

Peer Review

Review of: "The Pauli Spin Algebra, Related Clifford Algebra Representation, and the Dirac Hamiltonian Operator Derivation"

Jason Hanson¹

1. Computer Science, DigiPen Institute of Technology, Redmond, United States

(1) The creation operator identity in equation (1.10), while true for an exterior algebra, is incorrect for a Clifford algebra. It should be

$$a^+(v)a^+(w) + a^+(w)a^+(v) = 2\eta(v, w)1$$

because of the Clifford identity $v \cdot w + w \cdot v = 2\eta(v, w)1$. Here and throughout these comments, $\cdot$ denotes Clifford multiplication.

(2) The statements following equation (1.11) need clarification and are possibly incorrect. To say that the creation and annihilation operators are adjoint requires a metric to be defined on the Clifford algebra, which the author does not do. The canonical extension of the metric of the underlying vector space to the Clifford algebra is defined as

$$\eta(e, f) \doteq \langle \tau(e) \cdot f \rangle_0$$

for elements e, f of the Clifford algebra. Here $\langle e \rangle_0$ denotes the scalar part of e , and τ is the transposition involution. However, the creation and annihilation operators are not adjoint with respect to this metric:

$$\langle \tau(a^+(u)(e)) \cdot f \rangle_0 \neq \langle \tau(e) \cdot a(u)f \rangle_0$$

in general. Indeed, $a^+(u)u = \eta(u, u)1$. So that if $e=u$ and $f=1$, then the left--hand side is nontrivial, while the right--hand side is trivial.

(3) The statement of Proposition 2.2 is confusing and needs clarification. The paragraph following the proof of the proposition seems to imply that the 4--dimensional matrix representation ρ used in the

proposition maps $\sigma_j \in \text{Cl}_3(\sigma)$ to the outer product matrix $\sigma_1 \otimes \sigma_j$ acting on the tensor product space $C^2 \otimes C^2 \cong C^4$. In particular, the basis vectors of C^4 are not elements of the vector space $C\{\sigma_1, \sigma_2, \sigma_3\}$ from which the Clifford algebra $\text{Cl}_3(\sigma)$ is formed. So an operator like $b_{\pm}^+ = a^+(e_{\pm})$ does not make sense ($e_{\pm}$ a basis vector in C^4).

However, equation (2.7) uniquely determines $b_{\pm}^+$, $b_{\pm}$, and Γ . Namely,

$$b_{\pm}^+ = \frac{1}{2}(\rho(\sigma_1) + i\rho(\sigma_2)), \quad b_{\pm} = \frac{1}{2}(\rho(\sigma_1) - i\rho(\sigma_2)), \quad \Gamma = \rho(\sigma_3)$$

These equations could thus be used to *define* $b_{\pm}^+$, $b_{\pm}$, and Γ . On the other hand, it is not clear how $b_{\mp}^+$, $b_{\mp}$ would be defined.

In my reading of the text, equation (2.11) is incorrect. The Clifford algebra $\text{Cl}_3(\sigma)$ is split using the pseudo--scalar $\xi = -i\sigma_1 \cdot \sigma_2 \cdot \sigma_3$ according to the projections $P_{\pm} = \frac{1}{2}(1 \pm \xi)$. So that the splitting of the representation space $F_{\text{int}} = F_{\text{int}}^+ \oplus F_{\text{int}}^-$ is given by $F_{\text{int}}^{\pm} = \rho(P_{\pm})(F_{\text{int}})$. Because ξ commutes with the entire Clifford algebra,

$$(b_{\pm}^+ + b_{\pm})(F_{\text{int}}^{\pm}) = F_{\text{int}}^{\pm}.$$

The proof of Proposition 2.2 uses the projections $\pi_{\pm} = \frac{1}{2}(1 \pm \Gamma)$ in place of $P_{\pm}$. It is true that

$$(b_{\pm}^+ + b_{\pm})\pi_{\pm}(F_{\text{int}}) = \pi_{\mp}(F_{\text{int}})$$

as Γ anti--commutes with $b_{\pm}^+$ and $b_{\pm}$. However, $\pi_{\pm}(F_{\text{int}}) \neq F_{\text{int}}^{\pm}$.

Declarations

Potential competing interests: No potential competing interests to declare.