Review of: "Nonlinearity and Illfoundedness in the Hierarchy of Large Cardinal Consistency Strength"

Peter Holy[1]

1 Technische Universität Wien

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In this paper, the author presents various positions contrasting the widespread belief that all natural large cardinal axioms in set theory should turn out to be well-ordered in terms of consistency strength, arguing that this ordering neither needs to be linear, nor well-founded. A large number of counterexamples to this supposedly flawed belief are provided, all of which could be said to make use of some sort of self-referential statement. That mostly is the universal algorithm, which is one fixed algorithm that obeys any prescribed behaviour in some model of a fixed base theory (usually some extension of ZFC). Using this algorithm produces a partial function f between natural numbers for which statements such as “there are f(n) inaccessible cardinals” for various natural numbers n become incomparable in consistency strength. This is the case basically because the truth of f(n)<f(m) varies across different models with nonstandard natural numbers. Another key example is that of cautious enumerations of theories, that is we consider a theory such as ZFC only up to the point where we find a proof from what we have enumerated so far that our theory is in fact inconsistent, leading to variants of ZFC of strictly weaker consistency strength (which is again seen by considering models with nonstandard natural numbers). Continuing this process throughout the natural numbers then leads to ill-foundedness of this hierarchy.

Now many may complain that these are not natural examples of large cardinal notions at all, and in case you think you have seen too many of these examples at some point, you may (at least for a while, before possibly returning) want to skip forward to Section 7ff, where many of these possible objections are being addressed. Let me just mention few of them briefly. For one, many (hierarchies of) large cardinal notions were defined just so that they yield a well-ordered hierarchy by their very definition, for example by imposing stronger and stronger closeness assumptions with respect to the set theoretic universe on the target model of elementary embeddings, as is commonly done in the large cardinal hierarchy from measurability upwards. Also, for quite a few large cardinal notions, we simply do not know whether or not they fit into a linear hierarchy of consistency strength, perhaps with the strength of the existence of two strongly compact cardinals being a prime example. But I think an even stronger point is that the most common ways of producing set theoretic universes – passing to transitive inner models or to forcing extensions – do not change arithmetic truth, and thus simply cannot be used to provide instances of incomparability in consistency strength, for the statement that a theory is consistent is an arithmetic statement: to find such instances, one needs to consider models with differing natural number structures. So the reason that no natural examples of nonlinearity have yet been found may simply be that we do not have any reasonable methods to find them. Another very good point that the author makes is that constant self-reference lies at the very core of set theory – numerous large cardinals notions make use of some sort of self-referential statement, in particular some of the most popular ones like weakly compact, measurable or supercompact cardinals do. And, as the author argues, many of the most popular and supposedly natural set theoretic principles, for example the proper forcing axiom, are also very much self-referential, already the notion of properness is. Or going back further in history, to the starting points of set theory like Russel's paradox or Cantor's diagonal arguments, those are very much self-referential. So it seems odd that anybody working in set theory could reject a notion for the reason that it is self-referential. Finally, the author suggests that set-theorists should try and rescue their claim about the well-foundedness of the large cardinal consistency strength hierarchy by replacing the "empty" naturality talk and rather try identify what could be the desired features of large cardinal notions that might lead to an actual linearity or well-order phenomenon.

Summing up, I think this paper is a super-interesting read for anyone who at least occasionally deals with large cardinals in their mathematics. And for those who don't want to see all the many examples of non-linearity and non-wellfoundedness, it's well worth skipping ahead to Section 7 and reading the remainder of the paper from there.

I finally have a list of very minor comments and typos, that could well be deleted after these have been considered and possibly fixed by the author. Any numbers in the below refer to the current pdf version of the paper.

- Paragraph 4: There seems to be a typo in "suffice": in what I can see, the "ff" is missing.
- Paragraph after Lemma 5: Perhaps "sufficient" should be "sufficiently strong"?
- Statement of Theorem 8: It should perhaps be said that n ranges over the natural numbers.
- Paragraph right after the statement: Perhaps the definition of "strongly independent" should rather go before the statement of Theorem 8, for it is used there.
- Proof of Theorem 9, Line 6: An "is" is missing here.
- Paragraph before Theorem 10, Line -3: It should either be "set theorists" or "considers".
- Line -2: Should "inaccessible" be "measurable"?
- Proof of Theorem 10, Line 8: "t+1" should by "t" I think?
- Proof of Theorem 11, Line 7: What is a "strong theory"?
- Paragraph after Theorem 12, Line 4: "consistent" should be "consistency".
- Section 4, Definition of ZFC^0 and ZFC^{00}: For ZFC^0 you are asking whether there is a proof of inconsistency of ZFC in what you have enumerated so far, while for ZFC^{00} you are asking whether there is a proof in "ZFC" of the inconsistency of ZFC or … . I think it should also be a proof in "what you have enumerated so far" in the second case (that is, for ZFC^{00})? It might also make sense to make these key definitions a tiny bit more precise.
- Theorem 18: Perhaps this has happened earlier in the paper, but it seems this theorem is using the background assumption Con(ZFC+Con(ZFC)), right? So perhaps this should be mentioned in the statement of the theorem, or as a general assumption somewhere. Similar comments seem to apply to quite a few further results.
- Proof of Theorem 20, Line 2: You seem to be confusing the universal computable function f with the universal algorithm, refering to f as an algorithm, or inquiring whether f(n) has already halted three lines further down. This also happens in many other places in the paper. I'm not sure whether this is standard terminology.
- Proof of Theorem 20, Line -1: … the "countable" free Boolean algebra.

- Paragraphs after Theorem 23, middle of Page 23: Are you claiming here that a possible proof of inconsistency for strong cardinals would be shorter than for measurable cardinals? While this seems extremely plausible, it also seems kind of hard to verify.
- Statement of Theorem 24: …"consistency strengths" … …There is "a" computable function…
- Section 8, third paragraph: I guess it should be "To my way of thinking, …".
- Page 33, Line 2: "Entscheidungsproblem" is missing a "c".
- Line -9: …if "an" arithmetic statement…
- Line -8: …holds in V_kappa.
- Section 10, Line 13: "phenomena"
- Section 11, Paragraph -2, Line 1: "are" should be "is".
- Paragraph -1, Line -9: …of "a" uniform procedure…?