A NOTE ON A RECENT ATTEMPT TO PROVE THE IRRATIONALITY OF $\zeta(5)$

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Abstract. Recently, Shekhar Suman [5] made an attempt to prove the irrationality of $\zeta(5)$. Unfortunately, the proof is not correct. In this note, we discuss the fallacy in the proof.

Keywords: Irrationality, Odd zeta values

1. Introduction

The study of irrationality and transcendence of numbers is a classical topic in transcendental number theory. Recall the Riemann zeta values

$$\zeta(n) = \sum_{k\geq 1} \frac{1}{k^n} \quad \text{for } n \geq 2.$$

Euler proved that

$$\zeta(2) = \frac{\pi^2}{6}$$

and more generally

$$\zeta(2n) = -\frac{B_{2n}(2\pi i)^{2n}}{2(2n)!} \quad \text{for } n \geq 1,$$

where B_m is the m-th Bernoulli number. It is conjectured that the values of the Riemann zeta function at all positive integers are irrational.

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In 1882, Lindemann proved that the number π is transcendental (see, for example, [6, Theorem 1.2]). As a consequence, the even values $\zeta(2n)$ are irrational. In 1979, Apéry [1] famously demonstrated that the number $\zeta(3)$ is irrational. Subsequently, Beukers [3] provided an elementary proof of $\zeta(3)$ based on integral representations. In 2001, Keith Ball and Tanguy Rivoal [2, Théorème 1] applied Nesterenko's criterion to prove that the $\mathbb{Q}$-vector space spanned by odd zeta values is infinite-dimensional:

$$\dim_{\mathbb{Q}}\langle 1, \zeta(3), \zeta(5), \ldots, \zeta(2n+1)\rangle_{\mathbb{Q}} \geq \frac{1}{3}\log(2n+1).$$

In the same year, Zudilin [7] demonstrated that at least one of the four numbers

$$\zeta(5), \zeta(7), \zeta(9), \zeta(11)$$

is irrational. Incidentally, the proof of the irrationality of $\zeta(3)$ can also be found in Zudilin's recent book [8, Section 7].

To this day, it remains unknown whether $\zeta(5) \notin \mathbb{Q}$. It is also not known whether $\zeta(3) \notin \pi^3\mathbb{Q}$.

Recently, Shekhar Suman [5] claimed to have proved the irrationality of $\zeta(5)$. Unfortunately, the proposed proof is incorrect.

This note is organized as follows. In Section 2, we identify the logical flaws in the proof presented in [5]. In Section 3, we outline a standard approach to proving the irrationality of a number and highlight that the irrationality of $\zeta(2m+1)$ for general odd integers $2m+1$ remains an open and intriguing area of research.

2. Logical Fallacies in Suman's Proof

In this section, we analyze the flaws in Shekhar Suman's proof of [5, Theorem 1]. While the preliminary result, [5, Lemma 1], is correct and well-established, the proof of [5, Theorem 1] contains a critical logical error.

Suman claims that the linear Diophantine equation [5, Eq. (47)] has no integer solution, which he uses to derive the irrationality of $\zeta(5)$. However, we show that [5, Eq. (47)] indeed admits integer solutions, invalidating his conclusion.

Suman argues as follows:

> Since $(d_n, 2d_n) = d_n$, the above linear Diophantine equation has an integral solution if and only if $d_n \mid k_i b$. So we have
>
> $$d_n a - 2d_n b = -k_i b \quad \text{where } d_n \mid k_i b, \quad 1 \leq k_i \leq d_n - 1, \quad n \geq b.$$
>
> He then claims that the following cases are impossible:
>
> $$d_n a - 2d_n b = -k_i b \quad \text{where } d_n \mid k_i b, \quad 0 \leq k_i \leq d_n, \quad n \geq 1.$$

He employs induction on n to establish this claim. However, a closer inspection reveals a critical flaw in his argument at the base case $n = 1$.

For $n = 1$, the equation becomes:

$$a - 2b = -k_i b, \quad \text{where } 0 \leq k_i \leq 1 \text{ and } 1 \mid k_i b.$$

Solving this, Suman correctly concludes that the possible solutions are:

$$a - 2b = 0 \quad \text{or} \quad a - 2b = -b.$$

This implies that $a = 2b$ or $a = b$, both of which are valid integer solutions. However, instead of acknowledging the existence of these integer solutions as a refutation of his claim, Suman proceeds to argue that these solutions lead to:

$$\zeta(5) = \frac{a}{b} = 2 \quad \text{or} \quad \zeta(5) = 1,$$

which he deems "absurd" because it is well-known that $\zeta(5)$ is not an integer.

Here lies the critical error: the claim that equation [5, Eq. (48)] has no integer solutions is logically independent of whether $\zeta(5)$ is an integer or not. The well-established fact that $\zeta(5)$ is not an integer cannot be used to prove that [5, Eq. (48)] has no integer solutions. The existence of integer solutions for [5, Eq. (48)] at $n = 1$ directly invalidates the induction base case and, consequently, the entire proof.

To summarize, the error in [5, Theorem 1] lies in conflating two unrelated facts: the solvability of the Diophantine equation [5, Eq. (48)] and the irrationality of $\zeta(5)$. The former is a purely algebraic property of the equation, while the latter is a number-theoretic property of the zeta function. By assuming $\zeta(5)$ is not an integer to argue against the solvability of [5, Eq. (48)], Suman undermines the logical foundation of his proof. As a result, Theorem 1 in [5] is incorrect.

In addition to the errors in the proof of [5, Theorem 1], similar logical fallacies can be identified in the proof of [5, Theorem 2], which aims to establish the irrationality of $\zeta(2m+1)$ for integers $m \geq 2$. While the specific details of the argument in [5, Theorem 2] differ from [5, Theorem 1], the underlying issue remains the same: the reliance on assumptions or conclusions that are independent of the solvability of the corresponding Diophantine equations.

3. A Standard Approach to Proving the Irrationality of a Number

In this section, we review a standard method for proving the irrationality of a number α (the criterion for irrationality), and show that Suman's method does not meet the criterion.

Proposition 3.1. *Suppose that we can construct sequences of pairs of rational numbers a_n, b_n with the following properties:*

- *(1) There is a small number $0 < \varepsilon < 1$ such that*
$$0 < |a_n\alpha - b_n| < \varepsilon^n$$
for all sufficiently large n.
- *(2) Let $d_n \in \mathbb{N}$ be the common denominator of a_n, b_n:*
$$d_n a_n \in \mathbb{Z}, \quad d_n b_n \in \mathbb{Z}.$$
Assume that $d_n < D^n$ for some $D \in \mathbb{R}$.
- *(3) D is not too big:*
$$D\varepsilon < 1.$$

Then α is irrational.

Proof. (by contradiction) Suppose that α is rational, $\alpha = \frac{p}{q}$ where $p, q \in \mathbb{Z}, q > 0$. Assumption (1) then becomes

$$0 < \left|a_n \frac{p}{q} - b_n\right| < \varepsilon^n \quad \text{for large } n.$$

By multiplying through by q and d_n, we obtain

$$0 < |d_n a_n p - d_n b_n q| < q d_n \varepsilon^n < q D^n \varepsilon^n.$$

Since by assumption (3), $D\varepsilon < 1$, the right-hand side tends to zero. Thus, we can find a large n such that

$$0 < \left| \underbrace{(d_n a_n)}_{\in \mathbb{Z}} p - \underbrace{(d_n b_n)}_{\in \mathbb{Z}} q \right| < 1.$$

But by (2), this is an integer between 0 and 1, which is a contradiction. $\square$

Remark. Besides Beukers [3], other interesting applications of Proposition 3.1 can be found in, for example, Huylebrouck [4].

Numerical verification shows that the I_n in [5, Lemma 1] does not satisfy the condition (3) of Proposition 3.1. Therefore, I_n cannot be used to prove the irrationality of $\zeta(5)$ via Proposition 3.1. In fact, mathematicians have been searching for a suitable integral representation of the form $a_n + b_n\zeta(5)$ that converges to 0, but such a representation has not yet been found. Therefore, the irrationality of $\zeta(2m+1)$ for $m \geq 2$ still remains an open problem worth exploring. Finally, finding a good integral representation for $\zeta(5)$ or even for $\zeta(2m+1)$ that can be used to approximate them continues to be a worthwhile pursuit.

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