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Additive and Multiplicative Operations on Set of Polygonal Numbers

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Abstract:

In this paper, the focus on generating k-gonal numbers P (k, n) = $\begin{Bmatrix} \frac{n}{2}[(k-3)(n-1)+(n+1)] \; for \; k>2, n\geq 0 \\ \frac{n}{2}[(k-3)(n+1)+(n-1)] \; for \; k>2, n<0 \end{Bmatrix}$.

Also introduce to define additive (+) and multiplicative (*) operations on Sets of k-gonal numbers. In particular, concerning addition (+), $p(k,m)+p(k,n)=p(k,m+n)-(k-2)mn$ forsome integer $k>2$. Also, concerning multiplication (*), $p(k,m)*p(k,n)=p(k,mn)+\frac{(k-4)(k-2)mn}{4}(m-1)(n-1)$ forsome integer $k>2$.Also, summation of any two different k-gonal is $p(k_1,n)+p(k_2,n)=n(n+1)+\frac{n(n-1)}{2}[(k_1+k_2)-6]$ $for\; k_1,k_2>2.$ Also, I applied above properties on some Sets of Polygonal numbers, which are generated by replacing integer k with 3,4,5,6,7 and 8. Also, introduced to study of Repeated steps of Residues of the above Sets of Polygonal numbers generated by k with 3,4,5,6,7 and 8.

Keywords: Polygonal number, Triangular numbers, Residues, Non- Residues, k-gonal number.

Introduction:

Now we can go to generate sets of numbers of square, pentagonal, Hexagonal, Heptagonal, and octagonal with using of Triangular number $p(3,n-1)$. Also, by applying recursive results of Triangular numbers, generated k-gonal numbers are $\frac{n}{2}[(k-3)(n-1)+(n+1)]$ for **k > 2**.

We obtain following Polygonal numbers by replacing k as 3, 4 ,5 ,6 ,7 ,8.

For some positive integer n, the Triangular number is $\frac{n(n+1)}{2}(if\; k=3)$; the square number is $n^2(if\; k=4)$; the Pentagonal number is $\frac{n(3n-1)}{2}$ $(if\; k=5)$; the Hexagonal number is $n(2n-1)$ $(if\; k=6)$; the Heptagonal number is $\frac{n(5n-3)}{2}(if\; k=7)$; and the octagonal number is $n(3n-2)(\; if\; k=8)$.

Also, The formation of triangular numbers is 1,1+2,1+2+3,…etc. The formation of square numbers is 1,1+3,1+3+5,……etc. It implies, successive addition of Arithmetic Progression the formation of Pentagonal numbers1,1+4,1+4+7,….etc. it follows that k-gonal numbers have having common difference **k-2**.

Main work:

In this paper, the total work is classified into Three parts:

Part A: Focused on generating square, pentagonal, Hexagonal, Heptagonal and Octagonal numbers with using of Triangular number $p(3,n-1)(=T_{n-1})$.

Part-B: Introduce to define additive and multiplicative operations on Sets of Polygonal numbers.

Part-C: Repeated steps of Residues of above Polygonal numbers with integer modulo n from 0 to 10.

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Part-A: Generating of Polygonal numbers

k-gonal numbers are denoted by P (k, n), and defined by $\frac{n}{2}[(k-3)(n-1)+(n+1)]$ $for\ k>2$.

From Reference [1], Replace k = 3, we obtain Triangular Numbers, defined by $p(3,n)=\frac{n(n+1)}{2}, n\geq 0$.

In general Triangular numbers are denoted by $p(3,n)=T_n$.

Table 1: A set of Triangular numbers for n = 0 to n =8 as follows:

n	0	1	2	3	4	5	6	7	8	9	10
T_n	0	1	3	6	10	15	21	28	36	45	55

Now, we can go to generate k-gonal numbers from each integer k = 4 to k =8 with using of Triangular number $p(3,n-1)$ as follows:

Case1: **Generating Square numbers $p(4,n)$:**

The sum of two consecutive successive triangular numbers can form a square number.

Consider $p(4,n)=p(3,n-1)+p(3,n)=T_{n-1}+T_n=\frac{n(n-1)}{2}+\frac{n(n+1)}{2}=n^2$.

In general Square numbers are denoted by $p(4,n)=S_n$.

Table 2: Generated some Set of square numbers

n	0	1	2	3	4	5	6	7	8
$p(4,n)\ =n^2$	0	1	4	9	16	25	36	49	64
$p(3,n-1)=\frac{n(n-1)}{2}$	0	0	1	3	6	10	15	21	28
$p(3,n)=\frac{n(n+1)}{2}$	0	1	3	6	10	15	21	28	36
$p(3,n-1)+p(3,n)$	0	1	4	9	16	25	36	49	64

Case2: Generating Pentagonal numbers $p(5,n)$:

The sum of a triangular number $p(3,n-1)$ and a square number $p(4,n)$ can form a pentagonal number.

Consider $p(5,n)=p(3,n-1)+p(4,n)=\frac{n(n-1)}{2}+n^2=\frac{n(3n-1)}{2}$.

Also, from Case 1,$p(5,n)=2p(3,n-1)+p(3,n)=2T_{n-1}+T_n=2\left(\frac{n(n-1)}{2}\right)+\frac{n(n+1)}{2}=\frac{n(2n-2+n+1)}{2}=\frac{n(3n-1)}{2}$.

In general pentagonal numbers are denoted by $p(5,n)=P_n$.

Table 3: Generated Some Set of pentagonal numbers:

n	0	1	2	3	4	5	6	7	8
$p(5,n)=\frac{n(3n-1)}{2}$	0	1	5	12	22	35	51	70	92
$p(3,n-1)+p(4,n)$	0	1	5	12	22	35	51	70	92
$2p(3,n-1)+p(3,n)$	0	1	5	12	22	35	51	70	92

Case3: Generating Hexagonal numbers $p(6,n)$::

The sum of a triangular number $p(3,n-1)$ and a pentagonal number $p(5,n)$ can form a hexagonal number.

Consider $p(6,n) = p(3,n-1) + p(5,n) = \frac{n(n-1)}{2} + \frac{n(3n-1)}{2} = n(2n-1)$.

Also, from Case 2, $p(6,n) = 3p(3,n-1) + p(3,n) = 3T_{n-1} + T_n$

$$= 3\left(\frac{n(n-1)}{2}\right) + \frac{n(n+1)}{2} = \frac{n(3n-3+n+1)}{2} = n(2n-1).$$

In general hexagonal numbers are denoted by $p(6,n) = Hx_n$.

Table 4: Generated Some Set of hexagonal numbers :

n	0	1	2	3	4	5	6	7	8
$p(6,n) = n(2n-1)$	0	1	6	15	28	45	66	91	120
$p(5,n)$	0	1	5	12	22	35	51	70	92
$p(3,n-1)$	0	0	1	3	6	10	15	21	28
$p(3,n-1) + p(5,n)$	0	1	6	15	28	45	66	91	120
$p(3,n)$	0	1	3	6	10	15	21	28	36
$3p(3,n-1) + p(3,n)$	0	1	6	15	28	45	66	91	120

Case4: Generating Heptagonal numbers $p(7,n)$:

The sum of a triangular number $p(3,n-1)$ and a hexagonal number $p(6,n)$ can form a Heptagonal number.

Consider $p(7,n) = p(3,n-1) + p(6,n) = \frac{n(n-1)}{2} + n(2n-1) = \frac{n(5n-3)}{2}$

Also, from Case 3,$p(7,n) = 4p(3,n-1) + p(3,n) = 4T_{n-1} + T_n$

$= 4\left(\frac{n(n-1)}{2}\right) + \frac{n(n+1)}{2} = \frac{n(4n-4+n+1)}{2} = \frac{n(5n-3)}{2}$.

In general heptagonal numbers are denoted by $p(7,n) = Hp_n$

Table 5: Generated Some Set of Heptagonal numbers :

n	0	1	2	3	4	5	6	7	8
$p(7,n) = \frac{n(5n-3)}{2}$	0	1	7	18	34	55	81	112	148
$p(3,n-1)$	0	0	1	3	6	10	15	21	28
$p(6,n)$	0	1	6	15	28	45	66	91	120
$p(3,n-1) + p(6,n)$	0	1	7	18	34	55	81	112	148
$p(3,n)$	0	1	3	6	10	15	21	28	36
$4p(3,n-1) + p(3,n)$	0	1	7	18	34	55	81	112	148

Case5: Generating Octagonal numbers $p(8,n)$:

The sum of a triangular number $p(3,n-1)$ and a heptagonal number $p(7,n)$ can form an octagonal number.

Consider $p(8,n) = p(3,n-1) + p(7,n) = \frac{n(n-1)}{2} + \frac{n(5n-3)}{2} = n(3n-2)$.

Also, from Case 4, $p(8,n) = 5p(3,n-1) + p(3,n) = 5T_{n-1} + T_n = 5\left(\frac{n(n-1)}{2}\right) + \frac{n(n+1)}{2}$

$= \frac{n(5n-5+n+1)}{2} = \frac{n(6n-4)}{2} = n(3n-2)$. In general octagonal numbers are denoted by $p(8,n) = O_n$

Table 6: Generated Some Set of Octagonal numbers :

n	0	1	2	3	4	5	6	7	8
$p(8,n) = n(3n-2)$	0	1	8	21	40	65	96	133	176
$p(3,n-1)$	0	0	1	3	6	10	15	21	28
$p(7,n)$	0	1	7	18	34	55	81	112	148
$p(3,n-1)+p(7,n)$	0	1	8	21	40	65	96	133	176
$5p(3,n-1)+p(3,n)$	0	1	8	21	40	65	96	133	176

By observing above cases, we can go to generate k-gonal numbers using a Triangular number $p(3,n-1)$ is

$$p(k,n) = (k-3)\,p(3,n-1) + p(3,n)$$

$$= (k-3)\,T_{n-1} + T_n$$

$$= (k-3)\,\frac{n(n-1)}{2} + \frac{n(n+1)}{2}$$

$$= \frac{n}{2}[(k-3)(n-1) + (n+1)] \text{ for } k > 2.$$

Table 7: Generated Some Set of k-gonal numbers:

n	T_n	T_{n-1}	S_n	P_n	Hx_n	Hp_n	O_n
0	0	0	0	0	0	0	0
1	1	0	1	1	1	1	1
2	3	1	4	5	6	7	8
3	6	3	9	12	15	18	21
4	10	6	16	22	28	34	40
5	15	10	25	35	45	55	65
6	21	15	36	51	66	81	96
7	28	21	49	70	91	112	133
8	36	28	64	92	120	148	176
9	45	36	81	117	153	199	235
10	55	45	100	145	190	235	280

Table 8: Generating Polygonal numbers with using of Positive integers as follows:

n	$T_n = \frac{n(n+1)}{2}$	$S_n = n^2$	$P_n = \frac{n(3n-1)}{2}$	$Hx_n = n(2n-1)$	$Hp_n = \frac{n(5n-3)}{2}$	$O_n = n(3n-2)$
0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	3	4	5	6	7	8
3	6	9	12	15	18	21
4	10	16	22	28	34	40
5	15	25	35	45	55	65
6	21	36	51	66	81	96
7	28	49	70	91	112	133
8	36	64	92	120	148	176

9	45	81	117	153	199	235
10	55	100	145	190	235	280

Also, by observation, for each positive integer n, Square numbers are generated by $\frac{2n^2-(0)n}{2}$; Pentagonal numbers are generated by $\frac{3n^2-n}{2}$;Hexagonal numbers are generated by $\frac{4n^2-2n}{2}$; Heptagonal numbers are generated by $\frac{5n^2-3n}{2}$; Octagonal numbers are $\frac{6n^2-4n}{2}$;

Hence for generalisation of k-gonal numbers are generated by $\frac{(k-2)n^2-(k-4)n}{2}$ for some integer k >2.

Initially For k =3, we obtain Triangular numbers $\frac{n^2+n}{2}$. Similarly replace k as 4 ,5 ,6 ,7 ,8 to obtain square numbers, Pentagonal numbers, Hexagonal numbers, Heptagonal numbers and Octagonal numbers respectively.

Part-B:

Introduce to study Additive and Multiplicative operations on Set of k-gonal numbers

Now, I can go to introduce to some inherent properties of Sets of Polygonal numbers.

In particular, concerning addition (+), $p(k,m)+p(k,n)=p(k,m+n)-(k-2)mn$ forsome integer $k>2.$

Also, concerning multiplication (*),

$p(k,m)*p(k,n)=p(k,mn)+\frac{(k-4)(k-2)mn}{4}(m-1)(n-1)$ forsome integer $k>2$.

Also, verified these properties are well defined for following Sets of Polygonal numbers by replacing k values for 3,4,5,6,7 and 8.

From **Reference [1]**, Binary operations on a set of Triangular numbers are well defined

That is $T_m+T_n=T_{m+n}-mn, T_m * T_n = T_{mn} - \frac{mn}{4}(n-1)(m-1).$ Now I can go to extend this property to all k-gonal numbers as follows:

Case 6: Now we can introduce an additive ('+') operation on a **set of Triangular numbers** as follows

$p(3,m)+p(3,n)=p(3,m+n)-mn$

Table 9: Verification of additive operation on a Set of Triangular numbers:

m	n	$p(3,m)$	$p(3,n)$	$p(3,m+n)$	mn	$p(3,m)+p(3,n)$	$p(3,m+n)-mn$
1	2	1	3	6	2	4	4
2	3	3	6	15	6	9	9
4	5	10	15	45	20	25	25

Now we can introduce a multiplicative ('*') operation on a Set of Triangular numbers as follows, for some positive integers a, b $p(3,m)*p(3,n)=p(3,mn)-\frac{mn}{4}(m-1)(n-1).$

Table 10: Verification of multiplicative operation on a Set of Triangular numbers:

m	n	$p(3,m)$	$p(3,n)$	$\frac{mn}{4}(m-1)(n-1)$	$p(3,mn)$	$p(3,mn)-\frac{mn}{4}(m-1)(n-1)$	$p(3,m)*p(3,n)$
2	3	3	6	3	21	21-3 =18	18
3	4	6	10	18	78	78-18 = 60	60

4	5	10	15	60	210	210-60 = 150	150
5	6	15	21	150	465	465-150 = 315	315

Now we can extend this methodology to remaining all other polygonal numbers.

Case 7: Now we can go to define two types of Operations on a Set of Square numbers as follows

$\boldsymbol{S_n = n^2, S_m = m^2}$ then addition (+) and multiplication (*) operation are defined as follows

$\boldsymbol{S_n + S_m = S_{m+n} - 2mn}$ and $\boldsymbol{S_n * S_m = S_{mn}}$.

Proof: Consider $\boldsymbol{S_{m+n} - 2mn = (n+m)^2 - 2mn = n^2 + m^2 = S_m + S_n}$.

Hence $\boldsymbol{S_n + S_m = S_{m+n} - 2mn}$.

Table 11: Verification of additive operation on a Set of square numbers:

n	m	S_n	S_m	S_{n+m}	$S_{n+m} - 2mn$	$S_n + S_m$
1	2	1	4	9	5	5
2	3	4	9	25	13	13
4	5	16	25	81	41	41

Again consider $\boldsymbol{S_n * S_m} = n^2.m^2 = (nm)^2 = \boldsymbol{S_{nm}}$.Hence $\boldsymbol{S_n * S_m = S_{nm}}$.

Table 12: Verification of multiplicative binary operation on a Set of square numbers

n	m	S_n	S_m	S_{nm}	$S_n * S_m$
1	2	1	4	4	4
2	3	4	9	36	36
4	5	16	25	400	400

Under these operations, we can verify easily algebraic structure of $(\boldsymbol{S_n},*)$ is becomes as a Monoid, since this Nonempty Set is satisfying Closure axiom, Associate and Existence of Identity element is 1 concerning multiplication. Also, '*' is a binary operation on $\boldsymbol{S_n}$.

i.e Closure Axiom:$\boldsymbol{S_a * S_b = S_{ab}} \in \boldsymbol{S_n}$,for all $\boldsymbol{S_a} \in \boldsymbol{S_n}, \boldsymbol{S_b} \in \boldsymbol{S_n}$.

Associative axiom**:** $\boldsymbol{(S_a * S_b) * S_c = S_a * (S_b * S_c) = S_{abc}}$

Existence of identity**:** identity element $\boldsymbol{S_1 = 1 \in S_n}$,**such that** $\boldsymbol{S_n * S_1 = S_1 * S_n = S_n}$.

Hence $(\boldsymbol{S_n},*)$ is becomes a Monoid.

Case 8: Now we can go to define two types of Operations on a Set of Pentagonal numbers as follows

$P_n = p(5,n) = \frac{n(3n-1)}{2}, P_m = p(5,m) = \frac{m(3m-1)}{2}$ then additive (+), multiplicative (*) operations are defined as follows $P_n + P_m = P_{n+m} - 3mn, P_n * P_m = P_{nm} + \frac{3mn}{4}\ (n-1)(m-1)$

Proof: Consider $P_{n+m} - 3mn = \frac{(n+m)(3(n+m)-1)}{2} - 3mn = \frac{3(n+m)^2-(n+m)-6mn}{2}$

$= \frac{n(3n-1)}{2} + \frac{m(3m-1)}{2} = P_n + P_m$. Hence $P_n + P_m = P_{n+m} - 3mn$

Table 13: Verification of additive operation on a Set of Pentagonal numbers:

n	m	P_n	P_m	P_{n+m}	$P_{n+m} - 3mn$	$P_n + P_m = P_{n+m} - 3mn$
1	2	1	5	12	6	6
2	3	5	12	35	17	17
4	5	22	35	117	57	57

Again consider, $P_n * P_m - P_{nm} = \frac{n(3n-1)}{2}\frac{m(3m-1)}{2} - \frac{nm(3nm-1)}{2}$

$$= \frac{nm}{2}\left[\frac{9nm-3n-3m+1}{2} - \frac{3nm-1}{1}\right] = \frac{3mn}{4}\ (n-1)(m-1)$$

Hence $P_n * P_m = P_{nm} + \frac{3mn}{4}\ (n-1)(m-1)$.

Table 14: Verification of multiplicative operation on a Set of pentagonal numbers:

n	m	P_n	P_m	$P_{nm} + \frac{3mn}{4}\ (n-1)(m-1)$	$P_n * P_m$
1	2	1	5	5	5
2	3	5	12	60	60
4	5	22	35	770	770

It follows that the above binary operations are well-defined.

Case 9: Now we can go to define two types of Operation on a Set of Hexagonal numbers as follows

$Hx_n = n(2n-1)$, $Hx_m = m(2m-1)$ then additive (+), multiplicative (*) operations are defined as follows

$Hx_n + Hx_m = Hx_{n+m} - 4mn$ and $Hx_n * Hx_m = Hx_{nm} + 2mn(m-1)(n-1)$.

Proof: Consider $Hx_n + Hx_m - Hx_{n+m}$

$= n(2n-1) + m(2m-1) - (n+m)(2(n+m)-1) = -4mn$

Hence $Hx_n + Hx_m = Hx_{n+m} - 4mn$.

Table 15: Verification of additive binary operation on a Set of Hexagonal numbers:

n	m	Hx_n	Hx_m	Hx_{n+m}	$Hx_{n+m} - 4mn$	$Hx_n + Hx_m$.
1	2	1	6	15	7	7
2	3	6	15	45	21	21
4	5	28	45	153	73	73

Again consider $Hx_n * Hx_m - Hx_{nm}$

$= n(2n-1)\ m(2m-1) - mn(2mn-1)$

$= mn\ [(2n-1)\ (2m-1) - (2mn-1)] = 2mn(m-1)(n-1)$

Hence $Hx_n * Hx_m = Hx_{nm} + 2mn(m-1)(n-1)$

Table 16: Verification of multiplicative operation on a Set of Hexagonal numbers:

n	m	Hx_n	Hx_m	$Hx_{nm} + 2mn(m-1)(n-1)$	$Hx_n * Hx_m$
1	2	1	6	6	6
2	3	6	15	90	90

4	5	28	45	1260	1260

Case 10: Now we can go to define two types of Operations on a Set of Heptagonal numbers as follows $Hp_n = \frac{n(5n-3)}{2}, Hp_m = \frac{m(5m-3)}{2}$ then additive (+), multiplicative (*) operations are defined as follows

$Hp_n + Hp_m = Hp_{n+m} - 5mn$ and $Hp_n * Hp_m = Hp_{nm} + \frac{15mn}{4}(n-1)(m-1)$.

Proof: Consider $Hp_n + Hp_m - Hp_{n+m} = \frac{n(5n-3)}{2} + \frac{m(5m-3)}{2} - \frac{(n+m)(5(n+m)-3)}{2}$ = -5mn

Hence $Hp_n + Hp_m = Hp_{n+m} + 5mn$

Table 17: Verification of additive operation on a Set of Heptagonal numbers:

n	m	Hp_n	Hp_m	Hp_{n+m}	$Hp_{n+m} - 5mn$	$Hp_n + Hp_m$.
1	2	1	7	18	8	8
2	3	7	18	55	25	25
4	5	34	55	189	89	89

Again consider $Hp_n * Hp_m - Hp_{nm} = \frac{n(5n-3)}{2}\,\frac{m(5m-3)}{2} - \frac{mn(5mn-3)}{2} = \frac{nm}{2}\left[\frac{25nm-15n-15m+9}{2} - \frac{5nm-3}{1}\right]$

$= \frac{15mn}{4}(n-1)(m-1)$Hence $Hp_n * Hp_m = Hp_{nm} + \frac{15mn}{4}(n-1)(m-1)$

Table 18: Verification of multiplicative operation on a Set of Heptagonal numbers

n	m	Hp_n	Hp_m	Hp_{nm}	$Hp_{nm} + \frac{15mn}{4}(n-1)(m-1)$	$Hp_n * Hp_m$
1	2	1	7	7	7	7
2	3	7	18	81	126	126
4	5	34	55	970	1870	1870

Case 11: Now we can go to define two types of Operations on a Set of Octagonal numbers as follows $O_n = n(3n-2)$, $O_m = m(3m-2)$ then additive (+), multiplicative (*) operations are defined as follows

$O_n + O_m = O_{n+m} - 6mn$ and $O_n * O_m = O_{nm} + 6mn(m-1)(n-1)$

Proof: Consider $O_n + O_m - O_{n+m} = n(3n-2) + m(3m-2) - (n+m)(3(n+m)-2) = -6mn$.

Hence $O_n + O_m = O_{n+m} - 6mn$.

Table 19: Verification of additive operation on a Set of Octagonal numbers :

n	m	O_n	O_m	O_{n+m}	$O_{n+m} - 6mn$	$O_n + O_m = O_{n+m} - 6mn$
1	2	1	8	21	9	9
2	3	8	21	65	29	29
4	5	40	65	225	105	105

Again consider $O_n * O_m - O_{nm} = n(3n-2)m(3m-2) - nm(3nm-2)$

$= mn\ [(3n-2)\ (3m-2)-(3mn-2)]\ = 6mn(m-1)(n-1)$

Hence $O_n * O_m = O_{nm} + 6mn(m-1)(n-1)$.

Table 20: Verification of multiplicative operation on a Set of Octagonal numbers:

n	m	O_n	O_m	O_{nm}	$O_{nm} + 6mn(m-1)(n-1)$	$O_n * O_m = O_{nm} + 6mn(m-1)(n-1)$
1	2	1	8	8	8	8
2	3	8	21	96	168	168
4	5	40	65	1160	2600	2600

Hence, by observation, $\boldsymbol{p(k,m)+p(k,n)=p(k,m+n)-(k-2)mn}$ forsome integer $\boldsymbol{k>2}$.

Also, concerning multiplication (*), $\boldsymbol{p(k,m)*p(k,n)=p(k,mn)+\frac{(k-4)(k-2)mn}{4}(m-1)(n-1)}$ forsome integer $\boldsymbol{k>2}$.

Case 12: summation of any two different k-gonal is $\boldsymbol{p(k_1,n)+p(k_2,n)=n(n+1)+\frac{n(n-1)}{2}[(k_1+k_2)-6]}$.

Proof: we know that $\boldsymbol{p(k,n)=\frac{n}{2}[(k-3)(n-1)+(n+1)]\ for\ k>2}$.

Consider $\boldsymbol{p(k_1,n)+p(k_2,n)=\frac{n}{2}[(k_1-3)(n-1)+(n+1)]+\frac{n}{2}[(k_2-3)(n-1)+(n+1)]}$

$\boldsymbol{=n(n+1)+\frac{n(n-1)}{2}[(k_1+k_2)-6]=2T_n+[(k_1+k_2)-6]T_{n-1}}$.

E. g: Verify above result by taking some values of n, k_1, k_2.

Let $k_1 = 4, k_2 = 5$ and $n = 4$. Hence $T_4 = 10, T_3 = 6, S_4 = 16$ and $P_4 = 22$ implice $S_4 + P_4 = 38$.

Also, $n(n+1)+\frac{n(n-1)}{2}[(k_1+k_2)-6] = 20 + 18 = 38$.

Also,$2T_n + [(k_1+k_2)-6]T_{n-1} = 20 + 18 = 38$.

Part-C:

Repeated steps of Residues of Polygonal numbers

Property 1: Repeated steps of Residues of Triangular numbers as follows

Residues mod p repeats every p step if p is odd, and every $2p$ step if p is even.

Proof: $p(3,n)$ = {1,3,6,10,15,21,28,36,45,55,66,78,91,105,120, … … … . }.

Table 21: Some results are represented in the below table:

Integer Modulo of	Sequence of Residues	Repeated steps of Residues
2	(1,1,0,0,1,1,0, 0, … .)	4 steps
3	(1,0,0,1,0,0,1,0, 0, … .)	3 steps
4	(1,3,2,2,3,1,0,0,1,3,2,2,3,1,0, , … .)	8 steps
5	(1,3,1,0,0,1,3,1,0, 0, … … … .)	5 steps
6	(1,3,0,4,3,3,4,0,3,1,0,0,1,3,0,4,3, … .)	12 steps
7	(1,3,6,3,1,0,0,1,3,6,3,1,0, 0,…….)	7 steps

Table 22: Repeated steps of Residues with integer modulo 2 to 13 of a Set of Square numbers as follows p(4, n) ={0,1,4,9,16,25,36,49,64,81,100,121,144,169,196,225,256,289,324,,…..}

Integer modulo	Residues	Repeated steps of Residues	non Residues
2	0,1,0,1,0,1,……..	2	-
3	0,1,1,0,1,1,………	3	2
4	0,1,0,1,…………	2	2,3
5	0,1,4,4,3,0,1,4,4,3,……..	5	2
6	0,1,4,3,4,1,0,1,4,3,……..	6	2,5
7	0,1,4,2,2,4,1,0,1,4,……..	7	3,6
8	0,1,4,1,0,1,4,1,…….	4	3,6
9	0,1,4,0,7,7,0,4,1,0,1,3,0,7,7, 0,4,,1,0,1,4,……….	18	2,5,6,8
10	0,1,4,9,6,5,6,9,4,1,0,1,4,9,……..	10	2,3,7,8
11	0,1,4,9,5,3,3,5,9,4,1,0,1,4……	11	2,6,7,8
12	0,1,4,9,4,1,0,1,4,…………...	6	2,3,5,6,7,8,10,11
13	0,1,4,9,3,12,10,10,12,3, 9,4,1,0,1,4,…………	13	2,5,6,7,8,11

Table 23: Repeated steps of Residues with integer modulo 2 to 5 of a sequence of Set of pentagonal numbers p(5,n)={0,1,5,12,22,35,51,70,92,117,145,……} as follows

Integer modulo	Residues	Repeated steps of Residues	Non Residues
2	{0,1,1,0,1,1……}	3	-
3	{0,1,2,0,1,2,0,1,2,…..}	3	-
4	{0,1,1,0,2,3,3,2,0,1,1,….}	8	-
5	{0,1,0,2,2,0,1,0,2,2,…..}	5	3,4

Table 24: Repeated steps of Residues with integer modulo 2 to 7 of a sequence of Set of Hexagonal numbers p(6,n) ={0,1,6,15,28,45,66,91,120,153,190,……} as follows

Integer modulo	Residues	Repeated steps of Residues	Non Residues
2	{0,1,0,1,…….}	2	-
3	{0,1,0,0,1,0,0,1,…..}	3	2
4	{0,1,2,3,0,1,2,3,…..}	4	-
5	{0,1,1,0,3,0,1,1,….}	5	2,4
6	{0,1,0,3,4,3,0,1,0,…}	6	2,5
7	{0,1,6,1,0,3,3,0,1,6,…..}	7	2,4,5

Table 25: Repeated steps of Residues with integer modulo 2 and 3 of sequence of Set of Heptagonal numbers p(7,n) ={0,1,7,18,34,55,81,112,148,199,235……} as follows

Integer modulo	Residues	Repeated steps of Residues	Non Residues
2	{0,1,1,0,0,1,1,0,0,1,1,…}	4	-
3	{0,1,1,0,1,1,0,1,1,……}	3	2
4	{0,1,3,2,2,3,1,0,3,0,		

Table 26: Repeated steps of Residues with integer modulo 2 and 3 of a sequence of Set of Octagonal numbers p(8,n) ={0,1,8,21,40,65,……} as follows

Integer modulo	Residues	Repeated steps of Residues	Non Residues
2	{0,1,0,1,0,1…..}	2	-
3	{0,1,2,0,1,2,…..}	2	-

Theorem 1: Generalisation of k-gonal numbers with using of triangular numbers is $\boldsymbol{\frac{n}{2}[(k-3)(n-1)+(n+1)]}$ **for k > 2.** Successive Replacement of k values 3,4,5,….etc in $\frac{n}{2}[(k-3)(n-1)+(n+1)]$ we obtain Triangular numbers, Square Numbers, Pentagonal numbers… etc.

Proof: We know that nth term of Triangular number $\boldsymbol{p(3,n)=\frac{n(n+1)}{2}}$,

Also, generate all other Polygonal numbers with using of Triangular numbers as follows:

Square number $S_n = T_{n-1}+T_n$, Pentagonal number $P_n = T_{n-1}+S_n = 2T_{n-1}+T_n$

Hexagonal numbers $Hx_n = T_{n-1}+P_n = 3T_{n-1}+T_n$,

Heptagonal numbers $Hp_n = T_{n-1}+Hx_n = 4T_{n-1}+T_n$

Octagonal numbers $O_n = T_{n-1}+Hp_n = 5T_{n-1}+T_n$.

Hence, generalised k-gonal numbers $k_n = (k-3)T_{n-1}+T_n = \frac{n}{2}[(k-3)(n-1)+(n+1)]$ (if k>2).

Conclusion: In this paper proposed to generate some Polygonal numbers of nth term like Triangular number is $\frac{n(n+1)}{2}$; Square number is n^2; Pentagonal number is $\frac{n(3n-1)}{2}$; Hexagonal number is $n(2n-1)$; Heptagonal number is $\frac{n(5n-3)}{2}$; and octagonal number is $n(3n-2)$. generalised k-gonal numbers are obtained from $\frac{n}{2}[(k-3)(n-1)+(n+1)]\, for\; k>2$.

Also, by observation Square numbers are represented as $\frac{2n^2-(0)n}{2}$; Pentagonal numbers are represented as $\frac{3n^2-n}{2}$;Hexagonal numbers are represented as $\frac{4n^2-2n}{2}$; Heptagonal numbers are $\frac{5n^2-3n}{2}$; Octagonal numbers are $\frac{6n^2-4n}{2}$; Hence for generalisation of k-gonal numbers is $\frac{(k-2)n^2-(k-4)n}{2}$ for k >2

Also introduced Binary operations under addition and multiplication on a Set of Polygonal numbers as follows :

On a set of Triangular numbers $T_m+T_n = T_{m+n}-mn,\, T_m * T_n = T_{mn} - \frac{mn}{4}(n-1)(m-1)$.

On a Set of Square numbers $S_n + S_m = S_{n+m} - 2\sqrt{S_n S_m}$, $S_n * S_m = S_{nm}$.

On a Set of Pentagonal numbers $P_n + P_m = P_{n+m} - 3mn$, $P_n * P_m = P_{nm} + \frac{3mn}{4}(n-1)(m-1)$.

On a Set of Hexagonal numbers $Hx_n + Hx_m = Hx_{n+m} - 4mn, Hx_n * Hx_m = Hx_{nm} + 2mn(m-1)(n-1)$.

On a Set of Heptagonal numbers $Hp_n + Hp_m = Hp_{n+m} - 5mn$, $Hp_n * Hp_m = Hp_{nm} + \frac{15mn}{4}(n-1)(m-1)$.

On a Set of Octagonal numbers $O_n + O_m = O_{n+m} - 6mn$ and $O_n * O_m = O_{nm} + 6mn(m-1)(n-1)$.

particularly concerning addition, Binary operation of k-gonal numbers is

$k_m + k_n = k_{m+n} - (k-2)mn$. Also, concerning multiplication, Binary operation of k-gonal numbers is $k_n * k_m = k_{nm} + \frac{(k-4)(k-2)mn}{4}(n-1)(m-1)$.

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