, Origins Illuminated, *North Atlantic Books, Berkeley, CA* (1993 and 1999)

[6] G. Vasseur, *Limites Sur la Masse du Photon*, CEA preprint, Saclay, 1996.