Research Article

The Non-Reflexive Formulation of Quantum Mechanics: the Whys and the Hows

Décio Krause[1]

1. Graduate Program in Logic and Metaphysics, Universidade Federal Rural do Rio de Janeiro, Brazil

In the non-reflexive formulation of quantum physics, attention is given to a metaphysical view of quantum objects (q-objects) as non-individuals, characterised as entities to which the standard notion of identity lacks sense. The reference to the nature of q-objects is usually omitted in other interpretations and the most we can find is some vague reference to "to each [quantum] system we associate a Hilbert space", and nothing more is said about them; we need to infer from the theory their characteristics, and it gives us several alternatives. One of them, to be justified here in broad aspects, is the non-individual view. It is important to enlighten that we are not trying to just find a way to deal with quantum entities but to defend a possible ontology of non-individuals, which can be applied also to 'particles' arising from quantum fields. Our account is based on two assumptions: (i) the 'classical theory of identity' does not apply to q-objects, and (ii) their non-individuality must be attributed from the start, that is, as a primitive notion and not 'made a posteriori' say by assuming symmetry principles. The underlying logic is called non-reflexive because the reflexive law of identity $\forall x(x = x)$ is supposed not to hold for q-objects. Since all the details demand a lot of space, here we only sketch and justify our view. The references provide a more detailed account.

> "All physics is steeped in metaphysics."
>
> Mario Castagnino, Argentinian physicist

> "I beg to emphasise this and I beg you to believe it. It is not a question of our being able to ascertain the identity in some instances and not being able to do so in others. It is beyond doubt that the question of 'sameness', of identity [of elementary particles] really and truly

 doi.org/10.32388/NCXO2K

has no meaning.”

Erwin Schrödinger[1], pp. 121–122]

“[the] non-individuality [of elementary particles] must be attributed right at the start.”

Heinz Post[2]

“No logic can be imposed a priori to our systematisations.”

Newton da Costa[3], p.140]

1. Introduction

In the foreword of Max Jammer’s book ‘Concepts of Space’, Einstein made the following remark (my emphases):

> “The eyes of the scientist are directed upon those phenomena which are accessible to observation, upon their apperception and conceptual formulation. **In the attempt to achieve a conceptual formulation of the confusingly immense body of observational data, the scientist makes use of a whole arsenal of concepts which he imbibed practically with his mother’s milk; and seldom if ever is he aware of the eternally problematic character of his concepts.** He uses this conceptual material, or, speaking more exactly, these conceptual tools of thought, as something obviously, immutably given; something having an objective value of truth which is hardly ever, and in any case not seriously, to be doubted. How could he do otherwise? How would the ascent of a mountain be possible, if the use of hands, legs, and tools had to be sanctioned step by step on the basis of the science of mechanics? And yet in the interests of science it is necessary over and over again to engage in the critique of these fundamental concepts, in order that we may not unconsciously be ruled by them. This becomes evident especially **in those situations involving development of ideas in which the consistent use of the traditional fundamental concepts leads us to paradoxes difficult to resolve.**”[4], pp.18-19]

It is beyond doubt that one of these ‘imbibed with the mother’s milk’ concepts are those of identity and countability, being identity essential for countability.[1] Even in the case of a non-denumerable collection of things, in order to ‘count’ them, standardly we need to provide bijections, which presuppose the identity of their elements.

 doi.org/10.32388/NCXO2K

In quantum physics we do not deal with such a huge number of entities (an infinite quantity of things), but generally with just a finite number of them, although, sometimes, a huge number. But, even in this realm, some philosophers think that in order to attribute a number expressing a quantity of things (a cardinal to a set of things), say to the electrons in a neutral Sodium atom, we need that they can be discerned from one each other, that is, that they *have identity*[5][6][7].

But things cannot be taken so naively. In the electronic distribution of electrons in an atom, we consider the solutions of the Schrödinger Equation (the wave-functions) obeying conditions imposed by the potential in order to get the *quantum numbers*. These numbers, as well-known, provide the distribution of the electrons in the electronic levels *without particular individualisation of the electrons*. That is, what is taken into account is that the relevant things are entities of a kind (namely, electrons) and that they come in certain quantities.[2] No *identity* is presupposed, if by this term we understand that the electrons can be identified particularly (see[8], Chap.10]). Notice that even obeying Pauli's Exclusion Principle, which implies that no two electrons can share all the same quantum numbers, we are unable to specify *which is which*. As Dalla Chiara and Toraldo di Francia have said, the quantum realm is a land of anonymity, where proper names do not act as rigid designators[9] (see also[10]): if we call an electron 'Peter', this name will not identify *the same* electron again in other situations.

Bosons pose a still stronger way to challenge the standard notion of identity: in some situations, such as in a Bose-Einstein Condensate, a really great number of bosons behave in just one way, forming a Giant Matter Wave[11];[3] all the bosons behave as just one thing, sharing *all* their quantum numbers, so being *completely indiscernible* from one each other; no differences exist, something that would be a fact if the standard theory of identity applies to them. I think that nowadays no one questions the existence of entities that cannot be put apart or then, even if put apart, as in some 'practical situations' (see below), one is not able to tell which is which in a way that the identification remains in other contexts – call the two electrons in a neutral Helium atom 'Mike' and 'Ike'. It is impossible to state '*this* is Mike' and '*that* is Ike'; as emphasised by Hermann Weyl, we cannot find an alibi for electrons[12], p.241].

Notwithstanding, our logic and our mathematics are grounded on a 'Leibnizian' idea that in being more than one, the entities, whatever they are, must present some difference, either if we know it or not. In buying this idea, even if the distinguishing properties are unknown, we buy also the existence of some kind of *hidden variables* which could provide the distinctions. Consequently, in considering a metaphysical view according to which quantum entities are devoid of identity conditions, it would be interesting to find a compatible logic and mathematics in order to express this (to me) unavoidable quantum fact which, in my opinion, does not assume the existence of discerning properties (by the way, this would be against, say, the Bose-Einstein statistics — despite the claims of some, such as van Fraassen, that this is not so[13], chaps.11,12].[4] This is the *motto* of the non-reflexive formulation of quantum mechanics (NRFQM).

Thus, summing up, the NRFQM is grounded on two main lines: (1) in certain situations, we cannot attribute standard identity (given by the Standard Theory of Identity, STI) to quantum objects. This seems to be consonant with Schrödinger's view put in the epigraph, although he was not clear about the meaning of the word 'identity'. At that time, Schrödinger was speaking of orthodox quantum mechanics, but it is easy to extend the idea to the 'particles' in quantum field theories; (2) the absence of identity makes things *non-individuals*. Thus the second guideline: according to the philosopher of physics Heinz Post, this non-individuality should be attributed to the entities "right from the start"[2], and not made *a posteriori* as usually done (see below).

The lack of identity is related to the problem of individuation. But we do not superpose these notions; we are with Hume when he says that "one single object conveys the idea of unity, not that of identity"[14], p.200]. 'Individuation' needs to be discerned also from 'isolation' or 'identification'; something can be *isolated*, say in a quantum trap, but this does not make it an individual so as it does not provide it an identity. We shall need to qualify this claim, which we do below.

So, this paper is organised as follows. In the next section, we show how the standard theory of identity (STI) encompassed by classical logic and by the standard mathematics entails that every object (that obeys STI) is an *individual*. So, in assuming the above metaphysics that quantum entities can be viewed as non-individuals,[5] we can put classical logic in question and this motivates the introduction of non-reflexive logics.

The subsequent section introduces the main ideas of a non-reflexive 'set' theory, termed *quasi-set theory* termed '$\mathfrak{Q}$', which grounds the development of a formulation of quantum mechanics in $\mathfrak{Q}$-spaces, a kind of Fock spaces where there are no label for the considered entities, so avoiding the trick of assuming the identity and then going back and pretending that the entities are indistinguishable.

Due to the limitations of space, we cannot go into the details, to which we recommend the works in the References. The paper ends with some Conclusions.

doi.org/10.32388/NCXO2K

2 The Standard Theory of Identity, STI

Classical logic and standard mathematics (we assume the mathematics that can be founded in the ZF set theory) are 'Leibnizian' in the sense that whatever *two* objects present some difference.[6] We say that an object *does have identity* if it obeys STI to be formalised soon. In terms of sets, it is easy to grasp the intuition; being a and b with $a \neq b$, then the unitary sets $\{a\}$ and $\{b\}$, seen as the extensions of two properties I_a and I_b (the 'identities' of a and b respectively, defined as $I_y(x) \coloneqq x \in \{y\}$), grant that there are discerning properties (we shall be using this example below). In order to attribute the cardinal 2 to the set $\{a, b\}$, we need that its elements are *different* so that the bijection $f : \mathbf{2} \to \{a, b\}$ does exist, being $\mathbf{2} = \{0, 1\}$ the von Neumann's cardinal 'two'.

This can be generalised for any quantity n of objects (n is finite from now on). Some philosophers say that the distinguishing properties should not involve either the equality symbol or individual constants[15]. But in ZF we can define, as above, for any y, the predicate $I_y(x) \coloneqq x \in \{y\}$ and this condition is fulfilled, showing that in any realm described by a theory like ZF, *every object has identity*.

STI is formulated in first-order predicate logic taking '$=$' as a primitive binary predicate symbol subjected to the axioms of *reflexivity* $\forall x(x = x)$ and *substituvity*, $\forall x \forall y(x = y \to (\alpha(x) \leftrightarrow \alpha(y))$, being $\alpha(x)$ a formula with x free and $\alpha(y)$ got from substituting y in some free occurrences of x in $\alpha(x)$. From these postulates, one can prove that '$=$' is also symmetric and transitive.

In some situations, the predicate '$=$' can be defined if we find a formula $\alpha(x, y)$ in two free variables so that we can put $x = y \coloneqq \alpha(x, y)$ so that the above conditions can be obtained. This is the case with set theories, but we shall not enter the details[16], Chap.1]). In higher-order languages, we can define identity by Leibniz Law; let us exemplify with a second-order language. Being x and y individual variable and F a variable for predicates, then we put $x = y \coloneqq \forall F(F(x) \leftrightarrow F(y))$. This is essentially Whitehead and Russell's definition in their *Principia Mathematica*[17], §*13].

The semantics is usual, made in a theory like ZF. Considering first-order languages, there is a non-empty domain D so that the predicate of equality is interpreted in the *identity of* D (or the *diagonal* of D, namely, the set $\Delta_D = \{\langle x, x\rangle : x \in D\}$. But there is a challenge in this first-order schema; as exemplified in[18], there are elementary equivalent structures which also interpret our language and where the equality sign *is not* associated to the diagonal of the domain. So, we can say that, from the point of view of a first-order language, we never know whether we are dealing with individuals (elements of D) or with collections of

 doi.org/10.32388/NCXO2K

them. In other words, first-order languages cannot fix the diagonal or, saying with other words, do not define (standard) identity.

So, we have this: in a set-theoretical 'standard' (that is, grounded on classical logic) framework, such as ZF, *every object can be discerned from any other*, and this can be done *absolutely*, that is, using a monadic predicate:[7] they are individuals and cannot be completely indistinguishable (call entities of this kind '*indiscernible*').[8] Important to realise that we informally say that something is an *individual* if obeys the following two conditions: (1) it is a one-of-a-kind, a person, a cat, and (2) it can be re-identified as such in different contexts; the person we see now is *that* person we met yesterday. This second condition is precisely what quantum entities lack, so as portions of water and other kinds of natural kinds[19]. But these conditions are fulfilled for any object obeying STI; see the section 4 below.

We can *mimic* indiscernible things employing some tricks made *within* ZF, say by considering *deformable structures*. These are set-theoretical structures encompassing automorphisms other than the identity function. For instance, the natural numbers 2 and -2 are indiscernible within the structure of the additive group of the integers $\mathcal{Z} = \langle \mathbb{Z}, + \rangle$ since the application $h(x) = -x$ is an automorphism of this structure.

Within a deformable (or not-rigid) structure, things can look as if they were indiscernible as in the so-called Permutation Models in a set theory with atoms, where atoms *are made* to mimic indistinguishable elements (see the mentioned references). But this is no more than a subterfuge since *every ZF-structure can be extended to a rigid of not-deformable structure*, including the permutation models.[9] In a rigid structure, the only automorphism is the identity function, so an object is indiscernible only from itself. For instance, we can rigidify $\mathcal{Z}$ by adding the usual ordering '$<$' to the structure. In the same vein, permutation models treat atoms as indiscernible only inside the models, but they are *distinct* when seen from the whole universe of ZFA. It can be proven that the 'whole universe of sets', captured by the idea of the cumulative hierarchy $\mathcal{V} = \langle V, \in \rangle$, is rigid[20], p.66]. So, leaving the structures for broad ones, we realise that in the universe of standard sets, everything is an individual.[10]

Important to notice that the above holds for any other theory of sets such as NBG (von Neumann-Bernays-Gödel), KM (Kelley-Morse) and many others.

 doi.org/10.32388/NCXO2K

3. Non-reflexive logics

The term 'non-reflexive' comes from the violation of the Principle of Identity in first-order logic (PI-FOL) $\forall x(x = x)$ once it is assumed that it does not hold for all objects of the domain. This does not mean that there are objects which are not identical to themselves, but that expressions of the form '$x = y$' have no meaning for some entities. Since the referred principle is also called the Reflexive Law of Identity, the name arises.

So, we define non-reflexive logics as those systems that depart from STI in some aspect, even at the propositional level, where the Principle of Identity is written as '$p \to p$', being p a propositional variable (for a system violating it, see[21]). We can also consider the violation of the self-deduction $A \vdash A$ in non-reflexive systems[22]. But it will be with PI-FOL that we shall be occupied here.

The history of non-reflexive logics can be traced back to the beginning of the 20th century when some scholars have questioned the Principle of Identity in some way, including the Aristotelian '$A \text{is} A$', being 'is' the 'is' of identity[23], although this mentioned paper just discuss the issue without presenting a well-articulated logical system. The first non-reflexive logic was proposed by Newton da Costa in 1980[3], pp.117ff]. It is a first-order two-sorted system draw to deal with two kinds of entities, those of kinds 1 and 2. To the entities of kind 1, the expressions of the form '$x = y$' are not well-formed. The inspiration was to show that any logical principle can be questioned by a reasonable logical system where it does not hold; so, in da Costa's system, $\forall x(x = x)$ is not universally valid. This logic was extended to higher-order systems by the present author[24][25] (see[26][27]) and to set-theory with the formulation of a *theory of quasi-sets* which are collections of entities to some of which STI does not hold[28][29][30]. A quasi-set (qset) is a collection of entities such that to some of them STI does not hold; a qset may have a cardinal, its *quasi-cardinal*, but no ordinal is associated to it. We sketch the main lines of this theory below since it grounds the NRFQM.

The basic idea of the higher-order Schrödinger logics is to provide a way to define identity by Leibniz Law and to consider indistinguishability as the agreement concerning all properties, without any commitment to other things than properties and relations. That is, any form of substratum, *haecceity* and so on is avoided. We are within the scope of the Bundle Theories of Individuation[31].

But we face a problem. In defining identity by Leibniz Law, we become committed with *all* properties in the range of the universal quantifier. The same would happen with a definition of indiscernibility put in the same way: $x \equiv y := \forall F(F(x) \leftrightarrow F(y))$ and there would not exist any differences between these notions. The solution is to ground a semantics in the theory of quasi-sets, leaving identity holding for 'classical' objects but not for those which will represent quanta, and indiscernibility holding for all objects, with the proviso that $x \equiv y$ does not entail $x = y$.

As a consequence, we get interesting results we just refer to: we may have predicates, standing for *intensions*, which may have several 'distinct' extensions, something unimaginable in standard accounts. But such a fact shows that ideas such as those of Dalla Chiara and Toraldo di Francia in that the quantum world is a world of intensions, get a formal description. As they say, the predicate 'to be a collection with two electrons' can be realised (have extensions) of different kinds ('different' collections of electrons). This emphasises the importance of considering intensions in the quantum realm. The details are presented in[27][29].

But before going there, let us summarise the main motivation for our approach. After this, we advance some hints about the theory of quasi-sets.

4. Individuals and non-individuals

As anticipated earlier, the rough idea of an *individual* is the following: an individual is something that is a unity of a kind such as a chair or a person. But, fundamentally, they enable re-identifiability, that is, an individual can be recognised as such (as *that individual*) in different contexts. Julius Caesar was an individual, being just one person and *the same person* either when passing the Rubicon, fighting Pompey and staying with Cleopatra. Jonathan Lowe has a similar characterisation of this notion, and says that portions of water, for instance, are not individuals since they do not preserve the second condition[19]; when a cup of water is spilt in the sea, we no more will be able to identify 'our' portion of water again. The same happens with electrons after ionisation, when a neutral atom loses an electron in order to turn a cation; the expunged electron cannot be identified anymore, if there is a sense to say that it had an identity before (see below). This applies also to any quantum entity put in a trap; while in the trap, it can be said to be *isolated* or *individuated*, but not that it acquires an identity since it lacks re-identifiability.

Persons, chairs and all the usual objects of our surroundings are supposed to be individuals, although there are disputes even concerning them. David Hume, for instance, suggested the re-identification is something we make due to the habit, "[a] principle which determines me *to expect* the same for the future"[14], p.265, and Book I, *passim*], my emphasis. It seems that quantum entities, such as electrons, protons and so on lack the re-identifiability condition. Without entering the discussion about *what* are these entities precisely, since every particular theory has its own way of describing them,[11] we can consider that this seems to make complete sense; Schrödinger, for instance, said that

> If I observe a particle here and now and observe a similar one a moment later at a place very near the former place, not only cannot I be sure whether it is 'the same', but this statement has no absolute meaning. This **seems** to be absurd. For we are so used to thinking that at every moment between the two observations the first particle must have been **somewhere**, it must have followed a **path**, whether we know it or not. And similarly, the second particle must have come from somewhere, it must have **been** somewhere at the moment of our first observation. So in principle, it must be decided, or decidable, whether these two paths are the same or not—and thus whether it is the same particle. In other words, we assume—following a habit of thought that applies to palpable objects—that we could have kept our particle under **continuous observation**, thereby ascertaining its identity.[1], p.131]

Hermann Weyl also gave his pronunciation about the fact that quantum entities would not conform to the idea of individuals and called them precisely 'non-individuals', entities "without identity"[32], App.B]. Other important physicists who expressed the same feelings were Max Born and Werner Heisenberg (see[29] for the historical details).

These people used to say that these quantum entities have *lost their identities*, but in our point of view, nothing can lose what it lacks.[12] Quantum entities can be *discerned* from others in certain situations as being of different kinds (say electrons and protons) or even when they are of the same kind but are, say, located in different laboratories. But this is different from saying that they are *different*, this notion coming from STI, since this would entail that there is a property of one of them which is not shared by the other. Yes, you may be thinking of the spatial location, so let us consider this case. Let our two q-objects of the same kind (say two electrons) be located in Lab 1 and Lab 2 which are in different cities. Then we can describe the state of the first q-object by a wave-function ψ_1 and the state of the other by ψ_2. This seems to distinguish them, but not. When we consider the join system, and we are doing that once we are speaking of both of them, the state of the system q-objects 1 and 2 is given by an anti-symmetric state of the form $|\psi_{12}\rangle = A(|\psi_1\rangle\,|\psi_2\rangle - |\psi_2\rangle\,|\psi_1\rangle)$, being A a normalisation factor. In such a situation, despite the fact they obey Pauli's Principle, we cannot say which is which, so there is no precise sense in constructing a bijection from the collection of the two q-objects in the cardinal **2** (see the section 6.1 below). In fact, the value of $\|\psi_{12}\|^2$ involves an interference term which only for practical purposes can be dispensed with. Simply to eliminate it is a logical mistake similar to ignore the infinitesimals in the earlier calculus of Newton and Leibniz without going to Non-Standard Analysis; for a discussion on this specific topic, see[33][34].

Suppose we have a collection of such entities, that is, entities to which we cannot say that they are equal or different, but which count as more than one. Typical physically accepted cases are bosons in bosonic condensates and even fermions in entangled states, which despite differing in something (usually in their values of spin) due to Pauli's Exclusion Principle, do not enable us to say 'this is Peter' and 'this is Paul'. There is no 'which-is-which' in the quantum world. The question is how we can associate a quantity (a cardinal) with such a collection. Usually, a cardinal is a kind of ordinal[20] so it seems that we need to accept first that the elements of our collection can be ordered, hence discerned from one another. But this is not necessarily so as we shall see below.

5. Quasi-sets

It is time to make some general remarks about the theory of quasi-sets, $\mathfrak{Q}$. A quasi-set (qset) generalises the notion of a standard set of a chosen theory (we take ZFA) enabling two kinds of atoms; those that obey ZFA are called 'M-atoms', and those which do not are the 'm-atoms'; primitive monadic predicates M and m are assumed to express that. The second ones are drawn to mimic quantum non-individuals, so the identity of STI does not hold for them: if either x or y stand for an m-atom, then expressions such as '$x = y$' are not well-formed formulas.

A primitive binary relation '$\equiv$' of 'indistinguishability' is assumed as primitive, having the properties of an equivalence relation, but it is not a congruence, and this makes its difference to identity. A qset is something that is neither an m-atom nor an M-atom, obeying a defined predicate Q put as $Q(x) \coloneqq \neg m(x) \wedge \neg M(x)$. A defined notion of *extensional identity*, '$=_E$' is assumed to hold for M-atoms that belong to the same qsets or to qsets having the same elements (all of this is expressed in the language of the theory, as exposed in[30]). This extensional identity has all the properties of standard identity for the objects it applies, termed 'classical'. In this 'classical part' of the theory, we can develop all mathematics that can be built in ZFA.

Even devoid of identity conditions, those entities represented by the m-atoms can appear in 'species', or 'kinds' since we can attribute them properties; so we can speak of 'electrons', 'protons', and so on. But even those of the same kind can be assumed not to be 'completely' indiscernible from one another. As we have anticipated above, the attribute 'to be an electron of a collection of two electrons' can be paraphrased in the theory by a formula of the form 'x is a m-atom that has this and that characteristic and belongs to a qset whose quasi-cardinality (see below) is two'. The 'this and that' copes with the properties of electrons, such as a certain mass, a certain electric charge, etc and we know already that this predicate may have several extensions. The important thing is that even belonging to qsets of different natures (say one having other kinds of m-atoms as elements), the elements are such that they can be 'permuted' without altering the resulting collections. Let me explain.

In a certain way, we *distinguish* among m-atoms that belong to specific qsets but we cannot say that they do have identity or that they are individuals, since the theory is such that if some m-atom of a qset is substituted in some way (in the theory, by the 'qset-operations') by some 'other' indiscernible m-atom, the resulting qset will be indiscernible from the former (they will partake the relation of indiscernibility) — see[29], §7.2.6]. Notice that this is what happens if we take a neutral Helium atom and make ionisation, getting a cation He^{+}. The realised electron is lost and even if we capture 'another' electron, we will never be able to say that the captured electron is that which was realised nor that the new neutral atom 'is the same' as the original one. A principle such as the Axiom of Extensionality of ZFA does not hold here in full (but just for 'classical' things).

One of the core ideas is that of *quasi-cardinality*. The idea is to be able to associate to a qset (let us take a qset with n indiscernible m-atoms a natural number n expressing its quasi-cardinal. But in order to avoid the identification of the elements, it is adequate that the natural number is not an ordinal, that is, *it is not* taken from the model of the arithmetics that we can find in $\mathfrak{Q}$. In order to give a sense to this idea, we proceed as follows.

Let us consider the theory $\mathfrak{Q}$ and denote by $x, y, z, \ldots$ the qsets, some of them having absolutely indiscernible m-atoms as elements, and let n be a natural number. In short, we define the notion of 'the quasi-cardinal of the set x is the natural number n' as follows — see[35] for details. Let PA1 the first-order Peano arithmetics formalized with a signature $S = \langle 0, s, +, \cdot; 0, 1, 2, 2 \rangle$. Let $\mathfrak{Q}$' = $\mathfrak{Q}$ + PA1, meaning that we *add* the vocabulary of PA1 and its axioms to $\mathfrak{Q}$. Notice (again) that we *are not* taking a model of PA1 there may exist in $\mathfrak{Q}$, but we are considering that the arithmetics *runs in parallel* with $\mathfrak{Q}$, in the same sense that the theory of fields can 'runs in parallel' with the theory of vector spaces and group theory can run in parallel with the theory of vector spaces in representation theory. So, the natural numbers to be

doi.org/10.32388/NCXO2K

attributed to the sets of S are not taken from the set theory itself (that is, from the model of PA1), so *they are not ordinals.*

Let 0,1,2,...,n stand in $\mathfrak{Q}$' for the corresponding natural numbers, where $1 \coloneqq s0$, $2 \coloneqq ss0$ and so on; this can be done by adding new individual constants to the language of $\mathfrak{Q}$. Thus, in saying that a set x has a quasi-cardinal n, we write $qc(x, n)$ in the language of $\mathfrak{Q}$'. With this move, we can say that the set x has a cardinal n without committing it with any ordination of its elements.[13]

Thus we can attribute a quasi-cardinal to a qset without 'counting' its elements, just by *positing* a natural number to the collection by a way provided by the physical theory, as in the case of the electrons in the orbitals. For instance, chemistry teaches us that in the second energy level of an atom there can be eight electrons and the theory simply make reference to the number of electrons there. For some details, see again[34].

Notice that there is no counting process if by this we understand a process which could be like this: 'Peter and Paul, go to level 1s', 'Sarah, go to the second level', and so on. No, it is not this way that things happen as we have seen.

Quasi-set theory can be developed as a mathematical theory, but the interesting fact is its use to ground a formulation of quantum theory that avoids the use of particle labels. Let us have a look on the main ideas.

6. A glimpse on a non-reflexive quantum theory

Finally, having (so I hope) justified the reasons to adopt a non-reflexive formulation of quantum physics, let us briefly sketch how to approach quantum theory. The details can be seen in[36][37][30]. Let me emphasise that the main reasons are two: (1) to follow Schrödinger's view that identity (here taken from STI) does not apply to quantum entities and (2) that their indiscernibility is to be assumed 'right from the start' and not made by hand when some mathematical trick is done within a standard set theory.[14]

In the standard formalism via Hilbert spaces, it is assumed that quantum systems, when described jointly, are subjected to two types of pure states,[15] So, 'state' here means 'pure state' (but see the subsection 6.1), the other being considered as *surplus structures*, that is, as something that can be written in the formalism but are never realised since they would correspond to nothing in the real world[38]. The only admissible states are either *symmetric*, which hold for bosons or *anti-symmetric*, which are states of fermions. The Spin-Statistics Theorem says that the particles with entire spin are bosons and obey Bose-Einstein statistics while particles with half spin are fermions and obey Fermi-Dirac statistics; the

Symmetrization Postulate states that the states of a system containing n particles of the same kind (which physicists term 'identical') are either symmetric or anti-symmetric with respect to any permutation of the particle labels[39], p.595]. The labels are introduced due to the nature of the language, which is a language of individuals endowed with identity conditions, so that permutation conditions need to be assumed in order to pretend that they have no individuality, that is, that an exchange of labels do not alter the results. The language to be presented now, which grounds NRFQM does not need such a trick and the symmetry conditions arise 'naturally'. In the above-mentioned papers, it is shown how we can construct a quantum theory (NRQM for 'non-reflexive quantum mechanics') by means of a Fock space formalism which considers the notion of indistinguishability as a primitive notion and not introduced *a posteriori* as usually made, when the physicist assumes that only symmetric and anti-symmetric states are viable for quantum systems, that is, by using standard languages and logic and distinguishing q-objects by attributing them labels such as '1', '2' and them assuming symmetry conditions; for instance, the Hamiltonian for the two electrons in a Helium atom is usually written

$$H = \left(\frac{-\hbar^2}{2m}\nabla_1^2 - \frac{2e^2}{r_1}\right) + \left(\frac{-\hbar^2}{2m}\nabla_2^2 - \frac{2e^2}{r_2}\right) \tag{1}$$

which is invariant by the permutation of the labels '1' and '2'. As Eugen Merzbacher says,

> "The coordinates of the electrons are labelled 1 and 2 under **the provisional assumption** that the particles are in principle distinguishable. Of course, we know that this assumption is false but (...) with this assumption we can obtain the entire spectrum of the two-electron system."[40], pp.442-3], my emphasis

Thus, the NRQM approach departs from the standards and it is in this point that it gains its advantages.

The mathematical basis is the theory $\mathfrak{Q}$. Two 'Q-spaces' are build, one for bosons and another for fermions. Basically, they differ relatively to the inner product where odd and even permutations are used to simulate anti-symmetric and symmetric situations. Particles are not labelled in any step of the formal construction. This is possible because these spaces are constructed using the non classical part of $\mathfrak{Q}$, where we may refer to intrinsically indistinguishable entities. Vectors in these spaces are only distinguished by the *occupation number* in each (energy) level. With these tools and using the language of $\mathfrak{Q}$, the formalism of quantum mechanics may be completely rewritten giving a straightforward answer to the problem of giving a formulation of quantum mechanics in which intrinsic indistinguishability is taken into account from the beginning, without artificially introducing extra postulates.

As Domemech et al. have said in their final discussion,

> "We have shown that it is possible to construct the quantum mechanical formalism for indistinguishable particles without labelling them in any step. To do so, we have built a vector space with the inner product, the Q-space, using the non-classical part of $\mathfrak{Q}$(...). Vectors in Q-space refer only to occupation numbers and permutation operators act as the identity operator, reflecting in the formalism the fact of unobservability of permutations, already expressed in terms of the formalism of $\mathfrak{Q}$[29]. We have also argued that it is useful to represent operators (which are intended to represent observable quantities) as combinations of creator and annihilation operators, in order to avoid particle indexation in the expression of observable quantities. We have shown that creation and annihilation operators which act on Q-space can be constructed. We have proved that they obey the usual commutation and anti-commutation relations for bosons and fermions respectively, and this means that our construction is equivalent to that of the Fock-space formulation of quantum mechanics. (...) this implies that we can recover the n-particles wave equation using Q-space in the same way as in the standard theory. Though both formulations are equivalent 'for all practical purposes', when subjected to careful analysis, the conceptual difference turns very important. Our construction avoids the LTPSF [Redhead and Teller's 'Labelled Tensor Product Space Formalism'[38]] by constructing the state spaces using $\mathfrak{Q}$, a theory which can deal with truly indistinguishable entities, and so, it gives an alternative (and radical) answer to the problems posed in[38], so as (we guess) answers Manin's problem posed in[41]; Yuri Manin proposed the development of a 'new theory of **sets**' to cope with collections of quanta, and $\mathfrak{Q}$ is a serious candidate for taking such a place].
>
> This last point seems remarkable, for our construction incorporates intrinsical indistinguishability from the beginning. Thus, our approach fulfils not only Post's claim already mentioned [namely, that the non-individuality of quanta should be ascribed 'right from the start'], but also both Manin's claim that we should find an adequate 'set theory' for expressing collections of indistinguishable quanta (...)"

6.1. An alternative view

Of course, there are alternatives to the treatment of indistinguishable quantum entities. Bohmian mechanics treats them as individuals endowed with identity conditions, and when considering 'identical

doi.org/10.32388/NCXO2K

quanta' (in the physicist's jargon), the theory uses permutation symmetries as well (see[42], §6.1.4] for seeing the trick).

Dennis Dieks and Andrea Lubberdink also proposed to interpret quantum entities as *emergent* from certain quantum states[43][44]. To them, the primitive notion is that of *state*, and in some situations, the entities are so apart that *can be treated* as they were individuals obeying classical physics and hence the mathematics could be the usual (say, grounded in ZFC). With such a move, they are *ignoring* the interference terms that appear when we take the square of the wave-function in order to get the probabilities once, as they say, in general, these values can be neglected. As they say, this is in accordance with the practice of physics, and really physicists seem to act this way. Only when sufficiently close do the entities mix in a way that their states cannot be put apart, as when the state is entangled. In this case, they say, we can assume quantum entities as non-individuals.

So, there are two situations to be referred to. The first is that Dieks and Lubberdink don't get rid of non-individuals, accepting them in certain situations. The second point is that, as said above, they *ignore* the interference terms; this can be useful for the physicist, but not for someone occupied with logical foundations. As suggested above, disregarding the interference terms could be compared with the disregarding of infinitesimals in the old infinitesimal calculus. As pointed out ever since Berkeley in his *The Analyst*, from 1734, this leads to a logical contradiction, so we can guess that the physicist's action does it too, despite we can work as an engineer does when using 'infinitesimal elements of volume'. Within certain approximations, it works and for the applications, this may be enough. So, duly qualified, we see their 'Alternative View' as complementary to ours (termed the 'Received View' — see[29][45] for an exposition), and not as a competing one[34].

7. The challenge of the 'physical properties'

From the philosophical and logical foundational analysis to which we are dedicated, it is important to consider that some philosophers guess (truly, they affirm this categorically!) that the distinguishing properties, that is, those properties that could lead to a distinction among quanta would be 'physical', and we suppose we can say that they get rid the 'purely logical' ones. In this section, we shall consider this possibility.

First of all, we need to say what it to be 'physical' concerning some property.[16][16]16Notice that we are proposing a logical analysis of the situation, and not a physical one. In this second sense, a good

 doi.org/10.32388/NCXO2K

discussion is in[46], Chap.2]; see also[33] where the authors take the motivations from Bridgman. Let us use F, G, H for variables for properties of individuals (that is, they have type $\langle\tau\rangle$, where the individuals have type τ); the individuals (basic entities in the logical sense) are denoted by x, y, z. Furthermore, let us take a constant predicate P of order $\langle\langle\tau\rangle\rangle$. Notice that, in being a constant predicate, we shall not quantify over it, so we remain in a second-order language. Thus if X is a predicate variable, $P(X)$ says that X is (or stands for) a *physical property*. Then we could say that objects x and y are 'physically discerned' ($x \simeq y$) this way:

$$x \simeq y \coloneqq \forall X(P(X) \to (X(x) \leftrightarrow X(y))). \tag{2}$$

but this is just saying that x and y are indistinguishable by these physical properties, not excluding *by fiat* the existence of other properties that could state their differences. To be 'physically discernible', there must exist a X obeying P so that $X(x)$ but $\neg X(y)$ or the other way around.

But according to the standards, physical properties should be 'measurable'. But what could be the postulates of P to characterise 'measurable properties'? Of course, they would be described in the object language, and we can't imagine what they might be. You can say that this characterisation will depend on the physical theory, and this is right, but the problem is that there is no apparent reason to disregard properties such as 'purely logical' ones such as the identities I_y defined above.

Logic is mandatory, and its theorems are theorems of any theory grounded on it, and this fact should not be neglected. So, if we use ZFC or something encompassing STI in the background, there will be no escape: every entity is an individual, has identity and can be discerned *absolutely* from any other individual. If we don't wish to use alternatives involving mathematical tricks or meta-assumptions, the only way to treat absolutely indiscernible things is to change the logic.

Notes

This works surveys the subject of my invited talk at the 17th CLMPST, Buenos Aires, August 2013 and is dedicated to Newton da Costa for his 94th Anniversary.

Funding

Partially supported by CNPq.

Footnotes

1 But as we shall see, identity is not essential for attributing a quantity (a cardinal) to a given collection of objects, say electrons in a certain orbital.

2 This is more or less in consonance with S. Lavine's suggestion that quantum mechanics would not seem as atomistic theory, but as "a theory of kinds of stuff"[47] (and of course of their quantities). This is precisely what the *theory of quasi-sets* — to appear below — provides us: kinds and quantities, no individuation. In guessing that we may have non-individuals, we are not compromised with any form of atomism in the sense of accepting invariant units[48].

3 In 2007, scientists have to get BECs (Bose-Einstein Condensates) with *circa* 120×10^6 atoms, and presently surely there are more[49].

4 Recent experiments in quantum technology and quantum optics have shown the necessity to resort to the *complete indistinguishability* of quanta in order to explain the phenomena; see the Hang-On-Mandel Effect[[30], §5.5].

5 Notice that I am not saying that quantum entities *are* non-individuals. I suspend the judgment; there are other approaches where they do obey STI – for instance, Bohmian mechanics; see also[44] and the references therein.

6 We shall not consider other logics such as modal systems since usually, the physicist does not make use of them.

7 Philosophers term 'absolute discernibility' when objects can be discerned by monadic properties.

8 For a more detailed discussion, see[29][50][30]. See section 7 below.

9 In this case, or course we are speaking of a theory with atoms.

10 If we are assuming the Axiom of Choice, it is enough to add a well-ordering of the domain to the structure. In the finite case, the unitary sets provide rigidification.

11 At chapter 6 of his book[51], Brigitte Falkenburg traces the 'metamorphoses' of the concept of particle from classical physics til the standard model; this can be generalised to other concepts as well.

12 This claim differs from the acceptation of alternative approaches that see q-objects as endowed with identity. We acknowledge that there are such moves, as said already. But no approach can be said to

 doi.org/10.32388/NCXO2K

determine what q-objects are; all we have are our suppositions engraved in our theories.

[13] As in[34], we can use the following postulates for this new concept: 1. $\forall x(qc(x,0) \leftrightarrow x =_E \varnothing)$. 2. $\forall x \forall y(qc(x,N) \wedge qc(y,1) \wedge x \cap y =_E \varnothing \to qc(x \cup y, N+1))$ 3. $\forall x(qc(x,N) \to qc(\mathcal{P}(x), 2^N))$ 4. $\forall x(qc(x,N) \to \forall M(M < N \to \exists y(y \in \mathcal{P}(x) \wedge qc(y,M)))$

[14] Recently, A. Sant'Anna suggested that some permutation models of ZFA could play the role of expressing non-individuals. This idea may work FAPP — this is John S. Bell's term for 'for all practical purposes — but goes against the metaphysics of non-individuals, since indiscernible *are made* to be so only within the permutation model. See[52] and the criticisms in[53].

[15] We shall not discuss mixtures, although our arguments can be extended also to them.

References

1. a, b*Erwin Schrödinger. What is an elementary particle? In Elena Castellani, editor, Interpreting Bodies: Classical and Quantum Objects in Modern Physics, chapter 12, pages 197–210. Princeton University Press, Princeton, 1998.*
2. a, b*Heinz Post. Individuality and physics. Vedanta for East and West. 132:14–22, 1973.*
3. a, b*Newton C. A. da Costa. Ensaio sobre os Fundamentos da Lógica. HUCITEC-EdUSP, São Paulo, 1980.*
4. ^*Max Jammer. Concepts of Space: The History of Theories of Space in Physics. Dover Pu., New York, 3rd, enlarged. edition, 2013.*
5. ^*Tomasz Bigaj. Identity and Indiscernibility in Quantum Mechanics. New Directions in the Philosophy of Science. Palgrave Macmillan, Cham, Switzerland, 2022.*
6. ^*Mario Dorato and M. Morganti. Grades of individuality: a pluralistic view of identity in quantum mechanics and in the sciences. Philosophical Studies. 163(3):591–610, 2013.*
7. ^*Francesco Berto. Counting the particles: entity and identity in the philosophy of physics. Metaphysica. 18(1):69–89, 2017.*
8. ^*Bruce H. Mahan. University Chemistry. Addison Wesley, 3rd.ed. edition, 1975.*
9. ^*Maria Luisa Dalla Chiara and Giuliano Toraldo di Francia. Individuals, kinds and names in physics. In Giovanna Corsi, Maria Luisa Dalla Chiara, and Gian Carlo Ghirardi, editors, Bridging the Gap: Philosophy, Mathematics, and Physics, Boston Studies in the Philosophy of Science, 140, pages 261–284. Kluwer Ac. Pu., 1993.*
10. ^*Maria Luisa Dalla Chiara. Some foundational problems suggested by physics. Synthese. 62:303–315, 1985.*

11. ^Wolfgang Ketterle. Bose-Einstein condensation: identity crisis for indistinguishable particles. In James Evans and Alan S. Thorndike, editors, Quantum Mechanics at the Crossroad: New Perspectives from History, Philosophy, and Physics, The Frontiers Collection, pages 159–182. Springer, Berlin and Heidelberg, 2007.
12. ^Hernann Weyl. The Theory of Groups and Quantum Mechanics. Dover Pu., New York, 1950.
13. ^Bas van Fraassen. Quantum Mechanicsv: An Empiricist View. Oxford Un. Press, Oxford, 1991.
14. a, bDavid Hume. Treatise of Human Nature. Oxford University Press, l.a. selby-bigge edition, 1985.
15. ^Adam Caulton. Permutations. In Eleanor Know and Alastair Wilson, editors, The Routledge Companion to Philosophy of Physics, pages 578–594. Routledge, 2022.
16. ^Abraham A. Fraenkel, Yohoshua Bar-Hillel, and Azriel Levy. Foundations of Set Theory. Studies in Logic and the Foundations of Mathematics, 67. Elsevier, Amsterdam, 2nd.revised edition, 1973.
17. ^Alfred N. Whitehead and Bertrand Russell. Principia Mathematica to *56. Cambridge Mathematical Library. Cambridge Un. Press, 1997.
18. ^Wilfrid Hodges. Elementary predicate logic. In Dov M. Gabbay and F. Guethner, editors, Handook of Philosophical Logic, volume I, pages 1–131. D. Reidel, Dordrecht, 1983.
19. a, bJonathan E. Lowe. Non-individuals. In Alexander Guay and Thomas Pradeu, editors, Individuals Across the Sciences, pages 49–60. Oxford University Press, Oxford, 2016.
20. a, bThomas J. Jech. Set Theory: The Third Millennium Edition, Revised and Expanded. Springer Monographs in Mathematics. Springer-Verlag, Berlin and Heidelberg, 2003.
21. ^Richard Sylvan and Newton C. A. da Costa. Cause as an implication. Studia Logica. 47(4):413–428, 1988.
22. ^Décio Krause and Jean-Yves Béziau. Relativizations of the principle of identity. Logic Journal of the IGPL. 5:327–338, 1997.
23. ^Oliver Reiser. Non-aristotelian logics. The Monist. 45:100–117, 1935.
24. ^Décio Krause. Non-Reflexivity, Indistinguishability, and Weyl's Aggregates (in Portuguese). PhD thesis, Faculty of Philosophy, University of São Paulo, August 1990.
25. ^Décio Krause. O Problema de Manin: Elementos para uma Análise Lógica dos Quanta. Tese para professor titular, Departamento de Matemática, Universidade Federal do Paraná, Curitiba, 1995.
26. ^Newton C. A. da Costa and Décio Krause. Schrödinger logics. Studia Logica. 53:533–550, 1994.
27. a, bNewton C. A. da Costa and Décio Krause. An intensional schrödinger logic. Notre Dame J. Formal Logic. 38(2):170–194, 1997.
28. ^Décio Krause. On a quasi-set theory. Notre Dame J. Formal Logic. 33(3):402–411, 1992.

29. a, b, c, d, e, f, g *Steven French and Décio Krause. Identity in Physics: A Historical, Philosophical, and Formal Analysis. Oxford Un. Press, Oxford, 2006.*
30. a, b, c, d, e *José A. de Barros, Federico Holik, and Décio Krause. Distinguishing indistinguishabilities: Differences Between Classical and Quantum Regimes. Springer, forthcoming, 2023.*
31. ^ *Loux MJ. Metaphysics: A Contemporary Introduction. 3rd ed. New York and London: Routledge; 2006.*
32. ^ *Weyl H. Philosophy of Mathematics and Natural Science. Princeton: Princeton University Press; 1949.*
33. a, b *Dalla Chiara ML, Toraldo di Francia G. A logical analysis of physical theories. Rivista del Nuovo Cimento. 3(1):1–20; 1973.*
34. a, b, c, d *Krause D. On Dieks against the Received View. Preprint; 2024.*
35. ^ *Krause D. Quasi-cardinals in quasi-set theory. Preprint; 2023.*
36. ^ *Domenech G, Holik F, Krause D. Q-spaces and the foundations of quantum mechanics. Foundations of Physics. 38(11):969–994; 2008.*
37. ^ *Domenech G, Holik F, Kniznik L, Krause D. No labeling quantum mechanics of indiscernible particles. International J. Theoretical Physics. 49:3085–3091; 2010.*
38. a, b, c *Redhead M, Teller P. Particles, particle labels, and quanta: the tool of unacknowledged metaphysics. Philosophy of Physics. 21:43–62; 1991.*
39. ^ *Meeiah A. Quantum Mechanics: Two Volumes Bound as One. Garden City, NY: Dover Publications; 2014.*
40. ^ *Merzbacher E. Quantum Mechanics. 2nd ed. New York: John Wiley and Sons; 1979.*
41. ^ *Manin YI. Problems of present day mathematics, i: foundations. In: Browder FE, editor. Proceedings of Symposia in Pure Mathematics: Mathematics Arising from Hilbert Problems. Vol. 28. Rhode Island: American Mathematical Society; 1976. p. 36.*
42. ^ *Tumulka R. Foundations of Quantum Mechanics. Lecture Notes in Physics, 1003. Heidelberg: Springer; 2022.*
43. ^ *Dieks D, Lubberdink A. How classical particles emerge from the quantum world. Foundations of Physics. 41(6):1051–1064; 2011.*
44. a, b *Dieks D. Quantum individuality. In: Arenhart JRB, Arroyo RW, editors. Non-Reflexive Logics, Non-Individuals, and the Philosophy of Quantum Mechanics: Essays in Honor of the Philosophy of Décio Krause. Synthese Library, 476. Cham, Switzerland: Springer; 2023. p. 11–28.*
45. ^ *Krause D, Arenhart JRB, Bueno O. The non-individuals interpretation of quantum mechanics. In: Freire Jr. O, editor. The Oxford Handbook of the History of Quantum Interpretation. Oxford: Oxford University Press; 2*

 doi.org/10.32388/NCXO2K

022. p. 1135–1154.

46. ^*Dalla Chiara ML, Toraldo di Francia G. Le Teorie Fisiche: Un'Analisi Formale. Torino: Boringhieri; 1981.*
47. ^*Lavine S. Is quantum mechanics and atomistic theory? Synthese. 89:253–271; 1991.*
48. ^*Post H. The problem of atomism. British Journal for the Philosophy of Science. 26(1):19–26; 1975.*
49. ^*Van Der Stam KMR et al. Large atom number bose-einstein condensate of sodium. Review of Scientific Ins truments. 78(1); 2007.*
50. ^*Krause D. On identity, indiscernibility, and (non-) individuality in the quantum domain. Philosophical Tra nsactions of the Royal Society A. 2255(381); July 2023.*
51. ^*Falkenburg B. Particle Metaphysics: A Critical Account of Subatomic Reality. Berlin, Heidelberg, New York: Springer; 2007.*
52. ^*Sant'Anna AS. On the consistency of quasi-set theory. In: Arenhart JRB, Arroyo RW, editors. Non-Reflexive Logics, Non-Individuals, and the Philosophy of Quantum Mechanics. Essays in Honour of the Philosophy of Décio Krause. Vol. 476 of Synthese Library. Springer; 2023. p. 97–104.*
53. ^*Krause D. A new quasi-set theory without atoms: a reply to adonai sant'anna. Preprint; February 2024.*

Declarations

Funding: Partially supported by CNPq.

Potential competing interests: No potential competing interests to declare.