

# Trace anomaly Redefined in a Convention Leading to the Pontryagin Resolution

Hani W. Maalouf Physics Department Lebanese University

Faculty of Science II Fanar Lebanon

## Abstract:

It is sought a fundamental reason for the then fore contradictory or randomly correct outcomes. As due the dilemma borne out from the multiple methods for the Pontryagin Chiral Fermions density anomalies resolutions and their either zero or imaginary results.

Such is based on extracting the unitarity from a unity-scale diffeomorphic transformation, which is resulted from being a regularization of  $\gamma^0$ .

## §1- Introduction

Recalling the conventional definition of the trace anomaly, via functional derivatives of the energy-momentum tensor in a field with respect to the metric, being such

$$T_{\mu\nu}(x) = 2/\sqrt{|g|} \frac{dS}{dg^{\mu\nu}}.$$

That would follow after the introduction of the conformal transformation  $g_{\mu\nu} \rightarrow e^{2\sigma(x)} g_{\mu\nu}$ .

For its inverse, with an infinitesimal value of the parameter  $\sigma(x): g^{\mu\nu} \rightarrow [1 - 2\sigma(x)]g^{\mu\nu}$ ,

$$\delta S = \frac{1}{2} \int d^4x \sqrt{|g|} T_{\mu\nu} \delta g^{\mu\nu} = - \int d^4x \sqrt{|g|} \sigma(x) T_{\mu}^{\mu} \quad (1)$$

Then, and for an arbitrary  $\sigma(x)$ , the invariance of  $S$  under the above conformal transformation

requires that the trace of the energy-momentum tensor has to be  $T_{\mu}^{\mu} = 0$ .

This so far classical traceless identity is broken by quantum effects beyond tree and on shell levels, such

$$0 \neq \langle T_{\mu}^{\mu} \rangle \equiv A \quad (1)'$$

Where the defined quantity  $A$  is called the trace or conformal anomaly, [1].

On dimensional grounds and in four-dimension, the most general form for the trace anomaly was found to be, [2],

$$\langle T_{\mu}^{\mu} \rangle = aG + bR^2 + b'\square R + cF + e\epsilon^{\alpha\beta\gamma\delta}R_{\alpha\beta\mu\nu}R_{\gamma\delta}^{\mu\nu} \quad (2)$$

Where the Gauss-Bonnet term  $G = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} + 4R^{\alpha\beta}R_{\alpha\beta} + R^2$  yields the Euler invariant, and

$F = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 2R^{\alpha\beta}R_{\alpha\beta} + \frac{1}{3}R^2$  is the square of the Weyl tensor.

The last term is the Parity-odd Pontryagin density, [3].

In the case of chiral fermions being added to the system, due to its Parity-odd symmetry properties, and at that lowest composition level, only the Pontryagin term contributes to the anomaly that is then why it is labeled as of type I.

## § 2- A Missing Unitary Mapping

### 1- A Non-Anomalous Diffeomorphic Action Leading to Unitarity

It is proposed here a resolution to that anomaly that however appears to encounter a dilemma which originates as is shown below from the degeneracy of its eigenvalues in the decomposition.

One is then looking for the symmetries which contain the above variation that caused such an anomaly. And as the considered action is gravitational, one has to distinguish between flat and curved space-time metrics, [4].

For Flat space-time, with a scaling  $x \rightarrow x^c = e^{\sigma}x$ , so then for an arbitrarily chosen wave function

$\phi(x) \rightarrow \phi^c(x) = e^{\sigma\Delta}\phi(x)$ , with  $\Delta$  being a canonical mass dimension brought by

Conformal  $\supset$  Scale + Rotations + Boosts + Translations

That gets  $\sigma(x)$  as a function of the quadri-norm of the vector  $x$ .

For Curved Space-time.

Conformal: Same as above when reduced infinitesimally into flat space-time which under the existence of the Killing invariants.

Weyl : Besides the functional transformation, there should be a metric transformation such that

$$g_{\mu\nu}^c(x) = e^{2\sigma} g_{\mu\nu}(x)$$

Note here that the above diffeomorphism  $x \rightarrow x^c$  map will not cause any anomaly since it remains at the classical level, and that is easily seen from its exact Lorentzian pairing up with the metric.

But at the quantum level, it may be picked up some phases which may be too eliminated by remarking that such phases can be paired up as opposite under degenerate eigenvalues.

However now, acting in plus by the derivative on both the left and the right sides makes it in need of regularization since these derivatives have to pass through the chiral composition which is known to be made from effective operators so the need for regularizations starting from  $g_{\mu\nu}(x) \equiv$

$$\frac{\partial y_\mu}{\partial x^\nu} \text{ and ending with } T_{\mu\nu}(x) = 2/\sqrt{|g|} \frac{\delta S}{\delta g^{\mu\nu}}.$$

Due that the result is

$$0 \neq \langle T_\mu^\mu \rangle = \langle g^{\mu\nu} T_{\mu\nu} \rangle \neq g^{\mu\nu} \langle T_{\mu\nu} \rangle \quad (2)'$$

It has to be eliminated the common eigenvalues as since are due to diffeomorphism.

That can be done either by subtracting, [5, 6], following what was adopted by [2], as

$$g^{\mu\nu} \langle T_{\mu\nu} \rangle - \langle g^{\mu\nu} T_{\mu\nu} \rangle \equiv A_{reg}.$$

Or by 'diagonalizing' in a fully non-degenerate space.

Which is simpler, and clarifies its subtleties. So that what is done and for that purpose the degeneracy here is unconventionally non-trivial.

### § 3- The Regularization of $\gamma^0$

#### a- Saving Chirality and Unitarity

The degeneracy in the metric space is treated in next, see also Appendix.

It is based on finding the bounding condition for any metric to be unitary and non-degenerate. That would be on the 4-norm of a rescaling of  $g$ .

What is needed is a Jacobian for a common scaling variation to both of  $x$  and  $y$  in

$$g_{\mu\nu} = \frac{\partial y_\mu}{\partial x^\nu} \rightarrow f(y_\mu) f^{-1}(x^\nu) \frac{\partial y_\mu}{\partial x^\nu} = f^2 g_{ij} + f^2 (g_{0j} - g_{i0}) - f^2 g_{00} \quad (3a)$$

So a metric with  $F \equiv f^2$ , while being fully symmetric can be deduced for two sides scaling as

$$ds'^2 = F dt^2 + F dx^2 = F ds^2 \quad (3b)$$

Now, the above form of the metric is encountered as a solution for the orbital variation along a metric in a Gauss-Bonnet Gravity, [7].

Then, in the case the eigenvalues are searched, such a configuration leads merely to an operator's acting change along a modular form orbifold (or in its simplest form a torus).

Since the norm of  $f$  is less or equal to one that keeps overall infinitely acting operators convergent.

Plus, since in the above map only the zero components get opposed by sign, so there is a temporal twist (or a negative spatial twist). To break that degeneracy (of the metric), it is sufficient to vary, here a rescaling on the one side-coordinate supposed to be the zero one.

After absorbing  $F$  then being rescaled by a negative  $-\lambda$  as

$$F ds^2 \rightarrow -\lambda dt'^2 + dx^2 \rightarrow [(-\lambda - 1) + 1] dt'^2 + dx^2 \rightarrow \left(-1 + \frac{1}{\lambda+1}\right) dt'^2 + \frac{1}{\lambda+1} dx^2$$

As  $F \leq 1$  was arbitrary,  $\frac{1}{\lambda+1} \leq 1$  can be re-identified with  $F$ , so one got a dispersive map as

$$ds'^2 = Fds^2 + Fdt^2 \rightarrow ds''^2 = -(1-F)dt^2 + Fdx^2 \quad (3b)'$$

So, as was already used above as a property of elimination for the redundant eigenvalues, but with the supposed broken space, one has then to use the Jacobian not of  $f$ , but of

$$\Delta = \begin{vmatrix} [\varepsilon_t(f^2 - 1)]^{\frac{1}{2}} & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{vmatrix} \neq 0 \quad (4)$$

The coefficient  $\varepsilon_t = \pm$  is considered to express in the case of correlations between two space-time structures, existent in close neighborhoods the need of any possible extra time twists.

As proved in the Appendix the case  $\Delta = 1$  is the case of Unitary operators however with  $\varepsilon_t = -1$ .

And an operator  $\mathcal{T}^{3\text{-space}}(F, F^*, g_{ii}) \mapsto \frac{1 \pm i\sqrt{3}}{2} \frac{g_{ii}}{\sqrt{3}}$ .

If the problem is considered for the 4-spinors of Dirac, the action is given

$$S = \int \sqrt{g} \bar{\Psi} i \overset{/}{\tilde{D}} \Psi \Rightarrow W = -i \log \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{iS} = -i \log \det \sqrt{g} i \overset{/}{\tilde{D}} \quad (5a)$$

Standard diagonalization in compact manifolds or any unitarily equivalent manifold

$$\sqrt{g} i \overset{/}{\tilde{D}} \Psi_n = \lambda_n \Psi_n \Rightarrow \det \sqrt{g} i \overset{/}{\tilde{D}} = \prod_n \lambda_n \quad (5a)'$$

The use of the operator  $\mathcal{T}^{3\text{-space}}$  will have the impact of pairing the eigenvalues between  $F$  and space-wise inversion  $F^*$ .

So if  $\Psi \rightarrow \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$ . The 2<sup>nd</sup> space is that of the right chirality however with opposite eigenvalues.

Then, by Inverting the Time, the Parity Becomes Odd (5b)

Therefore, the procedure conserves the Dirac character for the spinors and it is specifically a plain regularization for its zero index Gamma matrix.

That logic can be confirmed by another means using the Atiyah-Singer index theorem for the Polyakov strings under the Liouville action, [11]. More theoretical justifications and also their direct implications are developed in the next subparagraph.

One can proceed into any of the usual regularizations and whose original result was advent by, [8].

That refutes the claim of Ref. [4], that the regularization of the Dirac fermions has no P-odd terms.

As since the problem in their case originated from the fact that  $W$  was ill defined in  $(1/2, 0)$  spin space as it goes to  $(0, 1/2)$ , and it was sufficient to regularize  $\Psi_L$  and  $W = -i \log \det i \sigma \cdot D$  to lead into  $\delta W^{Weyl}$ .

Also, the imposition of the Unitarity as an external condition is not necessary as claimed by [5].

### **b- $\gamma^0$ Regularization**

Then, as due to that in the Weyl representation  $\bar{\Psi}$  contains  $\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ , the above diffeomorphic rescaling is merely a regularization of  $\gamma^0$  in Weyl spaces, as can be verified.

In fact, when changing the representation from the Weyl to the Dirac-Pauli one,  $\gamma^0$  changes

into  $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$  which has spontaneously the minus sign for the adjoint spinors.

However, any dimensional operator including the non-chiral Dirac-Pauli in the Lagrangian has its space as doubled as well its added Hermitian conjugate, so the definite regularizing diffeomorphism will have being acted by twice, then only and only one minus sign remains.

Note about the regularization in non-gravity environment as it uses the Ward identity, [12], that instead involves functional integral forms with matrix elements so can be handled also as path integrals. The matrices can act on vertices and propagators.

The vertices if spinorial contain their spin polarizations which are acted on then trivially.

The propagators are non-trivial when neither Lorentzian nor Euclidean; a way which was suggested in the Wicked path integral, [13], as they are

$$\Delta(P) = \frac{i}{E^2 - p^2 - m^2 + i\epsilon} \cong \frac{i}{E^2(1 + i\epsilon) - p^2 - m^2}$$

With the not quite Minkowskian metric is  $\eta_\epsilon = (1 + i\epsilon, 1, 1, 1)$ , whose determinant is  $\sqrt{-\det\eta_\epsilon} = \sqrt{1 + i\epsilon}$ . That becomes for,  $\epsilon = 0, \eta_\epsilon \rightarrow \eta_L$  as Lorentzian and  $\epsilon = 2i, \eta_\epsilon \rightarrow \eta_E$  as Euclidean.

This method does not shift any of the above identities except when the operators are gravitational and defined on metrics with signatures somewhere between  $\eta_E$  and  $\eta_L$ .

One then has to deal with the side-shifted correlators, then shifts back via the unitary operator  $\mathcal{T}$  regularization.

### c- Theoretical Justifications and Consequences

To fully justify the above regularization, one has to find if the global domain of definition for the driven from Dirac spinors remains well-behaved.

What is concerned here and are seen from the whole spectrum:

Does the Hilbert space remain well defined.

That is true under the developed down conditions and in the Appendix.

Plus is there a change in the duality property, or what remains self-dual and what defies it.

Here, the trick applies as low in dimension as the 1+1 string models which was found to be self

dual with the permutation operator  $\epsilon^{i_1 \dots i_n}$  is becoming complex, [14], which under the canonical

gauging, [15], claimed to be associated with the Color charge remains self-dual, except however not

under space doubling since that leads to the pop of the Real Orthogonal Flavor group indices, [16].

What is noticed here is the similarity in the complex-real transition between their breaking self-duality and ours in breaking the corresponding determinants. Supposedly then from unitary into composite-symmetric or -unitary operators, famously eliminating triangular flavor anomalies. More clearly while remaining in the complex structure, that would be, and since the base space of start is finite and that has led to the self-duality, then (as that is the statement negation) that would be for its operators being made as adjoint and in an infinite base space as also noticed in the cited references, so any iteration would be redundant and odd then the outcome set tends to be fractal as mentioned in the Appendix.

#### **Appendix:**

Starting from the most elementary change, with  $\lambda \equiv \frac{1-F}{F}$ , is

$$ds'^2 = ds'^2 - dt^2 = \varepsilon(F - 1)d\tau^2 + Fd\chi^2 \quad (\text{A1a})$$

An isotropifying map operation with the time component, is done with  $(3 - \sigma)$  –space-like directions. Then, that mapping,  $s(t, x) \rightarrow s''(\tau = \varepsilon t, \chi = x)$  is rendered a  $(4 - \sigma)$  dimensional vector, but with any an additional acting as by an inversion operator  $\varepsilon = \pm$  as  $\varepsilon(t', x'_{\text{iso}})$ , is done as true all along on one side of the scalar product so via a vierbein sandwiching non-trivially only the anti-symmetric permutations in the spectral representation so the double derivative action would be proportional to the representation itself.

Mathematically, this is an affine form for the Friedrichs extension, [9], conserving then any Unitarity if proved existent. That is through its re-defined Hilbert spaces, and as e.g. those implicitly dealt with, alike the module-kink-cusp links exposed in [10].

By kerneling these modular forms it leads into a Jacobian, non-zero positive and bounded by one.



So defining  $F = f^2$ ,

$$0 \neq \begin{vmatrix} [\varepsilon_t(f^2 - 1)]^{\frac{1}{2}} & 0 & 0 \\ 0 & f & 0 \\ 0 & \dots & 0 \\ 0 & 0 & f \end{vmatrix} = [\varepsilon_t(f^2 - 1)]^{\frac{1}{2}} f^{3-\sigma} \leq 1 \Leftrightarrow 0 \neq \varepsilon_t(f^2 - 1)f^2(f^{2-\sigma})^2 \leq 1 \quad (A1b)$$

In solving such a system, there should exist a compact set where that isotropy can be applied. Also,

$$0 \neq \varepsilon_t(f^2 - 1)f^2(f^{2-\sigma})^2 \Rightarrow f \neq 0 \text{ And } f \neq 1$$

While  $f = 0$  is the trivial identity transformation,  $f = 1$  generates a specific co-dimension where the isotropy is broken.

Resolving, then, for transverse isotropy, i.e. by setting  $\sigma = 2$  so  $x \equiv r$  is a transverse symmetric orbital length,

$$\varepsilon_t(F - 1)F = 1 \Leftrightarrow \varepsilon_t F^2 - \varepsilon_t F - 1 = 0 \Leftrightarrow \omega^2 = 1 + 4\varepsilon_t \Rightarrow F = \frac{1 \pm \sqrt{1 + 4\varepsilon_t}}{2} \quad (A2)$$

The case  $\varepsilon_t = 1$  leads to the solution  $F = \frac{1 \pm \sqrt{5}}{2}$  which is the Golden Ratio representing the

emergence of Fractals within the system. While the case  $\varepsilon_t = -1$ , leads to the solution  $F = \frac{1 \pm i\sqrt{3}}{2}$  as

it is associated with the time reversal.

The operation of orbital skipping can be applied as a scaling product with its conjugate-like such

$$-FF'^* = -\frac{1-3}{4} = \frac{1}{2} \text{ A Counter to Pile-ups} \quad (A2)'$$

This upper limit result, has to be extracted to represent an eigenvalues' transfer from a 3-space

isotropic 2-disk into a 3-sphere rolling, view it is normalized such that  $\mathcal{T}^{3\text{-space}}(F, F^*) \propto$

$$\frac{Fg_{ii}}{\sqrt{3}} \text{ or } \frac{F^*g_{ii}}{\sqrt{3}}.$$

So however the regularizations impose what is being equivalent to a time ordering  $T$  for  $F$  and  $F^*$  as

$$\mathcal{T}(F_i, F_i^*) \propto \frac{T(F_i, F_i^*)}{\sqrt{3}} = \frac{F}{\sqrt{3}} \text{ if } t < t^* \text{ and } \frac{F^*}{\sqrt{3}} \text{ if } t > t^* \quad (A3a)$$

And, now, whose imaginary satisfies the Unitarity semi-equation at an internal loop variable time

$$-2Im\mathcal{T} = \mathcal{T}\mathcal{T}^* \Leftrightarrow -2Im[\sum_{i=1}^3 \mathcal{T}(F_i, F_i^*)] = \sum_{i=1}^3 \mathcal{T}(F_i, F_i^*)\mathcal{T}^*(F_i, F_i^*) + \sum_{i \neq j}^3 \mathcal{T}(F_i, F_i^*)\mathcal{T}^*(F_j, F_j^*),$$

But,  $i \neq j$  has a unique ordering so the minus sign showing in Eq. (6a) and due to the time reversal

does not show again so  $\mathcal{T}(F_i, F_i^*)\mathcal{T}^*(F_j, F_j^*) = -\mathcal{T}(F_j, F_j^*)\mathcal{T}^*(F_i, F_i^*)$ . Also,

$$\sum_{i=1}^3 \mathcal{T}(F_i, F_i^*)\mathcal{T}^*(F_i, F_i^*) \propto 3 \frac{F}{\sqrt{3}} \frac{F^*}{\sqrt{3}} + 3 \frac{F^*}{\sqrt{3}} \frac{F}{\sqrt{3}} = -\frac{1}{2} - \frac{1}{2} = -1 \text{ And } -2Im[\sum_{i=1}^3 \mathcal{T}(F_i, F_i^*)] = -1 \quad (A3b)$$

The sphere map acting on the 3-metric via  $F$  and  $F^*$ , is thus proved an equality

$$\mathcal{T}^{3\text{-space}}(F, F^*, g_{ii}) = \frac{T(F_i, F_i^*)g_{ii}}{\sqrt{3}}$$

Therefore, the operation satisfy the Optical Theorem and is Unitary.

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