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STOKES' THEOREM FOR SPIRAL PATHS

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The macroscopic flow along the boundary of a closed curve is equivalent to the cumulative sum of microscopic flows within the enclosed area. Green's theorem formalizes this relationship by connecting the counterclockwise flow within the surface of a two-dimensional manifold to the counterclockwise flow along its boundary. Building on this, Stokes' theorem (henceforth ST) extends the concept to three-dimensional manifolds. By converting a surface integral of the curl of a vector field over the surface into a line integral around the boundary, ST enables the assessment of surface flows based on their boundary flows. In this context, we propose a further generalization of ST to include helicoidal spiral paths. This extension is applicable to a wide range of physical and biological systems where spiral motion plays a significant role, providing a robust framework for in-depth analysis of complex dynamical systems across multiple disciplines.

KEYWORDS: helical dynamics; boundary analysis; vector field integration; flow topology.

INTRODUCTION

Green's Theorem (GT) states that the sum of all microscopic circulations within a region enclosed by a positively oriented, piecewise smooth, simple closed curve is equal to the macroscopic circulation along the curve's boundary (Green, 1828; Cauchy, 1846). This relationship is expressed as the integral of the two-dimensional vector field along the closed curve (Pontryagin, 1959; Craven, 1964; Arfken, 1985). The three-dimensional counterpart of GT, Stokes' Theorem (ST), asserts that for a continuously differentiable, orientable surface, the macroscopic circulation—represented by the integral of a differential form over the boundary of the surface—is equivalent to the microscopic circulation, depicted by the surface integral of the curl of the vector field perpendicular to that surface (Schey, 1997; Vines et al., 2021).
GT and ST have practical applications in areas such as scalar potentials' horizon values in black holes (Heusler, 1998; De Villiers, 2006), earthquakes forecast (Wapenaar et al., 2019), probing Earth's deep interior (Livermore et al., 2013; Snieder, 2015; Yang et al., 2023; You et al., 2023), Coriolis force, hemispherical flows (Aubert and Finlay, 2019), heat transfer, airflow circulation around wings. The theorems also apply to biological phenomena like blood flow and electrical activity of the human brain (Tozzi and Peters, 2023), surface heat flux, ecosystem modeling, the lowest-cost path in reaction–diffusion equation models (Bressan et al., 2022). By integrating ST and GT into practical applications, researchers continue to uncover connections across disciplines, suggesting unexplored physical and biological uses for both theorems.
We propose extending the application of ST to helicoidal spirals. A helicoidal spiral is a three-dimensional smooth curve that is simultaneously lifted and rotated at a constant distance along its fixed axis, with tangent lines maintaining a constant angle to the fixed axis. Spirals often result from vortical processes, representing paths or structures shaped by angular momentum conservation, growth efficiency and navigation strategies. They arise naturally in systems where rotational forces or constraints play a significant role in shaping motion, structure or energy flow (Donepudi et al., 2024).
Although GT and ST traditionally apply to closed curves, we present a novel version of GT that holds for open spiral paths involving objects or entities moving along a helical trajectory.

MATHEMATICAL TREATMENT

We aim to prove that, given a continuously differentiable, orientable helicoidal spiral vector field, the macroscopic circulation represented by the integral of a differential form over its surface equals the microscopic circulation represented by the volume integral of the curl perpendicular to the surface. The main challenge here is in in defining the notion of a boundary in case of an open helicoidal spiral path.

Stokes' Theorem (ST) from vector calculus relates the surface integral of the curl of a vector field over a surface **S** to the line integral of the vector field along the boundary curve ∂S of the surface. In its general form, it states that:

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

where:

- **F** is a a continuously differentiable two-dimensional vector field,
- ∂S is the closed boundary curve of the surface **S** that can be bended and stretched,
- d**r** is a differential element of the curve,
- d**S** is the differential element of the surface area, and
- $\nabla \times \mathbf{F}$ is the curl of the vector field, i.e., a vector operator that characterizes the infinitesimal circulation of vector fields in three-dimensional spaces, providing a powerful framework for linking local properties of a field to its global behavior.

In sum, ST turns line integrals of a form over a boundary into more straight-forward double integrals over the bounded region, regardless of the position of vector singularities (Zenisek 1999). For ST to apply, the normal vector representing the surface must be positively oriented with respect to the tangent vector representing the orientation of the boundary.

Consider a vector field **F** defined over a region in three-dimensional space. Let the surface *S* be a portion of a plane or a more general surface that is bounded by a spiral curve $\gamma(t)$. The goal is to use ST to evaluate the line integral over the spiral path in terms of the surface integral of the curl of **F**.
Let the spiral curve $\gamma(t)$, with $t \in [a, b]$, be parameterized as:

$$\gamma(t) = (r(t)\cos(\theta(t)), r(t)\sin(\theta(t)), z(t))$$

where $r(t)$, $\theta(t)$ and $z(t)$ describe respectively the radial, angular and vertical components of the spiral.
Let's assume that $\gamma(t)$ lies on a flat plane, say the *xy*-plane, so we can simplify the spiral path to:

$$\gamma(t) = (r(t)\cos(t), r(t)\sin(t), 0)$$

where $r(t)$ increases as the angle *t* increases.

When the surface **S** is a surface spanned by the curve $\gamma(t)$, **S** stands for a portion of the plane or surface generated by the spiral curve.

We are interested in computing the line integral of a vector field **F** along the spiral path. By ST, this line integral can be transformed into a surface integral involving the curl of **F**:

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

The line integral over the spiral path is:

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\gamma(t)) \cdot \frac{d\gamma(t)}{dt}\, dt$$

where $\frac{d\gamma(t)}{dt}$ is the tangent vector to the spiral path at each point *t*.

The surface integral involves the curl of **F**, given by $\nabla \times \mathbf{F}$, and the surface normal vector $\hat{n}$ associated with **S**:

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_S (\nabla \times \mathbf{F}) \cdot \hat{n}\, dS$$

The normal vector $\hat{n}$ depends on the orientation of the surface, while *d*S is the differential area element of the surface.

PROOF FOR SPECIFIC EXAMPLE

Let's take a specific example where we apply ST to the vector field $\mathbf{F} = (y, -x, 0)$, which is a vector field in the plane with components involving the *x*- and *y*-coordinates.

1. Calculate the Curl of F. The curl of $\mathbf{F} = (y, -x, 0)$ is computed as:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = (0, 0, -2)$$

Thus, the curl of **F** is constant and directed along the negative z-axis.

2. Apply Stokes' Theorem. Now, let's apply ST to this example. Suppose we have a spiral path $\gamma(t)$ on the *xy*-plane, starting at the origin and spiraling outwards. The surface **S** could be taken as a portion of the disk bounded by the spiral. The surface integral becomes:

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_S (-2)\, dS$$

where *d*S is the differential area element. Since $\nabla \times \mathbf{F}$ points in the z-direction, the area element $d\mathbf{S} = \hat{k}\, dA$. Thus:

$$\int_S (-2)\, dS = -2 \cdot \text{Area of } S$$

3. Compute the Line Integral Over the Spiral Path. For the line integral, we have:

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\gamma(t)) \cdot \frac{d\gamma(t)}{dt}\, dt$$

If $\gamma(t) = (r(t)\cos(t), r(t)\sin(t), 0)$, then the differential path element is:

$$d\gamma(t) = \left(\frac{dr}{dt}\cos(t) - r(t)\sin(t), \frac{dr}{dt}\sin(t) + r(t)\cos(t), 0 \right).$$

Substitute into the line integral and compute to obtain the result. The key idea is that the line integral along the spiral is connected to the surface integral over the disk, as shown by ST.

CONCLUSION

Using ST, we have related the line integral along a spiral path to the surface integral of the curl of the vector field. This formulation provides a way to compute the line integral for complex paths such as helicoidal or conic spirals occurring within three-dimensional volumes, by transforming the problem into a surface integral over a well-defined region.

This version of ST is applicable to a variety of physical dynamical systems, including spiral galaxies like the Milky Way, electrons in magnetic fields, black hole accretion disks, wingtip vortices in flight, tornadoes, hurricanes, ocean eddies (Blaser et al., 2024) as well as smoke rings, Kármán vortex streets, whirlpools. Quantum vortices also appear in superfluids and Bose-Einstein condensates, magnetic vortices in superconductors, solar plasma flows (Sachkou et al., 2019). In biology, spiral dynamical paths are observed in seed dispersal patterns such as those in maple seeds, plants' phyllotaxis (Reinhardt and Gola, 2022), sharks' hunting patterns (Kajiura et al., 2022), jellyfish propulsion, migratory

birds' wingtip vortices, moth flight paths, bacterial spiral motility (Liu et al., 2024), sperm cell propulsion, cellular chiral rotations during development, intracardiac flows (Mulimani et al., 2022), muscle fiber movements during contraction (Long et al., 2007).
These examples highlight the profound impact of vorticity and the subtending mathematical principles in understanding and modeling both physical and biological phenomena, revealing the pervasive influence of rotational dynamics across a wide range of natural systems. All this makes the use of the new version of the theorem a powerful tool for investigating numerous problems in both physics and biology.

DECLARATIONS

Ethics approval and consent to participate. This research does not contain any studies with human participants or animals performed by the Author.
Consent for publication. The Author transfers all copyright ownership, in the event the work is published. The undersigned author warrants that the article is original, does not infringe on any copyright or other proprietary right of any third part, is not under consideration by another journal, and has not been previously published.
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