Filtering out electromagnetic noise caused by the interaction of the classical field with the fibre phonons from the quantum field in an optical fibre

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June 19, 2024

Abstract

We consider the propagation of the electromagnetic field within an optical fibre. The field consists of a classical component and a quantum component. The classical component is large and it interacts with the matter within the fibre comprising of atoms that are vibrating, ie, phonons. This interaction causes scattering of the classical field component that interacts with the quantum component of the field, thereby altering the state of the quantum field. After constructing this model, we propose two methods for reducing this "classical-photon-phonon interaction noise". The first method is based on an optimal control algorithm wherein we generate a "control potential" in such a way that after incorporating this potential in the Schrodinger dynamics of the state the state tracks as closely as possible a desired "noiseless state". The second approach is based on Belavkin's quantum filtering theory wherein we model the classical photons as a quantum Bosonic white noise process and by taking non-demolition measurements on the system, we generate on a real-time basis, an estimate of the evolving state which by application of a control potential is made to track a desired state. Analysis of the spectrum of the transmitted quantum electromagnetic field is carried out by using a quantum stochastic differential model for the noise based on the Hudson-Parthasarathy quantum stochastic calculus. Further, the basic ideas of Cq channel capacities in quantum information theory are used to analyze the rate of information transmission through this optical fibre quantum mechanical channel. Transmission of other kinds of particles like non-Abelian matter and gauge particles are also discussed and the role of superstring theory in building a foundation for corrections to the Yang-Mills action is also discussed. We also highlight the use of the quantum effective action of a field when it interacts with other fields and with random current sources. The quantum effective action, obtained using Feynman's path integral methods for fields provides a firm foundation

https://doi.org/10.32388/MTH6D3

for corrections to the classical action by quantum effects and enables one to describe the quantum corrections to the field equations accurately.

1 The quantum mechanical model

Let

$$A_\mu(x) = A_{c\mu}(x) + A_{q\mu}(x) - - - (1)$$

denote the electromagnetic four potential, with $A_{c\mu}$ comprising the classical component and $A_{q\mu}$ the quantum field component. For simplicity, we assume that there are no charges within the fibre, so that the scalar potential $A_0 = 0$ and on adopting the Coulomb gauge, we have

$$divA_c = 0, divA_q = 0 - - - (2)$$

The classical interaction Hamiltonian of the field with the atomic phonons in the fibre has the form

$$H_{I1}(t) = \sum_a (P_a + eA_c(t, Q_a))^2/2m - eV(Q_a)) - - - (3)$$

where (P_a, Q_a) are the canonical momentum and position vectors of the electron in the a^{th} atom with $V(Q_a)$ denoting the binding potential between the electron of the a^{th} atom and its nucleus. The Hamiltonian of the quantum field has the general form

$$H_0 = \sum_{k=1}^{K} \omega(k)c(k)^*c(k) - - - (4)$$

where K is tne number of propagating modes, with the $c(k)'s$ being the annihilation operator of the k^{th} photon mode and $c(k)^*$ the corresponding creation operators. It should be noted that there is also an interaction energy between the quantum and the classical field because the total electromagnetic field energy is given by the standard form

$$(1/2)(\int curl(A_c + A_q)^2 + (\partial_t(A_c + A_q))^2)d^3x - - - (5)$$

The quantum component of this energy is

$$(1/2)(\int curl(A_q)^2 + (\partial_t(A_q))^2)d^3x = \sum_k \omega(k)c(k)^*c(k) - - - (6)$$

and the classical-quantum interaction component of this energy is

$$H_{I2}(t) = \int [(curlA_q, curlA_c) + (\partial_t A_q, \partial_t A_c)]d^3x$$

$$= \sum_k (f_k(t)c(k) + \bar{f}_k(t)c(k)^*) - - - (7)$$

where the $f_k(t)'s$ are complex-valued functions of time determined by the classical component A_c of the field. We note that the classical component of the field within the fibre can be expressed in the form

$$A_c(x) = A_c(t,r) = \sum_k f_k(t)u_k(r) - - - (8a)$$

and the quantum field as

$$A_q(x) = A_q(t,r) = \sum_k (c(k)v_k(t,r) + c(k)^*\bar{v}_k(t,r)) - - - (8b)$$

where the $u_k(r)'s$ are the modal functions of the spatial variables within the guide and are determined completely by the geometry of the fibre and the boundary conditions on the field and likewise for the $v_k(t,r)$. Writing down the total Hamiltonian as

$$H(t) = H_0 + H_{I1}(t) + H_{I2}(t)$$

$$= \sum_k \omega(k)c(k)^*c(k) + \sum_k (f_k(t)c(k) + \bar{f}_k(t)c(k)^*) +$$

$$\sum_a (P_a + e\sum_k f_k(t)u_k(Q_a))^2/2m - eV(Q_a)) - - - (9)$$

The dynamical variables in this Hamiltonian are $c(k), c(k)^*, Q_a, P_a$ which satisfy the standard commutation relations

$$[Q_a, P_b] = i\delta(a,b), [c(k), c(m)^*] = \delta(k,m), [c(k), c(m)] = 0, [c(k), Q_a] = [c(k), P_a] = 0 - - - (10)$$

The wave function for this problem can be expressed as

$$|\psi(t,Q)>= \sum_n \psi(t,n,Q))|n>, Q = ((Q_a))_{a=1}^N - - - (11)$$

where $\psi(t,n,Q)$ is a complex valued function, and $n = (n(1), n(2), ..., n(K))$ is a K-tuple of non-negative integers so that $|n>$ is a photon number operator state:

$$c(k)^*c(k)|n>= n(k)|n>, k = 1, 2, , ..., K - - - (12)$$

Note that from the standard harmonic oscillator algebra, we have

$$c(k)|n>= \sqrt{n(k)}|n - e_k>, c(k)^*|n>= \sqrt{n(k)+1}|n> - - - (13a)$$

with

$$e_k = [0, 0, ..., 0, 1, 0, ..., 0]^T - - - (13b)$$

being a vector of size K with a one at the k^{th} position and zeros at all the other positions. Substituting this expression for the wave function into the Schrodinger equation

$$i\partial_t|\psi(t,Q)>= H(t)|\psi(t,Q)> - - - (14a)$$

we get

$$i\partial_t\psi(t,n,Q) = (\sum_{k=1}^{K}\omega(k)n(k)\psi(t,n,Q)$$

$$+\sum_k (f_k(t)\sqrt{n(k)+1}\psi(t,n+e_k,Q)+\bar{f}_k(t)\sqrt{n(k)}\psi(t,n-e_k,Q))$$

$$+\sum_a [(-i\nabla_a+eA(Q_a))^2/2m+V(Q_a)]\psi(t,n,Q) --- (14b)$$

The aim is to introduce an additional control potential $V_0(t,Q)$ into this dynamics so that this differential equation gets modified to

$$i\partial_t\psi(t,n,Q) = (\sum_{k=1}^{K}\omega(k)n(k)\psi(t,n,Q)$$

$$+\sum_k (f_k(t)\sqrt{n(k)+1}\psi(t,n+e_k,Q)+\bar{f}_k(t)\sqrt{n(k)}\psi(t,n-e_k,Q))$$

$$+\sum_a [(-i\nabla_a+eA(Q_a))^2/2m+V(Q_a)]\psi(t,n,Q)+V_0(t,Q)\psi(t,n,Q) --- (15)$$

The control potential $V_0(t,Q)$ will be determined by a classical optimal control algorithm that causes the wave function $\psi(t,n,Q)$ to track a desired wave function $\psi_d(t,n,Q)$,ie, minimize

$$\sum_n \int_0^T |\psi_d(t,n,Q),\psi(t,n,Q)|^2 dQ --- (16)$$

More generally, we can think of introducing a control potential $V_0(t,Q)$ and also Lindblad coupling operators L_j between the field within the fibre and a bath so with control parameters $\theta(t)$ so that the mixed state of the field tracks a desired state. A brief summary of this methodology is as follows. Let $H_0(t)$ be the Hamiltonian of the quantum electromagnetic field and let $\delta H(t)$ be the unwanted Hamiltonian, ie, the Hamiltonian produced by the interaction between the classical field and the phonons with consequent scattering affecting the quantum field dynamics. The control potential is $V_0(\theta(t))$ and the Lindblad coupling operators are $L_j(\theta(t))$ so that the state of the field follows the dynamics of an open quantum system with control potential:

$$\rho'(t) = -i[H_0(t)+V_0(\theta(t)),\rho(t)]+[\delta H(t),\rho(t)]+K(\theta(t),\rho(t)) --- (17a)$$

where

$$K(\theta,\rho) = -(1/2)\sum_j (L_j(\theta)L_j(\theta)^*\rho+\rho.L_j(\theta)L_j(\theta)^*-2L_j(\theta)^*\rho.L_j(\theta)) --- (17b)$$

The desired state satisfies

$$\rho_d'(t) = -i[H_0(t), \rho_d(t)] - - - (18)$$

This means that we wish to adapt the control parameters $\theta(t)$ with time so that the counter-term $K(\theta(t), \rho(t)) - i[V_0(\theta(t)), \rho(t)]$ cancels out the effect of the disturbance $-i[\delta H(t), \rho(t)]$. One way to solve this problem is to assume that the disturbance $\delta H(t)$ is a random Hamiltonian and then choose a large set of "test states" $\rho_j, j = 1, 2, ..., N$ and control parameters $\theta(t)$ so that

$$\mathcal{E}(\theta(t)) = \sum_{k,j} w(k,j) < \| -i[\delta H(t), \rho_j] - i[V_0(\theta(t)), \rho_j] + K(\theta(t), \rho_j) \|^2 > - - - (19)$$

is minimized. However, this optimization problem requires knowledge of the statistics of the random Hamiltonian $\delta H(t)$. An alternate approach based on adaptive signal processing techniques is to use an adaptive algorithm like the stochastic gradient algorithm in which we design a TPCP map $T(.|\theta(t))$ and pass the state of the system $\rho(t)$ through this TPCP map with the control parameters $\theta(t)$ designed so that

$$\| T(\rho(t)|\theta(t)) - \rho_d(t) \|^2 - - -(20)$$

is minimized, ie, the algorithm reads

$$\theta(t+dt) = \theta(t) - \mu.dt \nabla_{\theta(t)} \| T(\rho(t+dt)|\theta(t)) - \rho_d(t+dt) \|^2 - - -(21)$$

The TPCP map T may as suggested above, be realized via a control potential and Lindblad operators. This can be made more precise, by using a differential version of the TPCP map: We minimize $\| dT_t(\rho(t)|\theta)/dt - d\rho_d(t)/dt \|^2$ where

$$dT_t(\rho(t)|\theta)/dt = -i[V_0(\theta), \rho(t)] + K(\theta, \rho(t)) - - - (22)$$

wherein, we substitute

$$d\rho_d(t)/dt = -i[H_0, \rho_d(t)] - - - (23)$$

The full algorithm then reads

$$d\theta(t)/dt = -\mu.\nabla_{\theta(t)} \| -i[V_0(\theta(t)), \rho(t)] + K(\theta(t), \rho(t)) + i[H_0, \rho_d(t)] \|^2 - - -(24)$$

This adaptive learning of the control parameters in fact defines a quantum neural network at the training stage. After the training run, we can use this learnt parameter trajectory to cancel out the disturbance in the same fibre with any other electromagnetic field input.

Of course, this methodology requires us to be able to measure the actual state $\rho(t)$ of the evolving system. However, this is a very difficult task in general owing to quantum uncertainty during the measuring process, ie, any measurement will cause the state to collapse to a different state determined by the outcome of the measurement and then further evolution of the state will begin at the collapsed state. Belavkin's theory of quantum filtering as perfected by John Gough, comes to our aid here. This involves assuming that the disturbance Hamiltonian $\delta H(t)$ is white quantum noise coupled to System Lindblad operators and based on nondemolition measurements, we can get a good real-time estimate of the state $\rho(t)$ and then use this in the adaptation algorithm in place of the true state $\rho(t)$. Alternately, suppose we parameterize the state $\rho(t)$ by classical parameters ϕ and estimate ϕ using the maximum likelihood method applied to a POVM and then use this estimated state and the control TPCP map to calculate the control parameters, then we would do well enough, but the problem with this approach is that our measurement scheme must be continuous in time which is the reason why Belavkin's method works much better. It involves modeling the disturbance Hamiltonian as quantum white noise comprising creation, annihilation and conservation component processes in the language of Hudson and Parthasarathy and then allowing input non-demolition noise again built out of the same creation, annihilation and conservation components to be incident upon the system, ie, the quantum field within the fibre and then take measurements of the output deflected noise, noting that the nondemolition property of the noise will not affect the future system dynamics, but it will get affected by the system dynamics and hence will contain information about the system state. The resulting estimate of the system state is then passed through the control TPCP map which along with an estimate of the desired system state, would yield optimal control parameter trajectories that can be used during the testing stage.

2 Some alternative approaches to removing classical electromagnetic noise

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The total magnetic vector potential has the decomposition

$$A(x) = A_c(x) + A_q(x) - - - (25)$$

where A_c, A_q are respectively the classical and the quantum components. The quantum field is small and hence dominant noise that corrupts the quantum field is caused by the classical field getting scattered by the phonons in the fibre and the resulting scattered classical field then interferes with the quantum field. We therefore assume that the incident classical field is $A_{c0}(x)$ and the scattered classical field is $A_{cs}(x)$ so that the total scattered field is their sum:

$$A_c = A_{c0} + A_{cs} - -(26)$$

Let $E_{c0}, E_{cs}, B_{c0}, B_{cs}, E_c, B_c$ denote the corresponding electric and magnetic fields:

$$E_{c0} = -\partial_t A_{c0}, E_{cs} = -\partial_t A_{cs} - - - (27a),$$

$$B_{c0} = curl A_{c0}, B_{cs} = curl A_{cs} --- (27b),$$

$$E_c = -\partial_t A_c = E_{c0} + E_{cs} --- (27c),$$

$$B_c = curl A_c = B_{c0} + B_{cs} --- (27d)$$

Let Q_a, Q'_a denote the position and velocity of the a^{th} atom carrying a charge Ze. Then we have with M denoting the mass of the atom,

$$Q''_a = (Ze/M)(E_c(t, Q_a) + Q'_a \times B_c(t, Q_a)) - \nabla U(Q_a - \xi_a) --- (28a)$$

where U is the binding potential associated with an atom and ξ_a is the atom's equilibrium position. Further, the classical field satisfies Maxwell's equations

$$\partial_t^2 A_c(t, Q) - c^2 \nabla^2 A_c(t, Q) = J_c = Ze \sum_a \delta^3(Q - Q_a) Q'_a --- (28b)$$

This has the retarded potential solution

$$A_{cs}(t, Q) = \sum_a ZeQ'_a(t_a)/(|Q - Q_a|(1 - (Q'_a(t_a), Q - Q_a(t_a))/|Q - Q_a(t_a)|) --- (29a)$$

$$V_{cs}(t, Q) = \sum_a Ze/(|Q - Q_a(t_a)|(1 - (Q'_a(t_a), Q - Q_a(t_a))/|Q - Q_a(t_a)|))) --- (29b)$$

with t'_a denoting the retarded time:

$$t_a = t - |Q - Q_a(t_a)|/c --- (29c)$$

If we neglect retardation, ie, we use a non-relativistic approximation, then

$$A_{cs}(t, Q) = \sum_a ZeQ'_a(t)/|Q - Q_a(t)| --- (29d)$$

$$V_{cs}(t, Q) = \sum_a Ze/|Q - Q_a(t)| --- (29e)$$

The electric and magnetic fields at the site of the a^{th} atom are then

$$E_c(t, Q_a) = E_{c0}(t, Q_a) + \int_0^t \sum_{b \neq a} curl_{Q_a} curl_{Q_a}(ZeQ'_b(s)/|Q_a - Q_b(s)|) ds --- (30)$$

where in the latter expression, we have used the Maxwell equation

$$curl B = curl curl A = \partial E/\partial t -- (31)$$

in a region free of currents, or equivalently,

$$E(t, Q) = \int_0^t curl curl A(s, Q) ds --- (32)$$

and

$$B_c(t, Q_a) = B_{c0}(t, Q_a) + \sum_{b \neq a} curl_{Q_a}(ZeQ'_b(t)/|Q_a - Q_b(t)|) - - - (33)$$

Solving $(a), (b), (c)$, we get $\{Q_a(t)\}_{a=1}^N, A_{cs}(t, Q)$. We require the scattered field A_{cs}. Assuming that we have solved the above equations for this, we obtain the interaction field energy between the quantum field A_q and the scattered classical field A_c as

$$H_I(t) = \int (curlA_q, curlA_{c0} + curlA_{cs})d^3x + \int (\partial_t A_q, \partial_t A_{c0} + \partial_t A_{cs} + \nabla V_{cs})d^3x = \delta H(t) - - - (34)$$

and we wish to cancel this interference component from the state dynamics of the quantum field:

$$\rho'(t) = -i[H_0 + \delta H(t), \rho(t)] - - - (35)$$

where

$$H_0 = (1/2) \int (curlA_q)^2 + (\partial_t A_q)^2)d^3x = \sum_k \omega(k)c(k)^* c(k) - - - (36)$$

This cancellation can be achieved by the methods of optimal control and filtering discussed above.

Remark: If $\delta H(t)$ behaves like zero mean white noise, then we can write approximately for small τ,

$$\rho(t + \tau) \approx \rho(t) + \tau.\rho'(t) + (\tau^2/2)\rho''(t) - - - (37)$$

$$= \rho(t) - i\tau[H_0 + \delta H(t), \rho(t)] + (\tau^2/2)(-i[\delta H'(t), \rho(t)] - [H_0 + \delta H(t), [H_0 + \delta H(t), \rho(t)]]) - - - (38)$$

Taking the statistical average on both sides, we get

$$< \rho(t+\tau) >=< \rho(t) > -i\tau[H_0, < \rho(t) >] - (\tau^2/2) < [\delta H(t), [\delta H(t), \rho(t)]] > - - - (39)$$

where we neglect $O(\tau^2)$ terms, noting that $< (\delta H(t)\tau)^2 >$ is $O(\tau)$, because $\delta H(t)$ is white noise. We further observe that

$$\tau^2 < [\delta H(t), [\delta H(t), \rho(t)]] >_{ij}$$

$$= \tau^2 < \delta H(t)^2 \rho(t) + \rho(t)\delta H(t)^2 - 2\delta H(t)\rho(t).\delta H(t) >_{ij}$$

$$= \tau^2[< \delta H(t)_{ik}\delta H(t)_{km} >< \rho_{mj}(t) > + < \rho_{ik}(t) >< \delta H_{km}(t)\delta H_{mj}(t) >$$

$$-2 < \delta H_{ik}(t)\delta H_{mj}(t) >< \rho_{km}(t) >] - - - (40)$$

with summation over the repeated indices k, m being implied. By expanding the Hermitian matrix

$$K(ij|km)) =< \delta H_{ij}(t)\delta \bar{H}_{km}(t) >=< \delta H_{ij}(t).\delta H_{mk}(t) > - - -(41)$$

using the spectral representation for self-adjoint matrices, we get the Lindblad form on noting that $\tau^2.K(ij|km)$ is $O(\tau)$, we obtain the Lindblad form:

$$(\tau^2/2) < [\delta H(t), [\delta H(t), \rho(t)]] >=$$

$$= (\tau/2)\sum_j [L_j L_j^* < \rho(t) > + < \rho(t) > L_j L_j^* - 2L_j^* < \rho(t) > L_j] - - - (42)$$

and hence calling $< \rho(t) >$ as $\rho(t)$, we obtain the master equation

$$\rho'(t) = -i[H_0, \rho(t) + \theta(\rho(t) - - - (43)$$

on taking $lim \tau \to 0$, where

$$\theta(\rho) = (-1/2)\sum_j [L_j L_j^*.\rho + \rho.L_j L_j^* - 2L_j^*.\rho.L_j] - - - (44)$$

In other words, the disturbance to the state of the quantum field caused by the classical field after getting scattered by the phonons appears in the form of the Lindblad term $\theta(\rho)$. The operator θ depends on the $L_j'w$ which in turn depends on the statistical correlations of the perturbing Hamiltonian $\delta H(t)$ which in turn, depends on the incident classical field and the structure of the phonon lattice of the optical fibre. In many cases, these statistics can be computed from basic classical physics (more precisely, classical statistical mechanics) along the lines indicated above. Specifically, making appropriate approximations, letting $U(Q)$, $Q = (Q_a)_{a=1}^N$ denote the binding potential energy of the phonon lattice, the approximate equations of motion of the lattice are

$$Q_a'' = (e_a/M)(E_{c0}(t, Q_a) + Q_a' \times B_{c0}(t, Q_a)) - \nabla_{Q_a} U(Q)$$

$$\approx (e_a/M)(E_{c0}(t, \xi_a) + Q_a' \times B_{c0}(t, \xi_a)) + \sum_b K(a, b)(Q_b - \xi_b) - - - (45a)$$

where

$$K(a, b) = -\nabla_{Q_a} \nabla_{Q_b}^T U(\xi) - - - (45b)$$

These form a system of linear second-order coupled time-varying differential equations and can be solved by the standard Dyson series method. Specifically, in matrix form, we can write

$$Q''(t) = A_1(t)Q'(t) - A_2 Q(t) + b(t) - - - (46)$$

where $Q(t) = ((Q_a(t))$ is an $N \times 1$ vector valued function of time, and $A_1(t)$ is a time varying $N \times N$ matrix, A_2 is a constant $N \times N$ matrix, $b(t)$ is an $N \times 1$ time varying vector. Here, the matrix $A_1(t)$ is a linear function of the magnetic field $(B_{c0}(t, \xi_a) : a = 1, 2, ..., N)$, while $b(t)$ is a linear function of the electric field $(E_{c0}(t, \xi_a) : a = 1, 2, ..., N)$ and A_2 has components $K(a, b), a, b = 1, 2, ..., N$. We can assume that the initial positions and velocities of the phonons in the lattice have the Gibbsian distribution with total energy $E(Q,Q') = \sum_{a=1}^{N} mQ_a'^2/2 + (1/2)Q^T A_2 Q$. Thus, the initial probability distribution of the positions and velocities of the phonons in the lattice at temperature $T = 1/k\beta$ is the multivariate normal distribution

$$f_0(Q(0),Q'(0)) = Z(\beta)^{-1}.exp(-\beta.E(Q(0),Q'(0))) - - - (47)$$

and then, writing the solution to the above linear differential equations as

$$(Q(t),Q'(t)) = \psi(t,Q(0),Q'(0)|E_{c0},B_{c0}) - - - (48)$$

with the inverse of this function being given by

$$(Q(0),Q'(0)) = \psi^{-1}(t,Q(t),Q'(t)|E_{c0},B_{c0}) - - - (49)$$

it follows that the joint probability density of $(Q(t),Q'(t))$ is given by

$$f_t(Q,Q') = f_0(\psi^{-1}(t,Q,Q'|C_{c0},B_{c0}))J_{\psi^{-1}}(t,Q',Q'|E_{c0},B_{c0}) - - - (50)$$

where $J_{\psi^{-1}}$ is the Jacobian determinant of ψ^{-1} w.r.t (Q,Q'). It should be noted in fact, that with the above approximations, ψ is a linear function of $(Q(0),Q'(0))$. In fact, the differential equation (a) can be cast in state variable form as

$$\xi'(t) = A(t)\xi(t) + \eta(t) - - - (51a)$$

where

$$\xi(t) = \begin{pmatrix} Q(t) \\ Q'(t) \end{pmatrix} - - - (51b)$$

and

$$A(t) = \begin{pmatrix} 0 & I_N \\ -A_2 & A_1(t) \end{pmatrix} - - - (51c),$$

$$\eta(t) = \begin{pmatrix} 0 \\ b(t) \end{pmatrix} - - - (51d)$$

and hence if $\Phi(t,s)$ is the state transition matrix corresponding to $A(t)$, ie, it satisfies

$$\partial_t \Phi(t,s) = A(t)\Phi(t,s), t \geq s \geq 0 - - - (52a),$$

$$\Phi(s,s) = I_{2N} - - - (52b)$$

then the solution is

$$\xi(t) = \Phi(t,0)\xi(0) + \int_0^t \Phi(t,s)\eta(s)ds = \psi(t,\xi(0)|E_{c0},B_{c0}) - - - (52c)$$

and thus the probability density of $\xi(t)$ is again a Gaussian density

$$f_t(\xi) = f_t(\xi|E_{c0},B_{c0}) =$$

$$f_0(\Phi(t,0)^{-1}(\xi - \int_0^t \Phi(t,s)\eta(s)ds))|\Phi(t,0)|^{-1} - - - (53)$$

Of course, if we do not make the linearized approximation in the equations of motion of the phonon lattice then, the positions and velocities of the phonon lattice will be nonlinear functions of their initial values and further the initial density of the phonon lattice will also be non-Gaussian:

$$f_0(Q(0),Q'(0)) = Z(\beta)^{-1}.exp(-\beta(\sum_a mQ'_a(0)^2/2 + U(Q))) - - - (54)$$

and then the non-Gaussian density f_t of $\xi(t) = (Q(t),Q'(t))$ will be given by (c). The statistics of the perturbing Hamiltonian $\delta H(t) = H_I(t)$ can now be readily computed. Specifically, we have that

$$\delta H(t) = \int curl(A_q(t,r), curl(A_{c0}(t,r) + \sum_a e_a Q'_a(t)/|r - Q_a(t)|))d^3r$$

$$+\int(\partial_t A_q(t,r), \partial_t(A_{c0}(t,r) + \sum_a e_a Q'_a(t)/|r-Q_a(t)|) + \nabla(\sum_a e_a/|r-Q_a(t)|))d^3r$$

$$= \delta H(t|Q(t),Q'(t)) = F(t,c(k),c(k)^*, k = 1,2,...,K|Q(t),Q'(t))$$

with

$$A_q(t,r) = \sum_k (c(k)u_k(t,r) + c(k)^*\bar{u}_k(t,r))$$

with $Q''_a(t)$ being given in terms of $Q(t),Q'(t)$ by the equations of motion

$$Q''_a(t) =\approx (e_a/M)(E_{c0}(t,\xi_a) + Q'_a \times B_{c0}(t,\xi_a)) + \sum_b K(a,b)(Q_b - \xi_b)$$

The statistics of the random operator $\delta H(t)$ is then obtained using the joint density f_t of $(Q(t),Q'(t))$ calculated above. Specifically, for example, the correlations of the perturbing Hamiltonian can be computed as

$$< \delta H(t) \otimes \delta H(t) >=$$

$$\int F(t,c,c^*|\xi) \otimes F(c,x^*|\xi)f_t(\xi)d^{6N}\xi$$

Basic method of optimal control: Suppose that the state dynamics of the noisy quantum system with the coupling of the system to the noisy bath comprising of the classical field interacting with the random photon lattice is described by the master equation

$$\rho'(t) = -i[H_0, \rho(t)] + \theta(\rho(t))$$

where H_0 is the Hamiltonian of the quantum field and θ is the Lindblad noise term which we wish to cancel by making $\rho(t)$ track $\rho_d(t)$. We shall achieve this tracking by introducing a control time varying potential term

$$V(u(t)) = \sum_{k=1}^N u_k(t)V_k$$

with the $V_k's$ being Hermitian matrices and $u_k(t)$ real control functions of time. The controlled system then becomes

$$\rho'(t) = -i[H_0 + V(u(t)), \rho(t)] + \theta(\rho(t)) = \phi(u(t), \rho(t))$$

ρ_d satisfies the noiseless equation

$$\rho'_d(t) = -i[H_0, \rho_d(t)]$$

and $u(t) = ((u_k(t)))$ shall be designed so that

$$\int_0^T (\| \rho(t) - \rho_d(t) \|^2 + u(t)^T Q u(t)) dt = \int_0^T L(\rho(t), u(t)) dt$$

is a minimum where Q is an $N \times N$ positive definite matrix. This cost function ensures that by spending minimum control energy $\int_0^T u(t)^T Q u(t) dt$, we are able to make the density track the desired density thereby enabling us to control the system so as to reduce the noisy effects significantly. To this end, we define

$$C(t, \rho(t)) = min_{u(s), t \leq s \leq T} \int_t^T L(\rho(s), u(s)) ds$$

Then, standard optimal control methods give

$$C(t, \rho(t)) = min_{u(t)}(L(\rho(t), u(t))dt + C(t + dt, \rho(t + dt)))$$

or equivalently,

$$min_u(L(\rho, u) + \sum_{i,j} (\frac{\partial C(t, \rho)}{\partial \rho_{ij}} \phi_{ij}(u, \rho))$$
$$+ \partial_t C(t, \rho) = 0$$

The algorithm for solving this Bellman-Hamilton-Jacobi equation is to start with $0 = C(T, \rho)$. Then, assuming that $C(t, \rho)$ is known, calculate $u(t) = u(t, \rho)$ so that

$$F(t, u, \rho) = (L(\rho, u) + \sum_{i,j} (\frac{\partial C(t, \rho)}{\partial \rho_{ij}} \phi_{ij}(u, \rho))$$

is minimized and then calculate $C(t - dt, \rho)$ using

$$C(t - dt, \rho) = C(t, \rho) = F(t, u(t, \rho), \rho)$$

Keep continuing this iteration with progressively decreasing time in steps of dt until time $t = 0$ is reached. A total of $N = [t/dt]$ optimizations will be required.

Getting an accurate value of the Lindblad operators for determining the map ϕ to implement the optimal control algorithm can sometimes be very hard as we just saw. However, we can use the results of filtering theory especially when the state dynamics is corrupted by noise, to get a filtered estimate, like a Belavkin filter estimate of $\rho(t)$ on a real-time basis. Specifically, if $Y(t)$ denotes the Abelian measurement family, the quantum filtered estimate of $\rho(t)$ which we will denote by $\hat{\rho}(t)$ will satisfy a stochastic Schrodinger equation of the form

$$d\hat{\rho}(t) = F_0(\hat{\rho}(t), u(t))dt + G(\hat{\rho}(t))(dY(t) - H(t, \hat{\rho}(t))dt)$$

and it is possible by the same methods of optimal control to minimize

$$\int_0^T L(\hat{\rho}(t), u(t))dt$$

The filtering method has in addition, certain advantages: For example, if the Lindblad noise operators used for constructing θ_1, ϕ are not accurately known, then we can introduce some additional parameters $\eta(t)$ in them and treat the extended state to be estimated as $[\rho(t), \eta(t)]$ where $\eta(t)$ will satisfy a classical stochastic differential equation of the form

$$d\eta(t) = d\epsilon(t)$$

with $\epsilon(t)$ being a classical vector-valued Brownian motion process. Then a combination of quantum and classical filtering can be used to estimate the extended state very much like a quantum generalization of the extended Kalman filter.

A simple way to look at this filtering-based algorithm from the viewpoint of classical stochastic filtering theory is to consider the controlled dynamics of $\rho(t)$ as a classical stochastic differential equation with white Gaussian noise added to the control potential or equivalently to the Hamiltonian so that this dynamics is

$$d\rho(t) = -i[(H_0 + V(u(t))dt + \sigma dB(t), \rho(t)] + \theta(\rho(t)|\eta(t))$$

$$d\eta(t) = d\epsilon(t) = \sigma.dB(t)$$

where $\eta(t)$ contains the unknown information in the Lindblad noise parameters. and taking measurements as the average of a vector of observables $X_k, k = 1, 2, ..., p$, so that the measurement process is

$$dz_k(t) = Tr(\rho(t)X_k)dt + \sigma_k dB_k(t), k = 1, 2, ..., p$$

where $B_k, k = 1, 2, ..., N, B$ are independent Brownian motion processes. The extended Kalman filtered estimate of $\rho(t), \eta(t)$ can be constructed given the measurements $Z(t) = \sigma(z_k(s) : k = 1, 2, ..., p, s \leq t)$ and in particular, the unknown information in the Lindblad operators can be estimated reliably. The basic idea here is that the effect of noise in the form of the classical field interacting with the phonon lattice on the state of the quantum field is reflected in the average values of the observables X_k being measured and hence such a measurement can be used to estimate this classical noise.

Mixture of classical and quantum filtering for estimating both the quantum state of the field and the classical parameters upon which the Lindblad operators depend.

The Hudson-Parthasarathy qsde for the joint unitary evolution operator of the field and the noisy bath has the form (assuming absence of the conservation noise process)

$$dU(t) = (-i(H(\eta(t)) + P(\eta(t))dt + L(\eta(t))dA(t) - L(\eta(t))dA(t)^*)U(t)$$

$$d\eta(t) = d\epsilon(t) = \sigma.dB(t)$$

Non-demolition measurements are

$$Y(t) = U(t)^*Y_i(t)U(t), Y_i(t) = cA(t) + \bar{c}A(t)^*$$

We take a system observable Xa and a function $f(\eta)$ of the parameter η and construct the conditional expectation

$$\pi_t(fX) = \mathbb{E}(f(\eta(t))j_t(X)|\eta_0(t)) = \mathbb{E}j_t(fX)|\eta_o(t))$$

where $j_t(fX) = f(\eta(t))j_t(X)$ and

$$\eta_o(t) = \sigma(Y_o(s) : s \leq t)$$

The stochastic differential equation satisfied by $\pi_t(fX)$ can be derived from the orthogonality principle, also called the reference probability approach due to J.Gough and C.Kostler, given by

$$\mathbb{E}(j_t(fX) - \pi_t(fX))C(t)) = 0$$

with

$$dC(t) = g(t)C(t)dY(t), t \geq 0, C(0) = 1$$

so that $C(t)$ can be expressed as a nonlinear functional of $\eta_o(t)$ which is an Abelian Von-Neumann algebra. The filter π_t may be assumed to satisfy a stochastic differential equation

$$d\pi_t(fX) = F_t(fX)dt + G_t(fX)dY(t)$$

with $F_t(fX), G_t(fX) \in \eta_o(t)$ obtained from the orthogonality principle above or equivalently, since g is an arbitrary real function of time, as

$$\mathbb{E}(dj_t(fX) - d\pi_t(fX)|\eta_o(t)) = 0,$$

$$\mathbb{E}(j_t(fX) - \pi_t(fX)|\eta_o(t)) + \mathbb{E}(dj_t(fX) - d\pi_t(fX))dY(t)|\eta_o(t)) = 0$$

The crucial step here is computing $dj_t(fX)$ or more precisely, $\mathbb{E}(dj_t(fX)|\eta_o(t))$ and $\mathbb{E}(dj_t(fX).dY(t)|\eta_o(t))$ using a combination of quantum and classical Ito's formula:

$$j_t(fX) = f(\eta(t))j_t(X) = f(\eta(t))U(t)^*XU(t)$$

so

$$dj_t(fX) = f'(\eta(t))j_t(X).d\eta(t) + f''(\eta(t))\sigma^2 dt.j_t(X)$$
$$+f(\eta(t))dj_t(X)$$

where

$$dj_t(X) = j_t(\theta_0(X))dt + j_t(\theta_1(X))dA(t) + j_t(\theta_2(X))dA(t)^*$$

This gives

$$\mathbb{E}(dj_t(fX)|\eta_o(t)) =$$
$$dt[(\sigma^2/2)\pi_t(f''X) + \pi_t(f\theta_0(X)) + \pi_t(f\theta_1(X)) + \pi_t(f\theta_2(X))]$$

Note that if we write

$$\pi_t(fX) = \int Tr(\hat{\rho}(t,\eta)X)d\eta$$

so that $\hat{\rho}(t,\eta)$ stands for the joint density of the quantum system and the classical parameter η, then

$$\mathbb{E}(j_t(fX)|\eta_0(t))/dt =$$
$$\int Tr[(\sigma^2/2)\partial_\eta^2\rho(t,\eta) + \theta_0^*(\rho(t,\eta)) + \theta_1^*(\rho(t,\eta) + \theta_2^*(\rho(t,\eta))f(\eta)X]d\eta$$
$$\int Tr[D_0(\hat{\rho}(t,\eta))f(\eta)X]d\eta$$

where D_0 is a second-order differential operator in η acting in the space of the tensor product of the space of all second-order real-valued differentiable functions on the parameter space with the space of linear operators in the system Hilbert space. Continuing in this way, we obtain the forms of $F_t(fX), G_t(fX)$ as

$$F_t(fX) = \int Tr(D_1(\hat{\rho}(t,\eta))f(\eta)X]d\eta, G_t(fX) = \int Tr[D_2(\hat{\rho}(t,\eta))f(\eta)X]d\eta$$

where D_1, D_2 are third-degree nonlinear second-order differential operators acting in the same space as discussed above. Thus, our Stochastic Schrodinger equation for $\hat{\rho}(t,\eta)$ has the form

$$d\hat{\rho}(t,\eta) = D_1(\hat{\rho}(t,\eta))dt + D_2(\hat{\rho}(t,\eta))dY(t)$$

Once $\hat{\rho}(t,\eta)$ has been solved for using this equation, we can compute the maximum likelihood estimate of $\eta(t)$ by maximizing its conditional pdf:

$$p(t,\eta|\eta_o(t)) = Tr(\hat{\rho}(t,\eta))$$

3 frequency, wavelength and bandwidth aspects of the quantum information transmission problem

Assume that the Hamiltonian of the field perturbed by the noisy classical field after it interacts with the photon lattice is as above, ie,

$$H(t) = H_0 + \delta H(t)$$

where

$$\delta H(t) = \sum_k f_k(t) V_k$$

with the $f_k(t)'s$ being real classical random processes in time and the $V_k's$ being self-adjoint operators specifically constructed out of the quantum photon creation and annihilation operators $c(k), c(k)^*$. Let X, Y be observables in quantum field space, ie, once again constructed as functions of the $c(k), c(k)^*$. For example, X, Y can be certain components of the electric and magnetic fields at time zero at a specific spatial point. The initial state of the quantum field is $\rho(0)$, for example, it could be the Gibbs Gaussian state $Z(\beta).exp(-\beta \sum_k \omega(k)c(k)^* c(k))$ with

$$Z(\beta)^{-1} = Tr(exp(-\beta. \sum_k \omega(k)c(k)^* c(k))$$

The observables X, Y after time t become according to Heisenberg matrix mechanics,

$$X(t) = U(t)^* X U(t), Y(t) = U(t)^* Y U(t)$$

where

$$U(t) = T\{exp(-i \int_0^t H(s)ds)\}$$

is the random Schrodinger unitary evolution after time t in accordance with the random Hamiltonian $H(t)$. It is well known (ie, by Dirac's interaction picture of the dynamics) that

$$U(t) = U_0(t)W(t), U_0(t) = exp(-itH_0)$$

and $W(t)$ satisfies

$$W'(t) = -i\tilde{\delta} H(t)W(t),$$

where

$$\tilde{\delta} H(t) = U_0(t)^* \delta H(t) U_0(t) = \sum_k f_k(t)\tilde{V}_k(t),$$

$$\tilde{V}_k(t) = U_0(t)^* V_k U_0(t)$$

We get the Dyson series solution

$$W(t) = I + \sum_{n\geq 1} (-i)^n \int_{0<s_n<s_{n-1}<...<s_1<t} \delta H(s_1)...\delta H(s_n) ds_1...ds_n$$

In particular, since the noise is small, we have approximately

$$W(t) \approx I - i\int_0^t \tilde{\delta}H(s)ds = I - \sum_k \int_0^t kf_k(s)\tilde{V}_k(s)ds$$

The quantum correlation between the processes $X(.), Y(.)$ is given by

$$R(t+\tau,t) = R_{XY}(t+\tau,t0 =< Tr(\rho(0)X(t+\tau).Y(t)) >$$

Actually, we should symmetrize this expression in order to get a real correlation function, ie, define it as

$$\tilde{R}(t+\tau,t) =< Tr(\rho(0)(X(t+\tau)Y(t) + Y(t)X(t+\tau))) >$$

Let us however consider the first non-symmetrized expression:

$$R(t+\tau,t) =< Tr(\rho(0)U(t+\tau)^*XU(t+\tau).U(t)^*YU(t)) >$$

$$=< Tr(\rho(0)W(t+\tau)^*U_0(t+\tau)^*XU_0(t+\tau)W(t+\tau)W(t)^*U_0(t)^*YU_0(t))W(t)) >$$

$$=< Tr(\rho(0)W(t+\tau)^*\tilde{X}(t+\tau)W(t+\tau)W(t)^*\tilde{Y}(t)W(t)) >$$

$$= Tr(\rho(0) < W(t+\tau)^*\tilde{X}(t+\tau)W(t+\tau)W(t)^*\tilde{Y}(t)W(t)) >)$$

Note that $< .,. >$ stands for the classical statistical average w.r.t the probability distribution of the $\delta H(.)$ or equivalently, of the processes $f_k(.)$. $\tilde{X}(t) = U_0(t)^*XU_0(t)$ and likewise for $\tilde{Y}(t)$, ie, these stand for the evolution of the observables X, Y under the unperturbed Hamiltonian H_0. Writing

$$\int_0^t \tilde{\delta}H(s)ds = Z_1(t),$$

$$\int_{0<s_2<s_1<t} \tilde{\delta}H(s_1)\tilde{\delta}H(s_2)ds_1ds_2 = Z_2(t),$$

upto quadratic approximations in the noise amplitude $\delta H(.)$, we can write, noting that $\delta H(.)$ is assumed to have zero mean,

$$W(t) = 1 - iZ_1(t) - Z_2(t)$$

and

$$W(t)^* = 1 + iZ_1(t) - Z_2(t)^*$$

and more generally,

$$W(t+\tau)W(t)^* = T(exp(-i\int_t^{t+\tau} \tilde{\delta}H(s)ds))$$

$$= 1 - iZ_1(t,t+\tau) + Z_2(t,t+\tau)$$

where

$$Z_1(t,t+\tau) = \int_t^{t+\tau} \tilde{\delta}H(s)ds,$$

$$Z_2(t,t+\tau) = \int_{t<s_2<s_1<t+\tau} \tilde{\delta}H(s_1)\tilde{\delta}H(s_2)ds_1ds_2$$

so that again up to quadratic orders in δH, we get

$$W(t+\tau)^*\tilde{X}(t+\tau)W(t+\tau)W(t)^*\tilde{Y}(t)W(t) =$$

$$= (1+iZ_1(t+\tau)-Z_2(t+\tau)^*)\tilde{X}(t+\tau)(1-iZ_1(t,t+\tau)-Z_2(t,t+\tau))\tilde{Y}(t)(1-iZ_1(t)-Z_2(t))$$

$$= \tilde{X}(t+\tau)\tilde{Y}(t)$$

$$+Z_1(t+\tau)\tilde{X}(t+\tau)Z_1(t,t+\tau)\tilde{Y}(t)$$

$$+Z_1(t+\tau)\tilde{X}(t+\tau)\tilde{Y}(t)Z_1(t)$$

$$-\tilde{X}(t+\tau)Z_1(t,t+\tau)\tilde{Y}(t)Z_1(t)$$

$$-Z_2(t+\tau)^*\tilde{X}(t+\tau)\tilde{Y}(t)$$

$$-\tilde{X}(t+\tau)Z_2(t,t+\tau)\tilde{Y}(t)$$

$$-\tilde{X}(t+\tau)\tilde{Y}(t)Z_2(t)$$

Therefore,

$$< W(t+\tau)^*\tilde{X}(t+\tau)W(t+\tau)W(t)^*\tilde{Y}(t)W(t) > -\tilde{X}(t+\tau)\tilde{Y}(t)$$

4 Analysis of the power spectrum of observables when the noisy perturbation is modeled in accordance with the Hudson-Parthasarathy quantum stochastic calculus

We assume the noise component in the Hamiltonian of the quantum field to be described by the Hudson-Parthasarathy generalized noise processes $\Lambda_b^a(t), a, b \geq 0$ with $\Lambda_0^0(t) = t$ only being the signal component. These generalized noise processes satisfy the quantum Ito formula

$$d\Lambda_b^a(t).d\Lambda_d^c(t) = \epsilon_d^a d\Lambda_b^c(t0$$

The generalized HPS qsde is then given by

$$dU(t) = (L_b^a d\Lambda_a^b(t))U(t)$$

with summation over the repeated indices $a, b = 0, 1, ..., N$ being implied. $L_0^0 = -iH_0$ is the unperturbed Hamiltonian of the quantum field so that the perturbing white quantum noisy Hamiltonian is given by

$$\delta H(t) = i. \sum_{a+b\geq 1} L_b^a (d\Lambda_a^b(t)/dt)$$

Writing

$$U(t) = U_0(t)W(t), U_0(t) = exp(-itH_0)$$

we get using standard interaction picture arguments:

$$W'(t) = \sum_{a+b\geq 1} L_b^a(t) d\Lambda_a^b(t) W(t)$$

where

$$L_b^a(t) = U_0(t)^* L_b^a U_0(t)$$

The solution is then the following chaos expansion

$$W(t) = I + \sum_{n\geq 1} \int_{0<t_n<...<t_1<t} L_{b(1)}^{a(1)}(t_1)...L_{b(n)}^{a(n)}(t_n) d\Lambda_{a(1)}^{b(1)}(t_1)...d\Lambda_{a(n)}^{b(n)}(t_n) dt_1...dt_n$$

If X, Y are two system observables, and we define

$$\tilde{X}(t) = U_0(t)^* X U_0(t), \tilde{Y}(t) = U_0(t)^* Y U_0(t)$$

then the correlation between the processes

$$X(t) = U(t)^* X U(t) = W(t)^* \tilde{X}(t) W(t), Y(t) = U(t)^* Y U(t) = W(t)^* \tilde{Y}(t) W(t)$$

is given by

$$R(t, t+\tau) = Tr(\rho(0) X(t+\tau) Y(t))$$

where

$$\rho(0) = \rho_s(0) \otimes |\phi(u)><\phi(u)|$$

with $\rho_s(0)$ being the initial system state and the noisy bath being assumed to be in the coherent state $|\phi(u)>$. This expression for the correlation can equivalently be expressed using the above interaction picture as

$$R(t, t+\tau) = Tr(\rho(0) W(t+\tau)^* \tilde{X}(t+\tau) W(t, t+\tau) \tilde{Y}(t) W(t))$$

where

$$W(t, t+\tau) = W(t+\tau) W(t)^* =$$

$$I + \sum_{n\geq 1} \int_{t<t_n<...<t_1<t+\tau} L_{b(1)}^{a(1)}(t_1)...L_{b(n)}^{a(n)}(t_n) d\Lambda_{a(1)}^{b(1)}(t_1)...d\Lambda_{a(n)}^{b(n)}(t_n) dt_1...dt_n$$

Since we are interested only in noise effects up to second order in their amplitude while computing the correlations and power spectra, we write

$$W(t) \approx 1 + W_1(t) + W_2(t)$$

where

$$W_1(t) = \int_0^t L_b^a(t_1) d\Lambda_a^b(t_1),$$

$$W_2(t) = \int_{0<t_2<t_1<t} L^{(a(1)}_{b(1)}(t_1)L^{a(2)}_{b(2)}(t_2)d\Lambda^{b(1)}_{a(1)}(t_1)d\Lambda^{b(2)}_{a(2)}(t_2)$$

$$W_1(t,t+\tau) = \int_t^{t+\tau} L^a_b(t_1)d\Lambda^b_a(t_1),$$

$$W_2(t,t+\tau) = \int_{t<t_2<t_1<t+\tau} L^{(a(1)}_{b(1)}(t_1)L^{a(2)}_{b(2)}(t_2)d\Lambda^{b(1)}_{a(1)}(t_1)d\Lambda^{b(2)}_{a(2)}(t_2)$$

and we can then write up to second order,

$$R(t,t+\tau) = Tr(\rho(0)[(1+W_1(t+\tau)^*+W_2(t+\tau))\tilde{X}(t+\tau)(1+W_1(t,t+\tau)+W_2(t,t+\tau))\tilde{Y}(t)(1+W_1(t)+W_2(t))])$$

$$= Tr(\rho_s(0)\tilde{X}(t+\tau)\tilde{Y}(t))$$

$$+Tr(\rho(0)...$$

as before. Consider, for example, the computation of the term

$$Tr(\rho(0)(W_1(t+\tau)^*\tilde{X}(t+\tau)W_1(t,t+\tau)\tilde{Y}(t))$$

$$= \int_{0<t_1<t+\tau, t<t_2<t+\tau} Tr(\rho_s(0)(L^a_b(t_1))^*\tilde{X}(t+\tau)L^c_d(t_2)\tilde{Y}(t)) < \phi(u)|d\Lambda^a_b(t_1)d\Lambda^d_c(t_2)|\phi(u) >$$

$$= \int_{0<t_1<t+\tau, t<t_2<t+\tau} Tr(\rho_s(0)(L^a_b(t_1))^*\tilde{X}(t+\tau)L^c_d(t_2)\tilde{Y}(t))u_a(t_1)u_d(t_2)\bar{u}_b(t_1)\bar{u}_c(t_2)dt_1dt_2$$

$$+ \int_{t<t_2<t+\tau} Tr(\rho_s(0)(L^a_b(t_2))^*\tilde{X}(t+\tau)L^c_d(t_2)\tilde{Y}(t)) < \phi(u)|\epsilon^a_c\Lambda^d_b(t_2)|\phi(u) >$$

where the last term on the rhs comes from the quantum Ito formula. This further evaluates to

$$Tr(\rho(0)(W_1(t+\tau)^*\tilde{X}(t+\tau)W_1(t,t+\tau)\tilde{Y}(t))$$

$$= \int_{0<t_1<t+\tau, t<t_2<t+\tau} Tr(\rho_s(0)(L^a_b(t_1))^*\tilde{X}(t+\tau)L^c_d(t_2)\tilde{Y}(t))u_a(t_1)u_d(t_2)\bar{u}_b(t_1)\bar{u}_c(t_2)dt_1dt_2$$

$$+ \int_{t<t_2<t+\tau} Tr(\rho_s(0)(L^a_b(t_2))^*\tilde{X}(t+\tau)L^c_d(t_2)\tilde{Y}(t)).\epsilon^a_c u_d(t_2)\bar{u}_b(t_2)dt_2$$

Now observe that writing the spectral decomposition of H_0 as

$$H_0 = \sum_n |n > E(n) < n|,$$

we get

$$\tilde{X}(t) = exp(itH_0).X.exp(-itH_0) = \sum_{n,m} exp(itE(nm)) < n|X|m > |n >< m|$$

where

$$E(nm) = E(n) - E(m)$$

and likewise, for $\tilde{Y}(t)$ as also for

$$L_b^a(t) = \sum_{n,m} exp(itE(nm)) < n|L_b^a|m > |n >>< m|$$

Therefore, the integrand of the first term on the rhs is given by

$$Tr(\rho_s(0)(L_b^a(t_1))^* \tilde{X}(t+\tau) L_d^c(t_2) \tilde{Y}(t))$$

$$\times u_a(t_1) u_d(t_2) \bar{u}_b(t_1) \bar{u}_c(t_2)$$

$$=< k|\rho_s(0)|m >< m|L_b^a|n >^* exp(it_1 E(mn)) < n|\tilde{X}(t+\tau)|r > exp(it_2 E(rs)) < r|L_d^c|s >< s|\tilde{Y}(t)|k >$$

$$\times u_a(t_1) u_d(t_2) \bar{u}_b(t_1) \bar{u}_c(t_2)$$

$$=< k|\rho_s(0)|m >< m|L_b^a|n >^* exp(it_1 E(mn)) exp(i(t+\tau)E(nr)) < n|X|r > exp(it_2 E(rs))$$

$$\times < r|L_d^c|s >< s|X|k > exp(itE(sk)) u_a(t_1) u_d(t_2) \bar{u}_b(t_1) \bar{u}_c(t_2)$$

with summation over the repeated indices $k, m, n, r, s, a, b, c, d$ being understood.
This expression can be rearranged as

$$< k|\rho_s(0)|m >< m|L_b^a|n >^* < n|X|r >< r|L_d^c|s >< s|X|k > u_a(t_1) u_d(t_2) \bar{u}_b(t_1) \bar{u}_c(t_2)$$

$$\times exp(it(E(nr) + E(sk))) exp(i\tau E(nr)).exp(it_1 E(mn)).exp(it_2 E(rs))$$

The integral w.r.t t_1, t_2 over the required range, ie, the first term on the rhs of the earlier equation is given by

$$= \int_{0<t_1<t+\tau, t<t_2<t+\tau} Tr(\rho_s(0)(L_b^a(t_1))^* \tilde{X}(t+\tau) L_d^c(t_2) \tilde{Y}(t))$$

$$\times u_a(t_1) u_d(t_2) \bar{u}_b(t_1) \bar{u}_c(t_2) dt_1 dt_2$$

$$= exp(it(E(nr)+E(sk))) exp(i\tau E(nr)) < k|\rho_s(0)|m >< m|L_b^a|n >^* < n|X|r >< r|L_d^c|s >< s|X|k >$$

$$\times \int_{0<t_1<t+\tau, t<t_2<t+\tau} u_a(t_1) u_d(t_2) \bar{u}_b(t_1) \bar{u}_c(t_2).exp(it_1 E(mn) exp(it_2 E(rs)) dt_1 dt_2$$

Define the complex functions

$$F(u, a, b, s, t, \omega) = \int_s^t u_a(t_1) \bar{u}_b(t_1) exp(i\omega t_1) dt_1, 0 \leq s \leq t$$

Then, we can express the above integral as

$$exp(it(E(nr)+E(sk))) exp(i\tau E(nr))(< k|\rho_s(0)|m >< m|L_b^a|n >^* < n|X|r >< r|L_d^c|s >< s|X|k >$$

$$\times F(u, a, b, 0, t+\tau, E(mn)).F(u, d, c, t, t+\tau, E(rs))$$

Likewise, we evaluate the second term, ie, the term arising from Quantum Ito's formula:

$$\int_{t<t_2<t+\tau} Tr(\rho_s(0)(L_b^a(t_2))^* \tilde{X}(t+\tau) L_d^c(t_2) \tilde{Y}(t)).\epsilon_c^a u_d(t_2) \bar{u}_b(t_2) dt_2$$

$$= exp(it(E(nr)+E(sk)))exp(i\tau E(nr))(< k|\rho_s(0)|m >< m|L_b^a|n >^* < n|X|r >< r|L_d^c|s >< s|Y|k >$$

$$\int_{t<t_2<t+\tau} \epsilon_c^a u_d(t_2)\bar{u}_b(t_2).exp(it_2(E(rs)+E(mn)))dt_2$$

$$= exp(it(E(nr)+E(sk)))exp(i\tau E(nr)) < k|\rho_s(0)|m >< m|L_b^a|n >^* < n|X|r >< r|L_d^c|s >< s|Y|k > \epsilon_c^a$$

$$\times F(u,d,b,t,t+\tau,E(rs)+E(mn))$$

Combining the two expressions, we get

$$Tr(\rho(0)(W_1(t+\tau)^*\tilde{X}(t+\tau)W_1(t,t+\tau)\tilde{Y}(t)) =$$

$$exp(it(E(nr)+E(sk)))exp(i\tau E(nr))(< k|\rho_s(0)|m >< m|L_b^a|n >^* < n|X|r >< r|L_d^c|s >< s|X|k >$$

$$\times F(u,a,b,0,t+\tau,E(mn)).F(u,d,c,t,t+\tau,E(rs))$$

$$+exp(it(E(nr)+E(sk)))exp(i\tau E(nr)) < k|\rho_s(0)|m >< m|L_b^a|n >^* < n|X|r >< r|L_d^c|s >< s|Y|k > \epsilon_c^a$$

$$\times F(u,d,b,t,t+\tau,E(rs)+E(mn))$$

Likewise, we evaluate the other terms below:

$$R(t,t+\tau) = Tr(\rho(0)[(1+W_1(t+\tau)^*+W_2(t+\tau))\tilde{X}(t+\tau)(1+W_1(t,t+\tau)+W_2(t,t+\tau))\tilde{Y}(t)(1+W_1(t)+W_2(t))])$$

In the second-order expansion of this, we consider the term

$$Tr(\rho(0)W_2(t+\tau)\tilde{X}(t+\tau)\tilde{Y}(t))$$

$$\int_{0<t_2<t_1<t+\tau} Tr(\rho_s(0)L_b^a(t_1)L_d^c(t_2)\tilde{X}(t+\tau)\tilde{Y}(t))$$

$$< \phi(u)|d\Lambda_a^b(t_1).d\Lambda_c^d(t_2)|\phi(u) >$$

$$=< k|\rho_s(0)|m >< m|L_b^a|n >< n|L_d^c|r >< r|X|s >< s|Y|k >$$

$$\times \int_{0<t_2<t_1<t+\tau} exp(i(E(mn)t_1+E(nr)t_2+E(rs)(t+\tau)+E(sk)t))u_b(t_1)\bar{u}_a(t_1)u_d(t_2)\bar{u}_c(t_2)dt_1dt_2$$

$$=< k|\rho_s(0)|m >< m|L_b^a|n >< n|L_d^c|r >< r|X|s >< s|Y|k >$$

$$\times exp(i(E(rs)(t+\tau)+E(sk)t)) \int_0^{t+\tau} exp(iE(mn)t_1)u_b(t_1)\bar{u}_a(t_1)F(u,d,c,0,t_1,E(nr))dt_1$$

Now, this expression can be cast in a more convenient form involving only the function F as follows:

$$F(u,b,a,s,t,\omega) = \int_s^t u_b(t_1)\bar{u}_a(t_1)exp(i\omega t)dt$$

so that taking the inverse Fourier transform after defining

$$\theta_{s,t}(x) = \theta(x-s) - \theta(x-t), s < t$$

where θ denotes the Heavyside step function, or alternately, $\theta_{s,t}(x) = 1$ if x$\in$ $[s,t]$ and zero otherwise, we get on inverse Fourier transforming,

$$u_b(t_1)\bar{u}_a(t_1)\theta_{s,t}(t_1) =$$
$$(2\pi)^{-1}\int_{\mathbb{R}} F(u,b,a,s,t,\omega)exp(-i\omega t_1)d\omega$$

so finally, we can write

$$Tr(\rho(0)W_2(t+\tau)\tilde{X}(t+\tau)\tilde{Y}(t)) =$$
$$=< k|\rho_s(0)|m >< m|L^a_b|n >< n|L^c_d|r >< r|X|s >< s|Y|k >$$
$$(2\pi)^{-1}exp(i(E(rs)(t+\tau)+E(sk)t))$$
$$\times\int_{\mathbb{R}}\int_0^{t+\tau} F(u,d,c,0,t_1,E(nr))F(u,b,a,0,t+\tau,\omega)exp(-i\omega t_1)dt_1d\omega$$
$$=< k|\rho_s(0)|m >< m|L^a_b|n >< n|L^c_d|r >< r|X|s >< s|Y|k >$$
$$(2\pi)^{-1}exp(i(E(rs)(t+\tau)+E(sk)t))$$
$$\times\int_{\mathbb{R}}\int G(u,d,c,0,t+\tau,\omega),E(nr))F(u,b,a,0,t+\tau,\omega)exp(-i\omega t_1).d\omega$$

where

$$G(u,d,c,0,t,\omega,\theta) = \int_0^t F(u,d,c,0,t_1,\theta)exp(-i\omega t_1)dt_1$$

Remark: The final aim of all these calculations would be to obtain the short-time Fourier transform (STFT) of the cross-correlation function, ie, with w a definite window function centred around the origin, we seek to evaluate

$$S(t,\omega|X,Y,u,w)\int R(t,t+\tau)w(\tau)exp(-i\omega\tau)d\tau$$

Note that the coherent state parameter $u = ((u(t) : t \geq 0)$ determines the mean value of the noisy Hamiltonian in the following sense:

$$< \phi(u)|\sum_{a+b\geq 1} L^a_b d\Lambda^b_a(t)|\phi(u) >$$
$$= \sum_{a+b\geq 1} L^a_b u_b(t)\bar{u}_a(t)dt$$

or equivalently, in terms of white quantum noisy Hamiltonians,

$$< \phi(u)|\sum_{a+b\geq 1} L^a_b d\Lambda^b_a(t)/dt|\phi(u) >=$$
$$= \sum_{a+b\geq 1} L^a_b u_b(t)\bar{u}_a(t)$$

This expression contains linear terms in the $u(t)$ occurring when either $a = 0$ or $b = 0$ in the summation, and also quadratic terms in the $u(t)$ occurring when $a, b \geq 1$.

5 The complete expansion of the quantum correlation function

Rather than making computations for each term separately, we now outline a procedure for getting all the terms, to all orders (ie, not only until the second order. To this end, we write

$$W(s,t) = sum_{n\geq 0}(-i)^n \int_{s<t_n<...<t_1<t} L^{a(1)}_{b(1)}(t_1)...L^{a(n)}_{b(n)}(t_n)d\Lambda^{b(1)}_{a(1)}(t_1)...d\Lambda^{b(n)}_{a(n)}(t_n)dt_1...dt_n$$

$$= \sum_{n\geq 0} W_n(s,t), 0 \leq s \leq t$$

where

$$W_0(s,t) = I, W_n(s,t) = (-i)^n \int_{s<t_n<...<t_1<t} L^{a(1)}_{b(1)}(t_1)...L^{a(n)}_{b(n)}(t_n)d\Lambda^{b(1)}_{a(1)}(t_1)...d\Lambda^{b(n)}_{a(n)}(t_n)dt_1...dt_n, n \geq 1$$

Then,

$$R(t+\tau,t) = Tr(\rho(0)W(0,t+\tau)^*\tilde{X}(t+\tau)W(t,t+\tau)\tilde{Y}(t)W(t))$$

$$= \sum_{n,m,k\geq 0} Tr(\rho(0)W_n(0,t+\tau)^*\tilde{X}(t+\tau)W_m(t,t+\tau)\tilde{Y}(t)W_k(t))$$

$$= \sum_{n,m,k} \int Tr(\rho_s(0)L^{a(n)}_{b(n)}(t_n)^*...L^{a(1)}_{b(1)}(t_1)^*\tilde{X}(t+\tau)L^{c(1)}_{d(1)}(t'_1)...L^{c(m)}_{d(m)}(t'_m)\tilde{Y}(t)$$

$$L^{e(1)}_{f(1)}(t''_1)...L^{e(k)}_{f(k)}(t''_k) < \phi(u)|d\Lambda^{a(n)}_{b(n)}(t_n)...d\Lambda^{a(1)}_{b(1)}(t_1)$$

$$\times d\Lambda^{d(1)}_{e(1)}(t'_1)...d\Lambda^{d(m)}_{a(m)}(t'_m).d\Lambda^{f(1)}_{e(1)}(t''_1)...d\Lambda^{f(k)}_{e(k)}(t''_k)|\phi(u) >$$

where the integral is over

$$0 < t_n < t_{n-1} < ... < t_1 < t+\tau,$$

$$t < t'_m < t'_{m-1} < ... < t'_1 < t+\tau,$$

$$0 < t''_k < .. < t''_1 < t$$

The various terms are now evaluated using the following formulae: Let $X_1, ..., X_p$ be system observables and let

$$X_k(t) = U_0(t)^* X_k U_0(t)$$

Then, for any times $t_1, ..., t_p$, we have

$$Tr(\rho_s(0)X_1(t_1)...X_p(t_p)) =$$

$$< n_{p+1}|\rho_s(0)|n_1 >< n_1|X_1|n_2 > ... < n_p|X_p|n_{p+1} > exp(i(E(n_1n_2)t_1+...+E(n_pn_{p+1})t_p))$$

with summation over the repeated indices $n_1, ..., n_{p+1}$ being implied. Secondly, for any times $t_1, ..., t_p$ consider the evaluation of the matrix element

$$< \phi(u)|d\Lambda^{a(1)}_{b(1)}(t_1)...d\Lambda^{a(p)}_{b(p)}(t_p)|\phi(u) >$$

In the next section, we outline an algorithm for evaluating this matrix element.

6 A problem in quantum stochastic calculus

Problem: Derive a recursion formula for evaluating the matrix element

$$S_n(a(1), b(1), t_1, ..., a(n), b(n), t_n) =< \phi(u)|d\Lambda^{a(1)}_{b(1)}(t_1)...d\Lambda^{a(n)}_{b(n)}(t_n)|\phi(u) >$$

where $t_1, ..., t_n$ are arbitrary non-negative real numbers, ie, no restriction is placed upon the order in which they occur.

hint: Use the following formulae:

$$d\Lambda^{a(n)}_{b(n)}(t_n) = dA_{b(n)}(t_n)^* dA_{a(n)}(t_n)/dt_n$$

and also by quantum Ito's formula,

$$[dA_a(s), dA_b(t)^*] = \delta(a, b)\delta(s, t)dt + dA_b(t)^* dA_a(s)$$

where $\delta(u, v)$ is the Kronecker delta symbol, ie $= 1$ when $u = v$ and $= 0$ otherwise. Also use

$$dA_a(t)|\phi(u) >= u_a(t)dt|\phi(u) >$$

Then define for $r = 0, 1, 2, ..., n$,

$$T_{n,r}(a(1), b(1), t_1, ..., a(r), b(r), t_r|b(r + 1), t_{r+1}, ..., b(n), t_n) =$$

$$< \phi(u)|d\Lambda^{a(1)}_{b(1)}(t_1)...d\Lambda^{a(r)}_{b(r)}(t_r)dA_{b(r+1)}(t_{r+1})^*...dA_{b(n)}(t_n)^*|\phi(u) >$$

Then, show that

$$S_n(a(1), b(1), t_1, ..., a(n), b(n), t_n) =< \phi(u)|d\Lambda^{a(1)}_{b(1)}(t_1)...d\Lambda^{a(n)}_{b(n)}(t_n)|\phi(u) >$$

$$= T_{n,0}(a(1), b(1), t_1, ..., a(n), b(n), t_n) =$$

$$< \phi(u)|d\Lambda^{a(1)}_{b(1)}(t_1)...d\Lambda^{a(n-1)}_{b(n-1)}(t_n)dA_{b(n)}(t_n)^*|\phi(u) > u_{b(n)}(t_n)$$

$$= u_{b(n)}(t_n)T_{n,1}(a(1), b(1), t_1, ..., a(n - 1), b(n - 1), t_{n-1}|b(n), t_n)$$

Further,

$$T_{n,r}(a(1), b(1), t_1, ..., (r), b(r), t_r|b(r + 1), t_{r+1}, ..., b(n), t_n) =$$

$$dt_r^{-1} < \phi(u)|d\Lambda^{a(1)}_{b(1)}(t_1)...d\Lambda^{a(r-1)}_{b(r-1)}(t_{r-1})dA_{b(r}(t_r)^* dA_{a(r)}(t_r)dA_{b(r+1)}(t_{r+1})^*...dA_{b(n)}(t_n)^*|\phi(u) >$$

$$= dt_r^{-1} < \phi(u)|d\Lambda^{a(1)}_{b(1)}(t_1)...d\Lambda^{a(r-1)}_{b(r-1)}(t_{r-1}).dA_{b(r}(t_r)^*[dA_{a(r)}(t_r), dA_{b(r+1)}(t_{r+1})^*...dA_{b(n)}(t_n)^*]|\phi(u) >$$

$$+dt_r^{-1} < \phi(u)|d\Lambda^{a(1)}_{b(1)}(t_1)...d\Lambda^{a(r-1)}_{b(r-1)}(t_{r-1})dA_{b(r}(t_r)^* dA_{b(r+1)}(t_{r+1})^*...dA_{b(n)}(t_n)^* dA_{a(r)}(t_r)|\phi(u) >$$

$$= (\sum_{k=r+1}^{n} \delta(a(r), b(k))\delta(t_r, t_k)).T_{n-1,r-1}(a(1), b(1), t_1, ..., a(r-1), b(r-1), t_{r-1}|b(r+1), t_{r+1}, ..., \hat{(}b(k), t_k), ..., b(n),$$

$$+u_{a(r)}(t_r) < \phi(u)|d\Lambda^{a(1)}_{b(1)}(t_1)...d\Lambda^{a(r-1)}_{b(r-1)}(t_{r-1})dA_{b(r}(t_r)^* dA_{b(r+1)}(t_{r+1})^*...dA_{b(n)}(t_n)^*|\phi(u) >$$

where the notation $\hat{(}b(k), t_k)$ means that that term is omitted. Thus, we obtain the important recursion:

$$T_{n,r}(a(1), b(1), t_1, ..., a(r), b(r), t_r|b(r+1), t_{r+1}, ..., b(n), t_n) =$$

$$= (\sum_{k=r+1}^{n} \delta(a(r), b(k))\delta(t_r, t_k)).T_{n-1,r-1}(a(1), b(1), t_1, ..., a(r-1), b(r-1), t_{r-1}|b(r+1), t_{r+1}, ..., \hat{(}b(k), t_k), ..., b(n),$$

$$+u_{a(r)}(t_r).T_{n,r-1}(a(1), b(1), t_1, ..., a(r-1), b(r-1), t_{r-1}|a(r), b(r), t_r, ..., a(n), b(n), t_n)$$

7 Quantum information theory applied to the noise removal problem while transmitting quantum electromagnetic signals through an optical fibre:Computation of maximum bit rate for data transmission using the Cq Shannon coding theory due to A.Winter and A.S.Holevo

The information transmission problem here can be viewed in two ways that are dual to each other: The first is the Heisenberg view wherein the state of the field (system) and the noise (bath) is fixed while the electromagnetic signals being transmitted vary with time and space, the second is the Schrodinger view wherein the state of the field evolves with time while the fields are fixed in time. For our purposes, the Schrodinger picture is better because while dealing with the Heisenberg view, we can study just one observable's evolution with time, say a given component of the electric or magnetic field at a given spatial point owing to the uncertainty principle. On the other hand, in the Schrodinger picture, we study the state of the entire field at different times and from the state at any given time, we can evaluate the quantum expectations of all the spatial moments of the field at the different spatial points at that time. We can also think of the interaction picture viewpoint wherein quantum observables such as the different components of the electric or magnetic field at a given spatial point evolve according to the unperturbed quantum field Hamiltonian while states evolve according to the noisy perturbing Hamiltonian after a unitary rotation applied to the latter by the unperturbed field Hamiltonian. Now, taking the Schrodinger viewpoint, let $\rho_s(0)\otimes|\phi(u)><\phi(u)|$ be the initial state of the field and the bath. If $U(t)=U_0(t)W(t)$ denotes the unitary evolution, then the state of the system at time t would be given by

$$\rho_s(t)=Tr_2[U(t)(\rho_s(0)\otimes|\phi(u)><\phi(u)|)U(t)^*]=U_0(t).Tr_2[W(t).(\rho_s(0)\otimes|\phi(u)><\phi(u)|)W(t)^*]U_0(t)=U_0(t).$$

where $\tilde{\rho}_s(t)$ is the evolved system state in the interaction picture. Now, consider the problem of transmitting a classical information-bearing sequence at the input of the optical fibre. We assume that the entire information-bearing sequence is encoded into a binary string of a very large length n. Let $(x_1x_2x_3...x_n), x_i\in\{0,1\}$ denote this binary string. If $2^{nR},(R<1)$ is the total number of information-bearing messages to be transmitted over the fibre, then each of these messages is encoded into a binary string of length n, and the total time taken for transmitting any one bit x_i after encoding it into a quantum state $\rho(x_i)\in\{\rho(0),\rho(1)\}$ is τ so that the total bit rate of transmission becomes R/τ bits per second. It should be noted that if $p(0)$ is the probability of transmitting a zero and $p(1)$ the probability of transmitting a one, then in order to transmit a given binary bit, we are transmitting $I(p,\rho)=H(p(0)\rho(0)+p(1)\rho(1))-p(0)H(\rho(0))-p(1)H(\rho(1))$ qubits of information where $H(W)=-Tr(W.log_2(W))$ is the Von-Neumann entropy of the state W. The reason for this comes from Schumacher's noiseless quantum compression theorem which roughly states that for large n, the maximum of $Tr(E)$ where E varies over all projections in $\mathcal{H}^{\otimes n}$ for which $Tr(W^{\otimes n}E)>1-\epsilon$ equals $2^{nH(W)}$. It should be noted that the projection E that maximizes the above will be a Bernoulli or entropy typical projection, namely, the projection onto the range spanned by the eigenvectors of $W^{\otimes n}$ corresponding to ϵ-typical sequences. Thus, $nH(W)$ can be regarded as the number of qubits required to store the quantum state $W^{\otimes n}$, or equivalently, since $Tr(E)=dim\mathcal{R}(E)$ is the size of the typical projection E, the number of messages in the optimally compressed data is $Tr(E)$ which require $log_2Tr(E)\approx nH(W)$ qubits for storage, so that for each transmitted classical bit x_i that has been encoded into the state W, the number of qubits transmitted equals $log_2Tr(E)/n=H(W)$. Now, in our situation, to transmit one classical bit X, we are transmitting one of two possible states $\rho(0)$ or $\rho(1)$ with probabilities $p(0),p(1)$. So the average state received is $Y=p(0)\rho(0)+p(1)\rho(1)$. The information transmitted thus over the channel in order to transmit one classical bit is thus in accordance with classical information theory $H(X)-H(X|Y)=H(Y)-H(Y|X)=I(p,\rho)$ with $H(Y)=H(p(0)\rho(0)+p(1)\rho(1))$ and $H(Y|X)=p(0)H(\rho(0))+p(1)H(\rho(1))$. The maximum rate at which information can be transmitted over the channel is thus C/τ where $C=max_pI(p,\rho)$. In order to be able to construct detection operators for asymptotically zero error for transmission of the 2^{nR} messages as $n\to\infty$, we know from the Cq capacity theorem of Winter and Holveo that $R<C$ must be satisfied, ie the rate of classical data transmission R/τ in bits per second must be smaller than the maximum rate of quantum information transmission C/τ in

qubits per second. More precisely, suppose we encode the $N(n)=2^{nR}$ messages into the distinct binary strings $u(l)=((u(l,m)))_m\in\{0,1\}^n, l=1,2,...,N(n)$. We denote this code by $\mathcal{C}_n$. Corresponding to the l^{th} string $u(l)$, we form the state $W(u(l))=\otimes_{m=1}^n W(u(l,m))$ where 0 is encoded into $W(0)$ and 1 into $W(1)$. We also construct $N(n)$ detection operators $D(l), l=1,2,...,N(n)$ satisfying $D(l)\geq 0, \sum_l D(l)=1$. In order for a code $\mathcal{C}_n$ and detection operators $D(l), l=1,2,...,N(n)$ to exist so that the average decoding error probability $Pr_n(error)=1-\frac{1}{N(n)}\sum_{l=1}^{N(n)} Tr(W(u(l))D(l))$ to converge to zero as $n\to\infty$, it is necessary and sufficient that $R=lim log_2(N(n))/n<C$. This is the essential content of the Winter-Holevo result.

If we prepare the input quantum field in a state $\rho_s(0|\theta)$ given by an appropriate function of the field creation and annihilation operators $c(k), c(k)^*, k=1,2,...,p$, then after time t, this state will get changed to $\rho_s(t|\theta)=Tr_2(U(t)(\rho_s(0|\theta)\otimes |\phi(u)><\phi(u)|)U(t)^*)$ where θ is one of the 2^{nR} possible messages to be transmitted. More precisely, we encode each of the 2^{nR} messages θ to be transmitted into a string of n classical bits $u(\theta)$ and transmit each of these classical bits by encoding it into a quantum state $\rho_s(0|x), x=0,1$. This state at time t is then received as $\rho_s(t|x)=Tr_2(U(t)(\rho_s(0|x)\otimes|\phi(u)><\phi(u)|)U(t)^*)$. The received state when the message θ is to be transmitted is then given by $\rho_s(t|\theta)=\otimes_{x\in u(\theta)}\rho_s(t|x)$ and θ can be decoded by applying an appropriate detection operator. The results of Cq coding theory state that the recovery error probability can be made to converge to zero as $n\to\infty$ provided that $R<C$ where $C=max_{p(0),p(1)}I(p,\rho)$ is the Cq capacity of the channel.

One way of performing the Cq encoding process is to choose two Gibbs states

$$W(k)=Z(\beta(k))^{-1}exp(-\beta(k).\sum_m \omega(m)c(m)^*c(m))), k=0,1$$

corresponding to two different temperatures $T(k)=1/K\beta(k), k=1,2$ so that any given sequence $u=((u(k))\in\{0,1\}^n$ is encoded into the state

$$W(u)=\otimes_{k=1}^n W(u(k))=(\Pi_{k=1}^n Z(\beta(u(k))).exp(-\sum_{k,n}^{\beta}(u(k))\omega(n)c(n,k)^*c(n,k))$$

where $(c(n,k),c(n,k)^*), k=1,2,...,n$ are tensor independent copies of $(c(n),c(n)^*)$, ie,

$$[c(n,k),c(m,j)]=0, [c(n,k),c(m,j)^*]=\delta(n,m)\delta(k,j)$$

We require to calculate the TPCP map generated by the optical fibre carrying electromagnetic waves in the presence of classical electromagnetic disturbance followed by the application of a correction control potential. First, observe that the quantum electromagnetic field in the fibre at time $t=0$ can be expressed as

$$[E(t,r)^T, B(t,r)^T]^T=\sum_n (c(n)g_n(t,r)+c(n)^*\bar{g}_n(t,r))$$

where the $g_n(t,r)'s$ satisfy the wave equation

$$(\partial_t^2 - \nabla^2)g_n(t,r) = 0$$

within the cavity with appropriate boundary conditions that lead to

$$g_n(t,r) = exp(-i\omega(n)t)g_n(r)$$

where $g_n(r)'s$ satisfy the Helmholtz equation

$$(\nabla^2 + \omega(n)^2)u_n(r), n = 1, 2, ...$$

with appropriate cavity boundary conditions for the fibre. The boundary conditions cause only a discrete set $\{\omega(n(\}$ of frequencies to propagate within the fibre cavity. The Hamiltonian of the cavity electromagnetic field is given by the standard formula making use of orthogonality of the eigenfunctions of the Laplace operator (since this operator is self-adjoint):

$$H_0 = \int_{cavity} (\epsilon/2)|E(t,r)|^2 + |B(t,r)|^2/2\mu)d^3r$$

$$= \sum_{n,m} c(n)^* c(m) \int_{cavity} g_n(t,r)^* g_m(t,r) d^3r$$

$$= \sum_{n,m} c(n)^* c(m)\omega(n)\delta(n,m) = \sum_n \omega(n)c(n)^* c(n)$$

It should be noted that alternative to using the wave equation to obtain the time dependence of the quantum electromagnetic field within the fibre, we can use Heisenberg's matrix mechanics in the form

$$\partial c(n,t)/\partial t = i[H_0, c(n,t)]$$

with equal time commutation relations

$$[c(n,t), c(m,t)^*] = \delta(n,m), [c(n,t), c(m,t)] = 0$$

so that

$$[H_0, c(n,t)] = [\sum_m \omega(m)c(m,t)^* c(m,t), c(n,t)] = -\omega(n)c(n,t)$$

yielding thereby

$$\partial c(n,t)/\partial t = -i\omega(n)c(n,t)$$

which gives

$$c(n,t) = c(n)exp(-i\omega(n)t)$$

Now, when the Hamiltonian gets perturbed by a random term $\delta H(t)$, we have seen that the state dynamics (Schrodinger picture) get modified to

$$\partial_t \rho(t) = -i[H_0 + \delta H(t), \rho(t)]$$

or after taking classical statistical averages,

$$\rho'(t) = -i[H_0, \rho(t)] + \theta(\rho(t)) = (-iad(H_0) + \theta)(\rho(t))$$

where

$$\theta(\rho) = (-1/2)\sum_m (L_k^* L_k \rho(t) + \rho(t) L_k^* L_k - 2L_k \rho(t) L_k^*)$$

Solving this gives

$$\rho(t) = U_0(t)\tilde{\rho}(t)U_0(t) = exp(-it.ad(H_0))(\tilde{\rho}(t)), U_0(t) = exp(-itH_0)$$

where

$$\tilde{\rho}'(t) = exp(it.ad(H_0)).\theta.exp(-itad(H_0))(\tilde{\rho}(t))$$
$$= \tilde{\theta}(t)(\tilde{\rho}(t))$$

where

$$\tilde{\theta}(t) = exp(it.ad(H_0)).\theta.exp(-itad(H_0))$$

Solving this gives the quantum dynamical TPCP evolution map for the density operator:

$$\rho(t) = T_t(\rho(0)),$$

where

$$T_t = exp(-it.ad(H_0)).S_t$$

with

$$dS_t/dt = \tilde{\theta}(t).S_t, t \geq 0, S_0 = 1$$

having solution

$$S_t = T\{exp(\int_0^t \tilde{\theta}(s)ds)\}$$

$$= 1 + \sum_{n\geq 1}\int_{0<t_n<...<t_1<t} \tilde{\theta}(t_1)...\tilde{\theta}(t_n)dt_1...dt_n$$

The bit $k \in \{0, 1\}$ is encoded into the state $W(k)$ so that after time T, the output state is

$$W(k, T) = T(W(k))$$

where $T = T_T$, and by the above discussion, the maximum bit rate allowed for error-free decoding is given by

$$C/\tau$$

where

$$C = max_p[H(p(0)T(W(0))+p(1)T(W(1)))-p(0)H(T(W(0))-p(1)H(T(W(1))]$$

The question then is, can we increase the maximum allowable bit rate by using one of the several methods outlined in this paper involving noise reduction by either optimal control methods or filtering methods? All these methods can be summarized in the form of applying another "Noise removal TPCP map" K, so that after applying this operator, the composite TPCP map becomes KoT. The maximum bit rate for error-free transmission after appropriate asymptotic encoding of the messages into bit strings followed by discrete memoryless Cq encoding is then given by

$$C_K = max_p[H(p(0)KoT(W(0))+p(1)KoT(W(1)))-p(0)H(KoT(W(0))-p(1)H(KoT(W(1))]$$

Now we observe that the quantum relative entropy between the two states appearing in the expression below is $(p(x), x]inA$ is a probability distribution on an alphabet A of size d and $|x>, x \in A$ is an orthonormal basis for $\mathbb{C}^d$, $x \to W(x)$ is a mapping from A into the space $\mathcal{S}(\mathcal{H})$ of states in a finite-dimensional Hilbert space $\mathcal{H}$)

$$D(\sum_x p(x)|x><x| \otimes W(x)|I_d/d \otimes \sum_x p(x)W(x)\otimes)$$

$$= -H(p) - \sum_x p(x)H(W(x)) + log(d) + H(\sum_x p(x)W(x))$$

$$\geq I(p, W)$$

since $log(d) \geq H(p)$ with equality iff $p(x) = 1/d \forall x \in A$. Now let K be any quantum operation, ie, TPCP map from $\mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ into itself. Then, $K' = I \otimes K$ is again a TPCP map from $\mathcal{S}(\mathbb{C}^d \otimes \mathcal{H})$ into itself. Hence, by monotonicity of quantum relative entropy (Hayashi, Quantum Information Theory), it follows that

$$D(K'(\sum_x p(x)|x><x| \otimes W(x))|K'(I_d/d \otimes \sum_x p(x)W(x)\otimes))$$

$$\leq D(\sum_x p(x)|x><x| \otimes W(x)|I_d/d \otimes \sum_x p(x)W(x)\otimes)$$

Using the above formula, this results in

$$I(p, K(W)) \leq I(p, W)$$

where $K(W)(x) = K(W(x)), x \in A$. This formula shows that if we try to apply a control TPCP map K to reduce the effects of the noisy Hamiltonian $\delta H(t)$ on the system dynamics, then the channel capacity will reduce, ie, we have to transmit at a smaller bit rate C_K/τ rather than C/τ. However, can we hope to reduce noise by applying TPCP maps that are not in the form of some TPCP map composed with the original TPCP map? Specifically, if the original dynamics is

$$\rho'(t) = \theta_1(\rho(t))$$

and we modify the dynamics to

$$\rho'(t) = \theta_1(\rho(t)) + \theta_2(\rho(t))$$

then the original TPCP map after time t is

$$T_t = exp(t\theta_1)$$

while the modified TPCP map is

$$S_t = exp(t(\theta_1 + \theta_2))$$

This is of the form $K_t o T_t$ with $K_t = exp(t\theta_2)$ only if θ_2 commutes with θ_1. If not, then we can write using the interaction picture method for quantum dynamical semigroups,

$$S_t = K_t o T_t$$

where

$$(\partial_t K_t) T_t + K_t o T_t \theta_1 = K_t o T_t o(\theta_1 + \theta_2)$$

or equivalently,

$$\partial_t K_t = K_t o T_t \theta_2 T_t^{-1}$$

so that

$$K_t = 1 + \sum_{n \geq 1} \int_{0<t_n<...<t_1<t} \theta_2(t_1)...\theta_2(t_n) dt_1...dt_n$$

where

$$\theta_2(t) = T_t o \theta_2 o T_t^{-1}$$

Now, T_t, S_t are TPCP for all $t \geq 0$. But there is no guarantee that $T_t^{-1} = T_{-t}$ will be CP for $t > 0$. Hence, $K_t = S_t o T_{-t}$ will be TP but not necessarily CP. So, we can hope to increase the bit rate while simultaneously reducing noise.

Some remarks:

1.Let T be a TPCP linear map acting as a space of square matrices of a certain size. T can be invertible as a linear operator in the space of matrices but its inverse need not be TPCP. If its inverse is also TPCP, then from the monotonicity of quantum relative entropy, we get

$$D(T(\rho)|T(\sigma)) = D(\rho|\sigma)$$

for all states ρ, σ in the given space of matrices.

2. Let T, K_1, K_2 be TCP maps. Then, so is $K_1 o T o K_2$. More generally suppose $T, K_i, L_i, i = 1, 2, ..., n$ be TPCP maps and $p(i), i = 1, 2, ..., n$ a probability distribution. Then, so is the map

$$S = \sum_{i=1}^{n} p(i) K_i o T o L_i$$

In particular, $K_1 o T o K_2$ corresponds to the system obtained by first applying a preprocessing operator K_2 to the input state, followed by transmitting the state through the fibre via the operation T and finally, followed by postprocessing the output state by K_1. It should be noted that $D(S(\rho)|S(\sigma))$ can be greater than $D(T(\rho)|T(\sigma))$ which means that we can increase the bit rate by using the preprocessor K_2. Note that post-processing alone can only decrease the bit rate. Applying the operation S above, corresponds to applying the pre and post-processing pair (L_i, K_i) with probability $p(i)$ while passing the input state through the fibre and we can use this method to increase the bit rate significantly.

8 Bit rate of data transmission from the viewpoint of quantum electrodynamics

Consider the matter of which the fibre is made to be a collection of electrons and positrons. The total Hamiltonian of the quantum electromagnetic field interacting with the matter can be expressed in terms of the free photon and Fermion creation and annihilation operators as well as the classical random electromagnetic field arising as discussed above by the scattering of the input classical electromagnetic field component by the random motion of the phonons within the fibre. The Feynman diagrammatical methods of computation of scattering, absorption and emission amplitudes in quantum electrodynamics become important if the initial and final states of the photon and Fermion fields consist of a finite number of particles with specified momenta and helicities. The computation of these amplitudes by the Feynman diagrammatic methods involves the use of the photon and electron propagator which acquire corrections due to their mutual interactions (The Dyson-Schwinger equations) as well as due to interactions with the classical random electromagnetic field. The total Lagrangian of the electron and photon field taking into account these interactions is then

$$L = L_1 + L_2 + L_{12} + L_{13} + L_{23}$$

where

$$L_1 = (-1/2)F_{\mu\nu}F^{\mu\nu}$$

is the Lagrangian of the quantum electromagnetic field,

$$L_2 = \bar{\psi}.(i\gamma^\mu\partial_\mu - m))\psi$$

is the Lagrangian of the quantum (ie, second quantized) Dirac field,

$$L_{12} = \bar{\psi}\gamma^\mu\psi A_\mu$$

is the interaction Lagrangian between the quantum electromagnetic field and the quantum Dirac field,

$$L_{13} = (-1/2)F^{\mu\nu}F_{c\mu\nu}$$

is the interaction Lagrangian between the quantum electromagnetic field and the classical random electromagnetic field and finally,

$$L_{23} = \bar{\psi}\gamma^\mu\psi.A_{c\mu}$$

is the interaction Lagrangian between the quantum Dirac field and the classical electromagnetic field. It should be noted that the classical electromagnetic field has a well-defined classical probability distribution and hence the unperturbed Lagrangian of the quantum electromagnetic field is to be taken as $L_1 + L_{13}$ and likewise, the unperturbed Lagrangian of the quantum Dirac field is to be taken as $L_2 + L_{23}$. Thus, the perturbing interaction Lagrangian between the quantum photons and the quantum Fermions is L_{23}. If we adopt the interaction picture, then the quantum fields evolve according to the unperturbed Lagrangian $(L_1 + L_{13}) + (L_2 + L_{23})$ while the states evolve according to the interaction Lagrangian. However, for reasons which will become clear subsequently, it is better to take the unperturbed Lagrangian as $L_1 + L_2$ and the perturbing Lagrangian as $L_{13} + L_{23} + L_{12}$. Thus, after integrating over the spatial variables, the total Hamiltonian of the field has the form

$$H(t) = \sum_k \omega_p(k)c(k)^*c(k) + \sum_k \omega_e(k)a(k)^*a(k)$$

$$+\sum_k (f_k(t)c(k) + \bar{f}_k(t)c(k)^*) + \sum_{k,j} (g_{1kj}(t)a(k)a(j) + g_{2kj}(t)a(k)^*a(j) + h.c)$$

$$+\sum_{kjm} (h_{1kjm}(t)c(k)c(j)a(m) + h_{2kjm}(t)c(k)^*c(j)a(m) + h_{3kjm}(t)c(k)^*c(j)^*a(m) + h.c)$$

where $c(k), c(k)^*$ are the photon annihilation and creation operators while $a(k), a(k)^*$ are the electron and positron annihilation and creation operators. Owing to our choice of the unperturbed Lagrangian in our interaction picture dynamics, the operators $c(k), a(k)$ evolve as $c(k)exp(-i\omega_p(k)t)$ and $c(k)exp(-i\omega_e(k)t)$ which explains the fact that the first two terms in $H(t)$ corresponding to the free photon and free electron fields are constant in time. The term

$$H_{13}(t) = \sum_k (f_k(t)c(k) + \bar{f}_k(t)c(k)^*)$$

represents the interaction Hamiltonian coming from L_{13} and hence $f_k(t)'s$ are random complex functions of time with known probability distribution while the term

$$H_{23}(t) = \sum_{k,j} (g_{1kj}(t)a(k)a(j) + g_{2kj}(t)a(k)^*a(j) + h.c)$$

represents the interaction term coming from L_{23} and hence $g_{lkj}(t)'s$ are random complex functions of time with a known probability distribution. Finally, the term

$$H_{12}(t) = \sum_{kjm} (h_{1kjm}(t)a(k)a(j)c(m) + h_{2kjm}(t)a(k)^*a(j)c(m) + h_{3kjm}(t)a(k)^*a(j)^*c(m) + h.c)$$

represents the interaction term coming from L_{12} and hence the functions $h_{lkjm}(t)$ are complex non-random functions of time. Let the initial state of the photons be

$$\rho_{s1}(0) = \chi_1(c, c^*),$$

and that of the electrons and positrons be

$$\rho_{s2}(0) = \chi_2(a, a^*)$$

The initial state of the photon and electron-positron field is taken as

$$\rho_{s12}(0) = \rho_{s1}(0) \otimes \rho_{s2}(0)$$

Note that the electron-positron operators satisfy the canonical anticommutation relations

$$\{a(k), a(j)\} = 0, \{a(k), a(j)^*\} = \delta(k, j)$$

where

$$\{a, b\} = ab + ba$$

Note that $H(t)$ above has been expressed as the Hamiltonian in the interaction picture, and this implies that $f_k(t)$ contains an extra factor proportional to $exp(-i\omega_p(t))$ apart from its original dependence on time coming from the dynamics of the classical field. Likewise, $g_{1kj}(t)$ contains an extra factor proportional to $exp(-i(\omega_e(k)+\omega_e(j))t)$ apart from its original dependence on time coming from the classical field. $g_{2kj}(t)$ contains an extra factor proportional to $exp(i(\omega_e(k)-\omega_e(j))t)$ On the other hand, H_{12} is the interaction Hamiltonian between two quantum fields containing no classical component. Thus, $h_{1kjm}(t) = h_{1kjm}(0)exp(-i(\omega_e(k) + \omega_e(j) + \omega_p(m))t)$, $h_{2kjm}(t) = h_{2kjm}(0)exp(i(\omega_e(k) - \omega_e(j) - \omega_p(m))t)$ and $h_{3kjm}(t) = h_{3kjm}(0)exp(i(\omega_e(k) + \omega_e(j) - \omega_p(m))t)$. The state of the electrons and positrons after time t in the interaction picture is then

$$\tilde{\rho}_{s12}(t) = \mathbb{E}(W(t)\rho_{s12}(0)W(t)^*)$$

where the expectation operator $\mathbb{E}$ is taken w.r.t the probability distribution of the classical random processes $f_k(s), g_{lkj}(s), s \leq t$. In this expression,

$$W(t) = 1 + \sum_{n \geq 1} \int_{0<t_n<...<t_1<t} \delta H(t_1)...\delta H(t_n)dt_1...dt_n$$

$$= T(exp(-i \int_0^t \delta H(s)ds))$$

where

$$\delta H(t) = H_{13}(t) + H_{23}(t) + H_{12}(t)$$

The state of the photons alone after transmission at time t is then given by

$$\rho_{s1}(t) = T_t(\rho_{s1}(0)) = Tr_2\tilde{\rho}_{s12}(t) = Tr_2\mathbb{E}(W(t)\rho_{s12}(0)W(t)^*)$$

in the interaction picture and in the Schrodinger picture, the photon state at time t is

$$U_{01}(t)T_t(\rho_{s1}(0))U_{01}(t)^*,$$

$$U_{01}(t) = exp9 - itH_{01}), H_{01} = \sum_k \omega_p(k)c(k)^*c(k)$$

9 Propagation of other kinds of particles through the optical fibre

The other particles that we have in mind are the non-Abelian gauge Bosons and Fermions that together form a super-Yang-Mills multiplet and also a super-gravity multiplet consisting of gravitons and gravitinos. The gauge Bosons that appear in the super-Yang-Mills action have the gauginos appearing as Fermionic super-partners of the former. The general form of the action functionals for super-Yang-Mills and super-gravity theories taking into account higher order correction terms can to a certain extent, be derived from superstring theory, in such way that the higher order corrections appear in powers of the string length parameter. The super-Yang-Mills and super-gravity Lagrangians can be derived using standard prescriptions of quantum mechanical string amplitudes calculated in analogy with classical operator theoretic methods that involve sandwiching of the string propagator between vertex functions followed by taking a product of any number of such sandwiched terms, and finally followed by taking the matrix element of such products between an initial and a final state with the initial and final states being either ground states of the string corresponding to massless or massive particles or even Tachyons. A typical quantum mechanical amplitude of this form would be expressible as products of the polarization vectors/tensors appearing in the vertex function and supplemented by momentum vectors. The product of a polarization vector or tensor with a momentum vector can be interpreted in field theoretic language as corresponding to the space-time partial derivative of the corresponding Yang-Mills gauge potential or the Yang-Mills field tensor without the nonlinear term. Higher order string theoretic corrections in such amplitude calculations would be expressible as higher order polynomials in the Yang-Mills field that can be interpreted in conventional quantum field theoretic language as coming from the quantum effective action of the low energy field theory, ie, if $S[\phi]$ is the classical action of the field, the corresponding quantum effective action in the tree approximation, would involve path integrating $exp(iS[\phi_0+\phi_1])$ w.r.t the "quantum fluctuation" ϕ_1 over all one-particle-irreducible subgraphs. Superstring theory promises to deliver all the higher order correction terms merely by the simple procedure of computing superstring amplitudes between two states of operators built out of the superstring propagator and Bosonic and Bosonic and Fermionic string vertex functions.

Examples: The super-Yang-Mills action has the form

$$(-1/4)\int F^{a\mu\nu}F^a_{\mu\nu}d^10x + \int \bar{\psi}\Gamma^a\mu D_\mu\psi d^{10}x$$

where Γ^μ are the 10 Dirac Gamma matrices in ten-dimensional space-time while $D_\mu\psi_a = \partial_\mu\psi^a + g.C(abc)A^b_\mu\psi^c$ is the gauge covariant derivative in the adjoint representation. Here A^a_μ is the gauge potential while $F^a_{\mu\nu}$ is the gauge field corresponding to the gauge potential. ψ^a is the gaugino field, namely the superpartner of the gauge field. It should be noted that ten is one of the finite number of critical dimensions of space-time at which supersymmetry of this Lagrangian is guaranteed, owing to some miraculous cancellations taking place, in particular, cancellation of the triple product term in the gaugino field when A_μ^a gets changed by a supersymmetry transformation proportional to ψ^a in the covariant derivative D_μ. This cancellation takes place owing to certain properties of the Dirac Gamma matrices in ten-dimensional space-time, or more precisely, in any one of a finite set of space-time dimensions. Now, we abbreviate the above super-Yang-Mill action to

$$S[A,\psi] = (-1/4)\int F.Fd^Dx + \int \bar{\psi}.(\partial + g.A).\psi.d^Dx$$

The quantum effective action is calculated as

$$\Gamma[A,\psi] = -ilog\int exp(iS[A+A_1,\psi+\psi_1])DA.D\psi_1$$

with the path integral being calculated after expanding the action functional in the integrand up to quadratic orders in A_1, ψ_1. Of course, we can consider better and better approximations by expanding the action up to higher degree terms in the quantum fluctuations. What is important is that the resulting quantum effective action can be expressed as a superposition of products of a finite number of terms of the form $A_\mu^a, A_{\mu,\nu}^a, \psi^a, \psi_{,\mu}^a$ which in the D-momentum domain, can be abbreviated as ξ_μ^a (Fourier transform of A_μ^a), $k_\nu \xi_\mu^a, u^a$ (Fourier transform of ψ^a), $k_\nu u^a$. Now the fundamentally important fact is that such finite product terms can also be arrived at in the form of quantum mechanical amplitudes of a superstring with the amplitudes being given by terms such as

$$< f|\Delta.V(k_1,z_1,\xi_1).\Delta.V(k_2,z_2,\xi_2).\Delta...V(k_M,z_M,\xi_M).\Delta|i >$$

where Δ is the superstring propagator and $V(k,z,\xi)$ is a superstring Boson or Fermion vertex function of the form $exp(ik.X(z))$ or $\xi.(X'(z)+\psi(1)^T R\psi(1))exp(ik.X(z))$ or $\xi.\psi(z).exp(ik.X(z))$. The first vertex function is the scalar Boson vertex function the second is a vector Boson vertex function while the third is a Fermion vertex function. Note that since the Bosonic string propagator $< T(X(z_1)X(z_2)) >$ becomes infinite when $z_1 = z_2$, a vertex function of of the form $\xi.X'(z).exp(ik.X(z))$ would make sense only if $\xi.k = 0$, ie, the vector Boson cannot have any longitudinally polarized component. Likewise, a vertex function of the form $k.X'(z).exp(ik.X(z))$ would make sense only if it represents massless vector particles, ie $k^2 = 0$. The above expression for the quantum mechanical matrix element is based on the situation that one usually encounters in conventional quantum field theory: if H_0 is the unperturbed Hamiltonian and V the perturbing Hamiltonian so that the total Hamiltonian is $H = H_0 + V$, then the unitary evolution under perturbation can be expressed as

$$U(t) = exp(-itH) = U_0(t)W(t), U_0(t) = exp(-itH_0)$$

where

$$W'(t) = -i\tilde{V}(t)W(t), \tilde{V}(t) = U_0(t)^* V U_0(t)$$

and hence if $O(t)$ is a Heisenberg observable at time t, then the quantum average of the product $O(t_1)...O(t_n)$ with $t_1 > ... > t_n$ in the state $|\phi_i>$ (In the Heisenberg picture, states do not evolve while observables evolve) is given by

$$< \phi_i|O(t_1)...O(t_n)|\phi_i >=$$

$$< \phi_i|W(t_1)^* \tilde{O}(t_1)W(t_1,t_2)\tilde{O}(t_2)...W(t_{n-1},t_n)\tilde{O}(t_n)W(t_n,0)|\phi_i >$$
$$< \phi_f|W(\infty,t_1)\tilde{O}(t_1)W(t_1,t_2)\tilde{O}(t_2)...W(t_{n-1},t_n)\tilde{O}(t_n)W(t_n,0)|\phi_i >$$

where

$$W(t_1,t_2) = W(t_1)W(t_2)^*, t_1 > t_2, \tilde{O}(t) = U_0(t)^* O U_0(t)$$

is the evolution operator for states in the interaction representation and

$$|\phi_f >= W(\infty,0)|\phi_i >$$

is the final state in the interaction picture. $\Delta(t_2,t_1) = exp(-i(t_2-t_1)H)\theta(t_2-t_1)$ is the quantum mechanical propagator, for it satisfies the differential equation

$$\partial_{t_2}\Delta(t_2,t_1) = -iH\Delta(t_2,t_1) + \delta(t_2 - t_1)$$

with formal solution

$$\Delta = i(i\partial_t - H)^{-1}$$

$\tilde{\Delta}(t_2,t_1) = W(t_2,t_1)\theta(t_2 - t_1)$ is the propagator in the interaction picture as it satisfies the differential equation

$$\partial_{t_2}\tilde{\Delta}(t_2,t_1) = -i\tilde{V}(t_2)\tilde{\Delta}(t_2,t_1) + \delta(t_2 - t_1)$$

with formal solution

$$\tilde{\Delta} = i(i\partial_t - \tilde{V}(t))^{-1}$$

Now consider an interaction action between the current and vector field as

$$S_i(J,A) = \int J^\mu(x)A_\mu(x)dx = \int J.A$$

Let $S_0(A)$ be the unperturbed action of the field A_μ. Then, the total action is given by

$$S[A,J] = S_0(A) + S_i(J,A)$$

and then the quantum effective action of A_μ would be given by

$$\Gamma[A] = Ext_J[-i.logZ(J) - \int J.A]$$

where

$$Z(J) = \int exp(iS_0(A) + iS_i(J,A))DA$$

Note that if $S_0(A)$ were a quadratic functional of A, then the integral defining $Z(J)$ is a Gaussian integral and hence evaluates to

$$Z(J) = exp(iS_0(A_0) + iS_i(J, A_0))$$

or equivalently,

$$-i.logZ(J) - \int J.A_0 = S_0(A_0)$$

apart from a multiplicative constant where A_0 extremises $S_0(A) + S_i(J, A)$ for fixed J, ie,

$$S_0'(A_0) + J = 0$$

Then,

$$-i.lnZ(J) - S_i(J, A_0) = S_0(A_0)$$

On the other hand, by the duality property of the Legendre transform,

$$Ext_A[\Gamma[A] + \int J.A] = -i.logZ(J)$$

or equivalently,

$$\Gamma[A_1] + \int J.A_1 = -i.logZ(J)$$

where A_1 satisfies

$$\Gamma'[A_1] + J = 0$$

Comparing the above equations, we get

$$A_1 = A_0, \Gamma[A_0] = S_0(A_0)$$

ie, in the special case when $S_0(A)$ is linear quadratic, the quantum effective action coincides with the classical action. Generally, the quantum effective action for arbitrary $S_0(A)$, not necessarily quadratic, can be computed using the series expansion for $Z(J)$:

$$Z(J) = \sum_{n \geq 0} i^n Z_n(J)$$

where

$$Z_n(J) = \int exp(iS_0(A))(\int J.A)^n DA$$

$$= \int exp(iS_0(A))(\int J(x_1)J(x_2)..J(x_n).A(x_1)...A(x_n)dx_1...dx_n)DA$$

$$= \int J(x_1)...J(x_n)dx_1...dx_n \int exp(iS_0(A))A(x_1)...A(x_n)DA$$

where assuming without loss of generality $x_1^0 > x_2^0 > ... > x_n^0 > 0$, we have

$$\int exp(iS_0(A))A(x_1)...A(x_n)DA/exp(iS_0(A))DA$$

$$< 0|U_0(\infty, x_1^0)A(x_1)U_0(x_1^0, x_2^0)A(x_2)...A(x_n)U_0(x_n^0, 0)|0 >$$

where we have assumed that the vacuum is invariant under the unperturbed Hamiltonian. This form of the general term in the series expansion of the quantum effective action in terms of sandwiched propagator between the fields at different times strongly suggests to us that the formula used in string theory for computing the amplitudes can be interpreted in terms of low energy field theory with string theoretic corrections.

10 Calculating scattering probabilities in quantum field theory in the presence of interactions of the field with classical random current source fields

Consider a vector $\phi(x) = (\phi_a(x))$ of fields having an action functional $S_0(\phi) = \int L_0(\phi(x), \phi_{,\mu}(x))d^4x$ and interacting with a random current source field in accordance with the interaction action $S_1(\phi, J) = \int F_a(\phi(x), \phi_{,\mu}(x))J_a(x)d^4x$ with summation over the repeated index a being implied.

An example: Consider the second quantized quantum electromagnetic field potential $A_\mu(x)$ interacting with the classical random electromagnetic field potential $A_{c\mu}(x)$ and also with a classical random current field $J_{c\mu}(x)$. Also present is the Dirac second quantized wave field $\psi(x)$ interacting with the quantum electromagnetic field $A_\mu(x)$ in accordance with the interaction action $e\int \bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)d^4x$ and also with the classical random electromagnetic field in accordance with the interaction action $e\int \bar{\psi}(x)\gamma^\mu\psi(x)A_{c\mu}(x)d^4x$. The total Lagrangian of qed modified by these random source interactions clearly has the form

$$L = (-1/4)F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\gamma.\partial - m)\bar{\psi}\gamma^\mu\psi.A_{c\mu}$$

The corresponding action is

$$S[A, \psi|A_c, J_c] = S_{01}(A) + S_{02}(\psi) + S_{12}(\psi, A) + S_{13}(A, A_c) + S_{14}(A, J_c)$$

$$+S_{23}(\psi, A_c)$$

The first three terms are the standard ones appearing in the usual qed. The last three terms involve extra terms coming from the interaction of the two quantum fields with the random source fields.

Coming now to the original general formulation, the scattering amplitude matrix element between the initial and final states $|i >$ and $|f >$ is given by the series expansion

$$< f|\int exp(i(S_0(\phi) + S_1(\phi, J))D\phi|i >$$

or equivalently noting that $|i>$ can be expressed as a superposition of state of the form

$$\Pi_{k=1}^{m}\phi_{a(k)}(x_k)|0>$$

and likewise $|f>$ as

$$\Pi_{k=m+1}^{m+n}\phi_{a(k)}(x_k)|0>$$

in order to calculate the above matrix element, it suffices to calculate the path integral

$$\int exp(iS_0(\phi)+iS_1(\phi,J_c))\Pi_{k=1}^{m+n}\phi_{a(k)}(x_k)D\phi$$

This method transforms a general quantum mechanical amplitude into a vacuum-to-vacuum amplitude. The total scattering probability from the initial state $|i>$ to the final state $|f>$ can therefore be expressed as linear combinations of (after averaging the scattering probability w.r.t the random current source)

$$\int \Pi_{k=1}^{n+m}\phi_{a(k)}(x_k)\Pi_{k=1}^{n+m}\psi_{a(k)}(y_k).exp(iS_0(\phi)-iS_1(\psi))\mathbb{E}_J[exp(i(S_1(\phi,J_c)-S_1(\psi,J_c)))]D\phi.D\psi$$

11 References

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