On the existence of precession of planets' orbits in Newtonian gravity

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May 17

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Abstract

In a paper published in the mainstream journal Physics of the Dark Universe (C. Corda, Physics of the Dark Universe 32 (2021) 100834) we have shown that, contrary to a longstanding conviction older than 160 years, the precession of planets' orbits exists in Newtonian gravity if ones correctly analyzes the situation without neglecting the mass of the planet. In any case, the predicted Newtonian result was too large with respect to the observational values, despite it was, coincidently, in good accordance with the observational value of the precession of Mercury's orbit. In this new paper the situation is reanalyzed in Newtonian physics. It will be indeed shown that, despite the orbit's precession does not occur when the reference frame of the Sun is approximated as being fixed with respect to the fixed stars, it occurs, instead, in the real (in Newtonian sense) non-inertial reference frame of the Sun and it is due to the well known fact that, in a Newtonian framework, the distance which is travelled by a body depends on the reference frame in which the motion of the body is analyzed. After reviewing the solution of the problem which analyzes the planet's orbit as a harmonic oscillator, it will be shown that the precession is due to the breakdown of the conservation of the Hamilton vector in the non-inertial reference frame of the Sun. This approach will give a value of the same order of magnitude of previous result, but larger than that one and again without consistency with the observational values. In other words, it will be shown that it is not correct that Newtonian theory cannot predict an anomalous rate of precession of planets' orbits. The real problem is instead that a pure Newtonian prediction is too large to be consistent with the observational values.

https://doi.org/10.32388/NGRS8Y

1 Introduction

Via astronomical observations, in the early 1600s Kepler found that the orbit described by a planet around the Sun is an ellipse, being the Sun one of its foci. In the approximation in which the planet is subject only to the gravitational attraction of the Sun, Kepler's result can be obtained in the framework of Newtonian gravity. As also the other planets gravitationally attract the planet in question, one studies the effect of their presence. By realizing a calculation which takes into account this complication, one finds that the gravitational attraction due to all the other planets of the solar system on the planet in question generates a precession of the orbit. The precession of the Earth's rotation axis generates a similar effect too. In particular, Mercury's perihelion advances of 5,600 arcseconds per century in the direction in which the planet rotates around the Sun. If one removes the contribution of the Earth's precession (5,025 arcseconds) the additional contribution due to the presence of the other planets, calculated using Newtonian physics, cannot correctly predict the residual observed value. One indeed finds that 43 arcseconds are missed. For more than 160 years various computations have shown that this residual advance of Mercury's orbit cannot be justified via Newtonian theory. This is the famous problem of the anomalous rate of precession of the perihelion of Mercury's orbit which was originally recognized by the French Astronomer Urbain Le Verrier in 1859 in terms of "an important astronomical problem" [1]. Le Verrier's research on this issue started in 1843 [2], when he reanalyzed various observations of the perihelion of Mercury's orbit from 1697 to 1848. He obtained that the rate of the advance of Mercury's orbit was not consistent with the framework of Newtonian gravity. He indeed found a residual of $38''$ arcseconds per tropical century. In 1882 the Canadian-American astronomer Simon Newcomb corrected the value found by Le Verrier to $43''$ [3]. This residual appeared till now as being impossible to be achieved via Newtonian gravity. A certain number of ad hoc solutions being all unsuccessful, have been proposed, with the sole result to introduce more problems [4, 5]. In the 19th century, a famous approach was the introduction of a perturbing effect due to an unknown planet, Vulcan, hitherto escaped observation. Vulcan should have been smaller than Mercury and closer than it to the Sun [6, 7]. However, Vulcan has never been found by astronomers. Albert Einstein solved the problem via his magnificent theory of general relativity in 1916 [8]. A recent value of the advance of Mercury's orbit resulting from general relativity is about $42,98''$ per tropical century [9]. By expressing the perihelion's advance in radians per revolution (polar coordinates will be used hereafter), the general relativistic value is [10]

$$\triangle\varphi \simeq \frac{24\pi^3 a^2}{T_0^2 c^2\left(1-e^2\right)} = \frac{3\pi r_g}{a\left(1-e^2\right)}, \tag{1}$$

being a is the semi-major axis of the orbit, T_0 Mercury's Newtonian orbital period, c the speed of light, r_g the gravitational radius of the Sun and e the orbital eccentricity. Then, the total angle swept per revolution by Mercury is

$$\varphi \simeq \varphi_0 \left(1+\frac{12\pi^2 a^2}{T_0^2 c^2 \left(1-e^2\right)}\right), \tag{2}$$

being $\varphi_0 = 2\pi$ the unperturbed (i.e. in absence of precession) total angle swept by Mercury during a complete revolution around the Sun. If one inserts the numerical values in Eq. (1) [11–13], one gets the well known value $\triangle\varphi \simeq 5.02 * 10^{-7}$ radians per revolution arising from general relativity, which corresponds to about $0,1$ arcseconds.

In our recent work [14], the precession of the perihelion of Mercury's orbit has been calculated in the Newtonian framework. Three different approaches have been considered and the analysis has shown that the orbit of Mercury behaves as required by Newton's equations with a very high precision if one correctly analyzes the situation without neglecting the mass of Mercury. General relativity remains more precise than Newtonian physics, because the results in [14] seem to be a mere coincidence. In fact, the Newtonian formula of the advance of planets' perihelion breaks down for the other planets [14]. The predicted Newtonian result is indeed too large for Venus and Earth [14]. In fact, in [14] it has been shown that corrections due to gravitational and rotational time dilation are necessary. By adding such corrections, the same result of general relativity is retrieved.

Hence, two interesting results have been obtained in [14]:

i) It is not correct that Newtonian theory cannot predict the anomalous rate of precession of the perihelion of planets orbit. The real problem is instead that Newtonian prediction is too large;

ii) Perihelion's precession can be achieved with the same precision of general relativity by extending Newtonian gravity through the inclusion of gravitational and rotational time dilation effects. This second result is in agreement with the recent interesting works [15, 16], but, differently from such works, in [14] the importance of rotational time dilation has also been highlighted.

In this new paper the situation is reanalyzed in Newtonian physics. It will be indeed shown that, despite the orbit's precession does not occur if the reference frame of the Sun is approximated as being fixed with respect to the fixed stars, it occurs, instead, in the real (in Newtonian sense) non-inertial reference frame of the Sun and it is due to the well known fact that, in a Newtonian framework, the distance which is travelled by a body depends on the reference frame in which the motion of the body is analyzed. After reviewing the exact solution of the problem by analyzing the planet's orbit as a harmonic oscillator, it will be shown that the precession is due to the breakdown of the conservation of the Hamilton vector in the non-inertial reference frame of the Sun.

The new approach in this paper permits to obtain a value of the same order of magnitude of the Newtonian precession, but larger than the previous one in [14] and again without consistency with the observational values. Thus, the most important result of this paper is that it is not correct that Newtonian theory cannot predict an anomalous rate of precession of planets' orbits. The real problem is instead that a pure Newtonian prediction is too large to be consistent with the observational values.

2 Distance travelled by a body in two different Newtonian references frames

Suppose one has a highway where there is a straight line without curves for hundreds of kilometers in the east-west direction and two cars, 1 and 2, driving 1 in an east-west direction and 2 in a west-east direction at speed, one says 40 km/h, and the same direction but the opposite direction with respect to a fixed observer on the road. Compared to an observer in car 1 (a passenger from car 1), car 2 will move at a speed of 80 km/h. Now, if one asks how many kilometers car 2 will travel in an hour with respect to the observer fixed on the road, the answer will obviously be $40 km/h$, but if one asks how many kilometers car 2 will travel in an hour with respect to the observer in car 1, the answer it will be $80 km/h$. Now, one cannot tell that the $80 km/h$ answer is wrong and the 40 km/h answer is right, because in Newtonian physics there is no observer, or preferential frame of reference. Both answers are correct. It will be shown that a similar issue works also in the planet's orbit problem. If viewed with respect to the non-inertial reference frame of the Sun there is precession, if viewed with respect to the reference frame of the Sun approximated as being fixed with respect to the fixed stars there is no precession. The reason for this is similar to the example of the two cars on the straight. Compared to the non-inertial reference frame of the Sun, the planet moves faster than if the reference frame of the Sun is approximated as being fixed with respect to the fixed stars.

3 Planet's orbit as harmonic oscillator

One takes the origin of the frame of reference in the center of the Sun. In the following G will be the gravitational constant, M the solar mass, m the mass of the planet and r the distance between the Sun and planet. Following [14, 17], one recalls that each central attractive force can produce an approximate circular orbit that should not necessarily be closed. It is closed if the radial oscillation period is a rational multiple of the orbit period. Now, let $F_{c0}(r)$ be the total central force. The equation of motion for the planet is given by [14, 17]

$$F_{c0}(r) = m\left(\ddot{r} - \omega_0^2 r\right). \tag{3}$$

The last term in Eq. (3) can be physically interpreted as a force centrifuge. Since the angular momentum J_0 is a constant of motion, one has that

$$J_0 = mr^2\omega_0. \tag{4}$$

Solving for ω_0 and substituting in Eq. (3), one gets

$$F_{c0}(r) = m\left(\ddot{r} - \frac{J_0^2}{m^2 r^3}\right). \tag{5}$$

In the case of a circular orbit of radius r_0, $\ddot{r} = 0$ and Eq. (5) reduces to

$$F_{c0}(r_0) = -\frac{J_0^2}{mr_0^3}. \tag{6}$$

If the particle is now slightly perturbed in the plane of its orbit and perpendicularly to its initial trajectory, it will oscillate around r_0 [14, 17]. Then, one introduces $x = r - r_0$ and expresses the radial equation of motion in terms of x. Therefore [14, 17]

$$\begin{aligned} F_{c0}\left(x + r_0\right) &= m\ddot{x} - \tfrac{J_0^2}{m(x+r_0)^3} \\ &= m\ddot{x} - \tfrac{J_0^2}{mr_0^3\left(1+\frac{x}{r_0}\right)^3}. \end{aligned} \tag{7}$$

Since $\frac{x}{r_0} \ll 1$, one can use series expansion for the term in parentheses, considering only the first order terms in $\frac{x}{r_0}$. Expanding the member on the left in Taylor series around the point $r = r_0$ one gets [14, 17]

$$F_{c0}\left(r_0\right) + F'_{c0}\left(r_0\right) x = m\ddot{x} - \frac{J_0^2}{\mu r_0^3}\left(1 - \frac{3x}{r_0}\right), \tag{8}$$

where prime means derivative with respect to x. Inserting Eq. (6) in Eq. (8) one obtains [14, 17]

$$\ddot{x} + m^{-1}\left[-\frac{3F_{c0}\left(r_0\right)}{r_0} - F'_{c0}\left(r_0\right)\right] x = 0 \tag{9}$$

One notes that this equation describes a simple harmonic oscillator if the term in parentheses is positive [14, 17]. If this term was negative, there would be an exponential solution and the orbit would not be stable [14, 17]. Thus, for stable orbits, the period of oscillation around $r = r_0$ is equal to the corresponding of circular motion [14, 17]

$$T_0 = 2\pi\left(\frac{m}{-\frac{3F_{c0}(r_0)}{r_0} - F'_{c0}\left(r_0\right)}\right)^{\frac{1}{2}}. \tag{10}$$

One defines the apse angle $\frac{\varphi_0}{2}$ as the angle swept by the radial vector between two consecutive points of the orbit where the radial vector itself takes on an extremal value [14, 17]. The time that the planet needs to travel this angle is $\frac{T_0}{2}$. Since the orbit can be considered approximately circular and being therefore constant r and equal to r_0, one solves Eq. (4) for ω_0 and finds [14, 17]

$$\frac{\varphi_0}{2} = \frac{T_0}{2}\omega_0 = \pi\left(\frac{m}{-\frac{3F_{c0}(r_0)}{r_0} - F'_{c0}\left(r_0\right)}\right)^{\frac{1}{2}} \frac{J_0}{\mu r_0^2}. \tag{11}$$

Furthermore, observing Eq. (6), one notes that the last factor of Eq. (11) can be rewritten as [14, 17]

$$\frac{J_0}{mr_0^2} = \left(-\frac{F_{c0}\left(r_0\right)}{\mu r_0}\right)^{\frac{1}{2}}. \tag{12}$$

Then, one gets [14, 17]

$$\varphi_0 = 2\pi\left[3 + \frac{F'_{c0}\left(r_0\right)}{F_{c0}\left(r_0\right)} r_0\right]^{-\frac{1}{2}}, \tag{13}$$

and, by setting $F_{c0} = F_G$ in Eq. (13), where F_G is the Newtonian gravitational force given by ($\hat{u}_r$ is the versor in the radial direction)

$$\overrightarrow{F}_G = -\frac{GMm}{r_0^2}\hat{u}_r, \tag{14}$$

one finds $\varphi_0 = 2\pi$.

On the other hand, the above computation has been implicitly made in an inertial reference frame where the Kepler problem of a central force works. But we set the origin of the frame of reference in the center of the Sun and the motion of the Sun with respect to the planet is not inertial, because the Sun is subjected to the planet's back reaction due to Newton's third law. For an external inertial Newtonian observer, the Newtonian equations of motion for the Sun and the planet are

$$Ma_s\hat{u}_r = \frac{GMm}{r_0^2}\hat{u}_r \Longrightarrow a_s\hat{u}_r = \frac{Gm}{r_0^2}\hat{u}_r \tag{15}$$

and

$$ma_p\hat{u}_r = -\frac{GMm}{r_0^2}\hat{u}_r \Longrightarrow a_p\hat{u}_r = -\frac{GM}{r_0^2}\hat{u}_r, \tag{16}$$

respectively, where a_s is the acceleration of the Sun and a_p is the acceleration of the planet. Thus, the equation of the relative acceleration between the planet and the Sun is

$$a\hat{u}_r = a_p\hat{u}_r - a_s\hat{u}_r = -\left(\frac{GM}{r_0^2} + \frac{Gm}{r_0^2}\right)\hat{u}_r = -\frac{G\left(M+m\right)}{r_0^2}\hat{u}_r. \tag{17}$$

Then, the equation of motion for the planet becomes

$$\overrightarrow{F}_{M\rightarrow m} = -\frac{G\left(M+m\right)m}{r_0^2}\hat{u}_r. \tag{18}$$

Hence, one can consider the weak force

$$-\frac{Gmm}{r_0^2}\hat{u}_r \tag{19}$$

due to the the non-inertial behavior of the Sun as a perturbation with respect to the central force (14). Following [14], in order to take into account the perturbation one has to make the following replacements in Eqs. from (3) to (10):

$$F_{c0}(r) \to F_c(r) = \left(1 + \frac{m}{M}\right) F_{c0}(r), \tag{20}$$

where now $F_c(r)$ is given by the force defined in Eq. (18),

$$\omega_0 \to \omega, \tag{21}$$

where now ω is the corresponding perturbed angular velocity, and

$$J_0 \to J = mr^2\omega. \tag{22}$$

In particular, Eq. (10) is now replaced by

$$\begin{aligned} T &= 2\pi \left(\frac{m}{-\frac{3F_c(r_0)}{r_0} - F_c'(r_0)} \right)^{\frac{1}{2}} \\ &= 2\pi \left(\frac{m}{3 + \frac{F_c'(r_0) r_0}{F_c(r_0)}} \right)^{\frac{1}{2}} \left(\frac{1}{-F_c(r_0)/r_0} \right)^{\frac{1}{2}} \end{aligned} \tag{23}$$

One notes that it is

$$F_c'(r_0) = \left(1 + \frac{m}{M}\right) F_{c0}'(r_0), \tag{24}$$

and consequently by using (20) one obtains

$$\frac{F_c'(r_0)}{F_c(r_0)} = \frac{F_{c0}'(r_0)}{F_{c0}(r_0)}. \tag{25}$$

Therefore, Eq, (23) becomes:

$$\begin{aligned} T &= 2\pi \left(\frac{m}{3 + \frac{F_{c0}'(r_0) r_0}{F_{c0}(r_0)}} \right)^{\frac{1}{2}} \left(\frac{1}{-F_{c0}(r_0)/r_0} \right)^{\frac{1}{2}} \frac{1}{\left(1 + \frac{m}{M}\right)^{\frac{1}{2}}} \\ &= 2\pi \left(1 + \frac{m}{M}\right)^{-\frac{1}{2}} \left(\frac{m}{-\frac{3F_{c0}(r_0)}{r_0} - F_{c0}'(r_0)} \right)^{\frac{1}{2}}. \end{aligned} \tag{26}$$

Thus, by confronting Eqs. (26) and (10), one immediately finds

$$T = \frac{1}{\left(1 + \frac{m}{M}\right)^{\frac{1}{2}}} T_0, \tag{27}$$

which implies

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{T_0} \left(1 + \frac{m}{M}\right)^{\frac{1}{2}} = \omega_0 \left(1 + \frac{m}{M}\right)^{\frac{1}{2}}, \tag{28}$$

and

$$\varphi = \omega T_0 = 2\pi \left(1 + \frac{m}{M}\right)^{\frac{1}{2}} \simeq 2\pi \left(1 + \frac{m}{2M}\right), \tag{29}$$

in radians per revolution, where in the last step the first-order approximation in $\frac{m}{M}$ has been used, that is $\left(1+\frac{m}{M}\right)^{\frac{1}{2}} \simeq 1+\frac{m}{2M}$, because it is $m \ll M$. Then, in each complete revolution around the Sun, the planet sweeps an angle larger than the unperturbed angle $\varphi_0 = 2\pi$, and the difference, in radians per revolution, is

$$\triangle\varphi = \varphi - \varphi_0 \simeq \frac{\pi m}{M}. \tag{30}$$

Thus, the precession of the planet's orbit occurs in the non inertrial reference frame of the Sun because, compared to the non-inertial reference frame of the Sun, the planet moves faster than the inertial reference frame.

4 Breakdown of the conservation of the Hamilton vector in the non-inertial reference frame of the Sun

The key point is that, differently from what happens in an inertial reference frame, in the non-inertial reference frame of the Sun the two planet and the Sun does not interact only by the Newtonian central force of Eq. (14), which is

$$\overrightarrow{F}_G = -\frac{GMm}{r^2}\hat{u}_r. \tag{31}$$

Instead, one must consider also the additional inertial force of Eq. (19) which is

$$-\frac{Gmm}{r^2}\hat{u}_r \tag{32}$$

Then, the total force is given by the force of Eq. (18) which is

$$\overrightarrow{F}_{M\to m} = -\frac{GMm}{r^2}\hat{u}_r - \frac{Gmm}{r^2}\hat{u}_r. \tag{33}$$

This is the well known case in which an additional central force is present in the interaction between the two bodies [18, 19], but, in our knowledge, till now nobody argued that this argument can be used in order to find planets' precession in Newtonian theory. Following [18], we will use not the standard Laplace-Runge-Lenz (LRL) vector which is traditionally used in this kind of analyses [18]

$$\vec{A} \equiv \vec{v} \times \vec{L} - GMm\hat{u}_r, \tag{34}$$

where $\vec{L}$ is the angular momentum vector and $\vec{v}$ is the relative velocity of the planet with respect to the Sun, but its less known cousin, the Hamilton vector [18, 20–23]

$$\vec{u} \equiv \vec{v} - \frac{GMm}{L}\hat{u}_\varphi, \tag{35}$$

where φ is the polar angle in the orbit plane. In [18] the Authors stressed that this very useful vector constant of motion of the Kepler problem was well known in the past, but mysteriously disappeared from textbooks after the first decade of the twentieth century [18, 20–23]. One has to recall that $\vec{A}$ and $\vec{u}$ are not independent constants of motion. There is indeed the following relation between them [18]

$$\vec{A} \equiv \vec{u} \times \vec{L}. \tag{36}$$

The magnitude of the LRL vector is [18]

$$A = GMme, \tag{37}$$

where e is the eccentricity of the orbit. Thus, one gets the magnitude of the Hamilton vector as [18]

$$u = \frac{A}{L}. \tag{38}$$

The total potential coresponding to the force of Eq. (33) is

$$U(r) = -\frac{GMm}{r} - \frac{Gmm}{r}, \tag{39}$$

which corresponds to the sum of the "traditional" Newtonian potential $-\frac{GMm}{r}$ and the additional potential

$$V(r) = -\frac{Gmm}{r} \tag{40}$$

due to the inertial force acting on the planet. In other words, the total potential $U(r)$ contains a small central-force perturbation $-\frac{Gm}{r}$ besides the Newtonian binding potential [18]. In this case the Hamilton vector (as well as the LRL) is no more conserved and begins to precess with the same rate as the LRL vector [18]. Eq. (36) guarantees indeed that the two vectors must be perpendicular. In order to calculate the precession rate of the Hamilton vector one starts to find its time derivative as [18]

$$\frac{d\vec{u}}{dt} = -\frac{1}{m}\frac{dV(r)}{dr}\hat{u}_r, \tag{41}$$

and, in order to obtain Eq. (41), one uses the Newtonian equation of motion for $\frac{d\vec{v}}{dt}$ and the equation [18]

$$\frac{d\hat{u}_\varphi}{dt} = -\frac{d\varphi}{dt}\hat{u}_r. \tag{42}$$

Then, one finds the precession rate of the vector $\vec{u}$ as [18, 24]

$$\vec{\omega} = \frac{\vec{u} \times \frac{d\vec{u}}{dt}}{u^2}. \tag{43}$$

Eqs. (35), (41) and (43) imply that only the tangential component $r\frac{d\varphi}{dt}$ of the velocity vector

$$\vec{v} = r\frac{d\varphi}{dt}\hat{u}_\varphi + \frac{dr}{dt}\hat{u}_r \tag{44}$$

contributes. Hence, one finds [18]

$$\begin{aligned}\vec{\omega} &= \tfrac{1}{mu^2}\left(r\tfrac{d\varphi}{dt}-\tfrac{GMm}{L}\right)\tfrac{dV(r)}{dr}\hat{u}_r\times\hat{u}_\varphi\\ &= \tfrac{L^2}{m(GMme)^2}\left(r\tfrac{d\varphi}{dt}-\tfrac{GMm}{L}\right)\tfrac{dV(r)}{dr}\,\hat{u}_z,\end{aligned} \tag{45}$$

where $\hat{u}_z$ is the versor in the $z-$direction. Thus, in each complete revolution around the Sun, the Hamilton vector and, in turn, the perihelion of the orbit, increases of an angle [18]

$$\triangle\varphi=\int_0^{T_0}\omega dt=\int_0^{2\pi}\frac{L^2}{m\left(GMme\right)^2}\left(r-\frac{GMm}{L\frac{d\varphi}{dt}}\right)\frac{dV(r)}{dr}d\varphi, \tag{46}$$

where T_0 and 2π are the unperturbed period and the unperturbed angle in each revolution around the Sun. Setting [18]

$$p\equiv\frac{L^2}{m\left(GMm\right)}$$

one gets

$$\triangle\varphi=\frac{p^2}{GMm^2e^2}\int_0^{2\pi}\left(r-\frac{GMm}{L\frac{d\varphi}{dt}}\right)\frac{dV(r)}{dr}d\varphi. \tag{47}$$

It is also $L=mr^2\frac{d\varphi}{dt}$. Thus, one obtains [18]

$$\frac{GMm}{L\frac{d\varphi}{dt}}=\frac{r^2}{p}.$$

As the perturbative potential is very weak with respect to the Sun's central field, one can use the unperturbed orbit equation

$$\frac{p}{r}=1+e\cos\varphi. \tag{48}$$

Thus,

$$\begin{aligned}\triangle\varphi &\simeq \tfrac{p^2}{GMme}\int_0^{2\pi}\tfrac{\cos\varphi}{(1+e\cos\varphi)^2}\tfrac{dV(r)}{dr}d\varphi=\\ &=\tfrac{m}{Me}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\varphi d\varphi-\tfrac{m}{Me}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos\varphi d\varphi=\tfrac{4m}{Me}.\end{aligned} \tag{49}$$

The reason of splitting the integral

$$\frac{p^2}{GMme}\int_0^{2\pi}\frac{\cos\varphi}{\left(1+e\cos\varphi\right)^2}\frac{dV(r)}{dr}d\varphi$$

in Eq. (49) in the two parts

$$\frac{m}{Me}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\varphi d\varphi$$

and

$$-\frac{m}{Me}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos\varphi d\varphi$$

is the following. The vector $\frac{dV(r)}{dr}\hat{u}_r$, where $\hat{u}_r$ is the unit vector in the radial direction, has the same direction of $\hat{u}_r$. But the direction of $\hat{u}_r$ for $-\frac{\pi}{2}\leq\varphi<\frac{\pi}{2}$ is opposite with respect to the direction of $\hat{u}_r$ for $\frac{\pi}{2}\leq\varphi<\frac{3\pi}{2}$. This means that if one assumes to be positive the sign of $\frac{dV(r)}{dr}$ for $-\frac{\pi}{2}\leq\varphi<\frac{\pi}{2}$, then one must assume as being negative the sign of $\frac{dV(r)}{dr}$ for $\frac{\pi}{2}\leq\varphi<\frac{3\pi}{2}$. One notes that the result of Eq. (49) has the same order of magnitude of the result of Eq. (30), and this cannot be a coincidence. The small difference between the two results is due to the fact that two different approximations were used to obtain them. The first in Section 1 was the circular orbit approximation. The second in this Section is the use of the unperturbed orbit equation. In any case, the existence of a precession of the planets' orbits in Newtonian gravity is confirmed.

However, neither the first nor the second result agrees with the observations in the Solar System. Let us apply Eq. (30) to Mercury. The NASA official data are $m\simeq 3.3*10^{23}Kg$ [12] and $M=1,99*10^{30}Kg$ [11]. Thus, one finds $\triangle\varphi\simeq 5.21*10^{-7}$radians per revolution. This corresponds to about $0,107$ arcseconds. The NASA official data also give the Mercury/Earth ratio of the tropical orbit periods as being 0.241 [13]. Hence, one finds a value of $44.39''$ per tropical century. This result shows a value of the contribution of Newtonian gravity to the advance of the perihelion of Mercury per tropical century which well approximates both the observational value of $43''$ per tropical century and the value of about $42,98''$ per tropical century of the general theory of relativity [9]. This is a mere coincidence. Let us apply Eq. (30) to the trajectory of Venus. The planet's mass is $m_V\simeq 4.87*10^{24}Kg$ in the NASA official data [25]. This gives a value of $\triangle\varphi\simeq 7.68*10^{-6}$ radians per revolution corresponding to about 1.6 arcseconds. The Venus /Earth ratio of the tropical orbit periods results to be $0,615$ [25]. Then, one finds $258,16''$ per tropical century. This results 30 times larger than the value of $8.62''$ which results from the observations [14, 26]. If one considers Earth's data one finds analogous results. The Earth's mass is $m\simeq 5.97*10^{24}Kg$. This gives a value of $\triangle\varphi\simeq 9.42*10^{-6}$ radians per revolution for the precession, corresponding to about 1.94 arcseconds. The value becomes $194''$ per tropical century, being about 50 times larger than the value of $3.83''$ which results from the observations [14, 26]. On the other hand, one gets

$$\frac{4m}{Me}>\frac{\pi m}{M} \tag{50}$$

for all the planets in the Solar System. This means that if the result of Eq. (30) is not consistent with the observations in the Solar System also the result of Eq. (49) cannot be consistent with the observations in the Solar System

Thus, one confirms the result in [14] that, differently from a longstanding conviction which is older than 160 years, the real problem of the Newtonian framework which concerns the anomalous rate of the precession of the perihelion of planets orbit is not the absence of a result. Instead, the real problem with Newtonian gravity is that such a result is too large.

5 Conclusion remarks

The recent result in [14] showed that, contrary to a belief of over 160 years, the precession of planets' orbits exists in Newtonian gravity if ones correctly analyzes the situation without neglecting the mass of the planet. On the other hand, the predicted Newtonian result was too large with respect to the observational values, despite it was, coincidently, in good accordance with the observational value of the precession of Mercury's orbit. In this new paper the situation has been reanalyzed in Newtonian gravity. The result has been that, despite the orbit's precession does not occur when the reference frame of the Sun is approximated as being fixed with respect to the fixed stars, it occurs, instead, in the real (Newtonian) non-inertial reference frame of the Sun and it is due to the well known fact that, in Newtonian theory, the distance which is travelled by a body depends on the reference frame in which the motion of the body is analyzed. The first step has been reviewing the solution of the problem which analyzes the planet's orbit as a harmonic oscillator. Then, it has been found that the precession is due to the breakdown of the conservation of the Hamilton vector in the non-inertial reference frame of the Sun. This approach gives a value of the same order of magnitude of the result in [14], but larger than that one and again without consistency with the observational values. In other words, the main result in this paper is that it is not correct that Newtonian theory cannot predict an anomalous rate of precession of planets' orbits. The real problem is instead that a pure Newtonian prediction is too large to be consistent with the observational values.

6 Acknowledgements

The Author thanks Domenico Annunziata for stimulating discussions on the possibility to solve the anomalous rate of precession of Mercury in a Newtonian framework.

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