Qeios

Trace anomaly Redefined in a Convention Leading to the Pontryagin Resolution

Hani W. Maalouf Physics Department Lebanese University

Faculty of Science II Fanar Lebanon

Abstract:

It is sought a fundamental reason for the then contradictory or randomly correct outcomes, as due to the dilemma borne out from the multiple methods for the Pontryagin Chiral Fermions density anomalies resolutions and their either zero or imaginary results.

Such is based on extracting the unitarity from a unity-scale diffeomorphic transformation, which resulted in the being of a γ^0 regularization.

§1- Introduction

Recalling the conventional definition of the trace anomaly, via the energy-momentum tensor in the field being such

$T_{\mu\nu}(x) = 2/\sqrt{|g|}\frac{\delta S}{\delta g^{\mu\nu}}$.

That would follow after the introduction of the conformal transformation $g_{\mu\nu} \rightarrow e^{2\sigma(x)}g_{\mu\nu}$.

For its inverse with an infinitesimal value of the parameter $\sigma(x)$: $g^{\mu\nu} \rightarrow [1-2\sigma(x)]g^{\mu\nu}$,

$$\delta S = \frac{1}{2}\int d^4x\,\sqrt{|g|}T_{\mu\nu}\delta g^{\mu\nu} = -\int d^4x\,\sqrt{|g|}\sigma(x)T^{\mu}_{\mu} \qquad (1)$$

Then, and for an arbitrary $\sigma(x)$, the invariance of S under the above conformal transformation requires that the trace of the energy-momentum tensor be $T^{\mu}_{\mu} = 0$.

https://doi.org/10.32388/NGTNRS

This so far classical traceless identity is broken beyond tree and on-shell levels by quantum effects, such

$$0 \neq \langle T_\mu^\mu \rangle \equiv A \qquad (1)'$$

Where the defined quantity A is called the trace or conformal anomaly, [1].

On dimensional grounds and in four dimensions, the most general form for the trace anomaly was found to be, [2],

$$\langle T_\mu^\mu \rangle = aG + bR^2 + b'\Box R + cF + e\epsilon^{\alpha\beta\gamma\delta} R_{\alpha\beta\mu\nu} R_{\gamma\delta}^{\mu\nu} \qquad (2)$$

Where the Gauss-Bonnet term $G = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} + 4R^{\alpha\beta}R_{\alpha\beta} + R^2$ yields the Euler invariant.

$F = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 2R^{\alpha\beta}R_{\alpha\beta} + \frac{1}{3}R^2$ is the square of the Weyl tensor.

The last term is the Parity-odd Pontryagin density, [3].

In the case of chiral fermions being added to the system, due to the Parity property, only the Pontryagin term contributes to the anomaly.

§ 2- A Missing Unitary Mapping

1- A Non-Anomalous Diffeomorphic Action Leading to Unitarity

It is proposed here a resolution to that anomaly that appears to encounter a dilemma which originates, as is shown below, from the degeneracy of its eigenvalues in the decomposition.

One is then looking for the symmetries that contain the above variation that caused such an anomaly. And if the action is gravitational, one has to distinguish between flat and curved space-time metrics, [4].

For flat space-time, scale $x \to x' = e^\sigma x$

So, for an arbitrarily chosen wave function $\phi(x) \to \phi'(x) = e^{\sigma\Delta}\phi(x)$

With Δ being a canonical mass dimension, plus Conformal $\sqsupset$ Scale + Rotations + Boosts +Translations. Then, one gets $\sigma(x)$ as a function of the quadri-norm of the vector x.

For Curved Space-time:

Conformal: Same as above when reduced infinitesimally to flat space-time, which under the existence of the Killing invariants.

Weyl: Besides the functional transformation, there should be a metric transformation such that

$$g'_{\mu\nu}(x) = e^{2\sigma} g_{\mu\nu}(x)$$

Note here that the diffeomorphism $x \to x'$ map will not cause any extra anomaly besides that of the metric as well as those of the Lorentzian, since it remains at the classical level. However, since at the quantum level they may pick up a phase which can be eliminated by remarking that such phases can be paired up as opposite with a degenerate eigenvalue.

Also, acting by the derivative on both the left and the right sides makes it in need of regularization since these derivatives have to pass through the chiral composition which is known to be made from effective operators, so the need for regularizations starting from $g_{\mu\nu}(x) \equiv \frac{\partial y_\mu}{\partial x^\nu}$ and ending with $T_{\mu\nu}(x) = 2/\sqrt{|g|}\frac{\delta S}{\delta g^{\mu\nu}}$.

Due to that the result

$$0 \neq \langle T^{\mu}_{\mu} \rangle = \langle g^{\mu\nu} T_{\mu\nu} \rangle \neq g^{\mu\nu} \langle T_{\mu\nu} \rangle \qquad (2)'$$

Then, it has to be eliminated the common eigenvalues as since they are due to diffeomorphism. That can be done either by subtracting, [5,6], following what was adopted by [2], as $g^{\mu\nu}\langle T_{\mu\nu}\rangle - \langle g^{\mu\nu} T_{\mu\nu}\rangle \equiv A_{reg}$.

Or by 'diagonalizing' in a fully non-degenerate space.

Which is simpler and clarifies its subtilities. So that is what is done, and for that purpose, the degeneracy here is unconventionally non-trivial.

2- The Regularization of γ^0

a- Saving Chirality and Unitarity

The degeneracy in the metric space is treated in next; see also Appendix.

It is based on finding the bounding condition for any metric to be unitary and non-degenerate. That would be on the 4-norm of a rescaling of g.

What is needed is a Jacobian for a common scaling variation to both x and y in

$$g_{\mu\nu} = \frac{\partial y_\mu}{\partial x^\nu} \rightarrow f(y_\mu) f^{-1}(x^\nu) \frac{\partial y_\mu}{\partial x^\nu} = f^2 g_{ij} + f^2 (g_{0j} - g_{i0}) - f^2 g_{00} \tag{3a}$$

so the metric, $F \equiv f^2$,

$$ds'^2 = Fdt^2 + Fdx^2 = Fds^2 \tag{3b}$$

Now, the above form of the metric is encountered as a solution for the orbital variation along a metric in Gauss-Bonnet Gravity.

Then, in the case the eigenvalues are searched, such a configuration leads merely to a change operator acting along a modular form orbifold (or, in its simplest form, a torus).

Since the norm of f is less or equal to one to keep overall infinitely acting operators convergent, plus since in the above map only the zero components get opposed by sign, so there is a temporal twist (or a negative spatial twist).

To break the degeneracy of the metric, it is sufficient to perturb F on one coordinate, supposed to be the zero one; such a dispersion can be rescaled as $Fds^2 \rightarrow Fds^2 - dt^2$

$$ds"^2 = -(1 - F)dt^2 + Fdx^2 \tag{3b)'}$$

So, as was already used above as a property of elimination of the redundant eigenvalues, but with the supposed broken space, one has then to use the Jacobian not of f, but of

$$\Delta= \begin{vmatrix} [\varepsilon_t(f^2-1)]^{\frac{1}{2}} & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{vmatrix} \neq 0 \qquad (4)$$

The coefficient ε_t is considered to express, in the case of correlations between two space-time points, existent in close neighborhoods, the need for a possible one-time twist.

As proved in the Appendix, the case $\Delta= 1$ the case of unitary operators; however, with $\varepsilon_t = -1$.

And an operator $\mathrm{T}^{3-\mathrm{space}}(\mathrm{F}, \mathrm{F}^*, g_{ii}) = \frac{g_{ii}}{\sqrt{3}}$.

If the problem is considered for the 4-spinors of Dirac, the action is given

$$S = \int \sqrt{g}\bar{\Psi} i \overset{/}{\widetilde{D}} \Psi \Rightarrow W = -ilog \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\, e^{iS} = -ilogdet\sqrt{g} i \overset{/}{\widetilde{D}} \qquad (5a)$$

Standard diagonalization in compact manifolds or any unitarily equivalent manifold

$$\sqrt{g} i \overset{/}{\widetilde{D}} \Psi_n = \lambda_n \Psi_n \Rightarrow det\sqrt{g} i \overset{/}{\widetilde{D}} = \prod_n \lambda_n \qquad (5a)'$$

The use of the operator $\mathrm{T}^{3-\mathrm{space}}$ will have the impact of pairing the eigenvalues between F and space-wise inversion F^*.

So if $\Psi \rightarrow \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$, the 2nd space is that of the right chirality, however with opposite eigenvalues.

Then, by inverting the time, the Parity becomes odd (5b)

Therefore, the procedure conserves the Dirac character for the spinors, and it is specifically a plain regularization for its zero index Gamma matrix.

That logic can be confirmed by another means using the Atiyah-Singer index theorem for the Polyakov strings under the Liouville action, [9]. More theoretical justifications and also their direct implications are developed in the next subparagraph.

One can proceed into any of the usual regularizations, and so the result of deWitt is correct. That refutes the claim of Ref. [4] that the regularization of the Dirac fermions has no P-odd terms. As the problem in their case originated from the fact that W was ill defined in (1/2,0) spin space as it goes to (0,1/2), it was sufficient to regularize Ψ_L and $W = -ilogdeti\sigma.D$ to lead into δW^{Weyl} .

Then, and that is due to the fact that in the Weyl representation $\overline{\Psi}$ contains $\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, the above diffeomorphic rescaling is merely a regularization of γ^0 in Weyl spaces.

Also, the imposition of the unitarity as an external condition is not necessary, as claimed by [5].

b- Theoretical Justifications and Consequences

To fully justify the above regularization, which in fact acts on the Dirac matrices, one has to find if the global domain of definition for the spinors driven from Dirac spinors remains well-behaved.

What is concerned here and is seen from the whole spectrum:

Does the Hilbert space remain well defined?

That is true under the conditions developed below and in the appendix.

Plus, is there a change in the duality property, or what remains self-dual and what defies it?

Here, the trick applies as low in dimension as the 1+1 string models, which was found to be self-dual with the permutation operator $\epsilon^{i_1 \cdots i_n}$, is becoming complex, [10], which under the canonical gauging, [11], claimed to be associated with the color charge, remains self-dual, except, however, not under space doubling since that leads to the pop of the real orthogonal flavor group indices, [12].

What is noticed here is the similarity in the complex-real transition between their breaking self-duality and ours in breaking the corresponding determinants. Supposedly, then, from unitary into composite-symmetric or unitary operators, famously eliminating triangular flavor anomalies. More clearly, while remaining in the complex structure (that is, since the base space of the start is finite and that has led to the self-duality), then (as that is the statement negation), that would be for its operators to be made adjoint and in an infinite base space, as also noticed in the cited references, so any iteration would be redundant and odd, and the outcome set tends to be fractal, as mentioned in the Appendix.

Appendix:

Starting from the most elementary change,

$$ds'^2 = ds^2 - dt^2 = (F-1)dt^2 + Fdx^2 \qquad \text{(A1a)}$$

An isotropifying map operation with the time component is done with $(3-\sigma)$ −space-like directions. Then, that mapping, $s(t,x) \rightarrow s'(t' = t, x')$, is rendered a $(4-\sigma)$–dimensional vector, but with any an additional acting as by an inversion operator $\varepsilon = \pm$ as $\varepsilon(t, x'_{iso})$, is done as true all along on one side of the scalar product, so via a vierbein sandwiching non-trivially only the anti-symmetric permutations in the spectral representation, the double derivative action would be proportional to the representation itself.

Mathematically, this is an affine form for the Friedrichs extension, [7], conserving then any unitarity if proved existent. That is through its re-defined Hilbert spaces, and as, e.g., those implicitly dealt with, alike the module-kink-cusp links exposed in [8].

By kerneling these modular forms, one leads to a Jacobian non-zero, positive, and bounded by one.

So defining $F = f^2$,

$$0 \neq \begin{vmatrix} [\varepsilon_t(f^2-1)]^{\frac{1}{2}} & 0 & 0 \\ 0 & f & \\ 0 & & \begin{matrix} \cdots & 0 \\ 0 & f \end{matrix} \end{vmatrix} = [\varepsilon_t(f^2-1)]^{\frac{1}{2}} f^{3-\sigma} \leq 1 \Leftrightarrow 0 \neq \varepsilon_t(f^2-1)f^2\left(f^{2-\sigma}\right)^2 \leq 1 \qquad \text{(A1b)}$$

In solving such a system, there should exist a compact set where isotropy can be applied. Also,

$$0 \neq \varepsilon_t(\mathrm{f}^2-1)\mathrm{f}^2\left(\mathrm{f}^{2-\sigma}\right)^2 \Rightarrow \mathrm{f} \neq 0 \text{ And } \mathrm{f} \neq 1$$

While $f = 0$ is the trivial identity transformation, $f = 1$ generates a specific co-dimension where the isotropy is broken.

Resolving, then, for transverse isotropy, i.e., by setting $\sigma = 2$ so $x \equiv r$ is a transverse symmetric orbital length,

$$\varepsilon_t(F-1)F = 1 \Leftrightarrow \varepsilon_t F^2 - \varepsilon_t F - 1 = 0 \Leftrightarrow \omega^2 = 1 + 4\varepsilon_t \Longrightarrow F = \frac{1 \pm \sqrt{1+4\varepsilon_t}}{2} \qquad \text{(A2)}$$

The case $\varepsilon_t = 1$ leads to the solution $F = \frac{1\pm\sqrt{5}}{2}$ which is the Golden Ratio, representing the emergence of Fractals within the system. While the case $\varepsilon_t = -1$, leads to the solution $F = \frac{1\pm i\sqrt{3}}{2}$ as it is associated with the time reversal.

The operation of orbital skipping can be applied as a scaling product with its conjugate-like counterpart,

$$-FF'^{*} = -\frac{1-3}{4} = \frac{1}{2} \text{ A Counter to Pile-ups} \qquad \text{(A2)'}$$

This upper limit result has to be extracted to represent an eigenvalues' transfer from a 3-space isotropic 2-disk into 3-sphere rolling; view it as normalized such that $\mathcal{T}^{3-\mathrm{space}}(\mathrm{F}, \mathrm{F}^*) \propto \frac{Fg_{ii}}{\sqrt{3}}$ or $\frac{\mathrm{F}^* g_{ii}}{\sqrt{3}}$

.

So, however, the regularizations impose what is equivalent to a time ordering T for F and F^* as

$$\mathcal{T}(\mathrm{F}_i, \mathrm{F}_i^*) \propto \frac{T(F_i, \mathrm{F}_i^*)}{\sqrt{3}} = \frac{F}{\sqrt{3}} \; if \; t < t^* \; and \; \frac{F^*}{\sqrt{3}} \; if \; t > t^* \qquad \text{(A3a)}$$

And now, whose imaginary satisfies the unitarity semi-equation at an internal loop variable time

$$-2Im\mathcal{T} = \mathcal{T}\mathcal{T}^* \Leftrightarrow -2Im[\textstyle\sum_{i=1}^{3} \mathcal{T}(\mathrm{F}_i, \mathrm{F}_i^*)] = \sum_{i=1}^{3} \mathcal{T}(\mathrm{F}_i, \mathrm{F}_i^*)\mathcal{T}^*(\mathrm{F}_i, \mathrm{F}_i^*) + \sum_{i\neq j}^{3} \mathcal{T}(\mathrm{F}_i, \mathrm{F}_i^*)\mathcal{T}^*(\mathrm{F}_j, \mathrm{F}_j^*),$$

But, $i \neq j$ has a unique ordering so the minus sign showing in Eq. (6a) and due to the time reversal does not show again so $\mathcal{T}(\mathrm{F}_i, \mathrm{F}_i^*)\mathcal{T}^*(\mathrm{F}_j, \mathrm{F}_j^*) = -\mathcal{T}(\mathrm{F}_j, \mathrm{F}_j^*)\mathcal{T}^*(\mathrm{F}_i, \mathrm{F}_i^*)$. Also,

$$\textstyle\sum_{i=1}^{3} \mathcal{T}(\mathrm{F}_i, \mathrm{F}_i^*)\mathcal{T}^*(\mathrm{F}_i, \mathrm{F}_i^*) \propto 3\frac{\mathrm{F}}{\sqrt{3}}\frac{\mathrm{F}\prime^*}{\sqrt{3}} + 3\frac{\mathrm{F}\prime^*}{\sqrt{3}}\frac{\mathrm{F}}{\sqrt{3}} = -\frac{1}{2} - \frac{1}{2} = -1 \;\; \text{And} \; -2Im[\sum_{i=1}^{3} \mathcal{T}(\mathrm{F}_i, \mathrm{F}_i^*)] = -1 \quad \text{(A3b)}$$

The sphere map, acting on the 3-metric as a unitary operator via F and F*, is thus proved to be an equality

$$\mathcal{T}^{3-\text{space}}(\mathrm{F}, \mathrm{F}^*, g_{ii}) = \frac{T(F_i, \mathrm{F}_i^*)g_{ii}}{\sqrt{3}}$$

References:

1- L. Bonora, P. Pasti and M. Tonin J. Math. Phys. 27, 2259 (1986)

2- M. J. Duff, Class. Quant. Grav. 11, 1387-1404, hep-th/9308075

3- Y. Nakayama, Nucl. Phys. B 859, 288-298 (2012) hep-th/1201.3428

4- R. Larue, J. Quevillon, R. Zwicky JHEP 12 (2023) 064, hep-th/2309.08670

5- L. Bonora, S. Giaccari and B. Lima de Souza JHEP 07, 117 (2014) hep-th/1403.2606

6- Chang-Yong Liu Nucl. Phys. B 980 (2021) 115840

7- N.I. Akhiezer and I.M. Glazman, Theory of Linear Operators in Hilbert Space, Pitman, 1981

8- H.W. Maalouf Int. J. Mod. Phys. A Vol. 36, No. 10, 2150039 (2021)

9- C. Itzykson and J-M. Drouffe Statistical Firld Theory, Cambridge p. 787-781

10- H. Leutwyler Phys. Letters Vol. 153B. n 1,2 1985

11- W. Siegel Nucl. Phys. B238 (1984) 307

12- M. Ademollo, L. Brink, A. D'Adda, R. D'Auria, E. Napolitano and S. Siuto, E. Del Giudice, P. Di Vechhia, S. Ferrara, F. Gliozzi, R. Musto and R. Pettorino Nucl. Phys. B114 (1976) 297-316