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On the Essence of the Riemann Zeta Function and Riemann Hypothesis

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Abstract

Riemann's functional equation $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\left(\frac{1}{2}-\frac{s}{2}\right)}\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\zeta(1-s)$ is valid on the vertical line $s = 1/2 + it$. Each side is a real-valued function. The Riemann's Xi function is also a real-valued function along the vertical line of $s = 1/2 + it$. Through the holomorphic extensions of the Riemann zeta function, starting from the real-valued function at $s = 1/2 + it$ into the both sides of $\sigma < 1/2$ and $\sigma > 1/2$, we can get two versions of the zeta functional equation, eq. (45). The key property of the scaling and rotational factors g(s) and g(1-s) behave as multiplicative inverses in the complex plane, eq. (48). It is deduced that the Zeta function also has multiplicative inverses, the symmetric point is at (1/2,0) in the complex plane. The moduli behave as a hyperbola. Especially, along the vertical line $\sigma = 1/2 + it$, the amplitudes of both function g(s) and g(1-s) are equal to 1, its arguments have opposite signs. If $\sigma \neq 1/2$, the amplitudes of $\zeta(s)$ and $\zeta(1-s)$ are not equal to each other, because of their multiplicative inversion relationship. It is deduced that the non-trivial zeros can only be on the vertical line of $s = 1/2 + it$. A gamma function vector field is given in Appendix B, and some moduli of gamma function at special points are given. Finally, another variation of the Zeta function is provided in an integral form in Appendix D. The asymptotes behave as a c_8 cyclic group for the large t values.

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1. Euler's product and zeta function are reciprocal relationship

The Riemann zeta function is defined by the following infinite series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \cdots \qquad (1)$$

The Euler product formula and the Riemann zeta function have the following relation:

$$\dots\left(1 - \frac{1}{11^s}\right)\cdot\left(1 - \frac{1}{7^s}\right)\cdot\left(1 - \frac{1}{5^s}\right)\cdot\left(1 - \frac{1}{3^s}\right)\cdot\left(1 - \frac{1}{2^s}\right)\cdot\zeta(s) = 1 \qquad (2)$$

This can be written more concisely as an infinite product over all primes *p*:

$$\prod_{\underbrace{}_{prime}} (1 - p^{-s}) \cdot \zeta(s) = 1 \qquad (3)$$

This equation shows that Euler's product and zeta function have a reciprocal relationship.

If we plot $\prod_{\underbrace{}_{prime}} (1 - p^{-s})$ on the y-axis and $\zeta(s)$ on the x-axis, the curve will be a hyperbola with asymptotes along the axes. This hyperbola is symmetric with respect to the line y=x.

Dividing both sides by everything but the $\zeta(s)$ we obtain:

$$\zeta(s) = \prod_{\underbrace{}_{prime}} (1 - p^{-s})^{-1} = \prod_{\underbrace{}_{prime}} \left(\frac{1}{1 - p^{-s}}\right) = f(p^s) \qquad (4)$$

The Euler product formula is an alternative expression of the Riemann ζ function in terms of prime numbers. We define it as a function of $f(p^s)$.

Originally the function was defined for real arguments of $s = \sigma$. It is convergent when σ is greater than 1. It was Riemann who extended the real-valued function to be a complex function with a complex variable $s = \sigma + it$ rather than a real variable σ. Through the extension onto the entire complex plane $\mathbb{C}$, the Riemann zeta function is now expressed as:

$$\zeta(\sigma + it\ ) = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma+it}} = 1 + \frac{1}{2^{\sigma+it}} + \frac{1}{3^{\sigma+it}} + \frac{1}{4^{\sigma+it}} + \frac{1}{5^{\sigma+it}} + \cdots \quad (5)$$

2. Fourier series and vector dot product expressions

Each term in the eq. (5) can be expressed as:

$$n^{-\mathrm{s}} = n^{-\sigma-it} = n^{-\sigma} \cdot (\cos(t\omega_n) - i \cdot \sin(t\omega_n)) \quad (6)$$

where $\omega_n = \ln(n)$.

The complex conjugate of $n^{-\sigma-it}$ is:

$$\overline{n^{-\mathrm{s}}} = n^{-\sigma+it} = n^{-\sigma} \cdot (\cos(t\omega_n) + i \cdot \sin(t\omega_n)) \quad (7)$$

Combining the Complex Power Function and its Conjugate

$$n^{-\mathrm{s}} + \overline{n^{-\mathrm{s}}} = n^{-\sigma+it} = n^{-\sigma} \cdot 2\cos(t\omega_n) \quad (8)$$

Dividing by $2\cos(t\omega_n)$:

$$n^{-\sigma} = \frac{n^{-s}}{2\cos(t\omega_n)} + \frac{\overline{n^{-s}}}{2\cos(t\omega_n)} \tag{9}$$

With the help of the following relations:

$$\begin{cases} n^{-s} = n^{-\sigma-it} = n^{-\sigma} \cdot e^{-i(t\omega_n)} \\ \overline{n^{-s}} = n^{-\sigma+it} = n^{-\sigma} \cdot e^{i(t\omega_n)} \end{cases} \tag{10}$$

Hence, the Riemann zeta function for real arguments can be expressed as:

$$f(p^{\sigma}) = \zeta(\sigma\,) = \sum_{n=1}^{\infty} \frac{n^{-\sigma}}{2\cos(\omega_n t)} e^{-i\omega_n t} + \sum_{n=1}^{\infty} \frac{n^{-\sigma}}{2\cos(t\omega_n)} e^{i\omega_n t} \tag{11}$$

where, $t\omega_n \neq k\frac{\pi}{2}$, k is the positive integers, $k = 1,2,3,\cdots$.

We can also express the eq. (11) more concisely as

$$2f(p^{\sigma}) = 2\sum_{n=1}^{\infty} n^{-\sigma} = \sum_{n=-\infty}^{\infty} \frac{|n|^{-\sigma}}{\cos(\omega_n t)} e^{i\omega_n t} \tag{12}$$

When $t\omega_n = \frac{\pi}{2}$, we have

$$e^{i\frac{\pi}{2}} = i; \quad e^{-i\frac{\pi}{2}} = -i \tag{13}$$

Equation (10) can be expressed as:

$$\begin{cases} n^{-s} = n^{-\sigma - i\frac{\pi}{2}} = -n^{-\sigma} \cdot i \\ \overline{n^{-s}} = n^{-\sigma + i\frac{\pi}{2}} = n^{\sigma} \cdot i \end{cases} \tag{14}$$

Namely:

$$n^{s} + \overline{n^{-s}} = 0 \tag{15}$$

Thus, the eq. (11) or (12) still holds.

With the Euler formula

$$2\cos(\omega_n t) = e^{i\omega_n t} + e^{-i\omega_n t} \tag{16}$$

eq. (11) can also be rewritten as:

$$f(p^{\sigma}) = \sum_{n=1}^{\infty} \left(\frac{n^{-\sigma}}{e^{i\omega_n t} + e^{-i\omega_n t}} \right) e^{-i\omega_n t} + \sum_{n=1}^{\infty} \left(\frac{n^{-\sigma}}{e^{i\omega_n t} + e^{-i\omega_n t}} \right) e^{i\omega_n t} \tag{17}$$

If we define a basis vector for angular frequencies

$$\vec{e} = [\ldots \quad e^{-i\omega_2 t} \quad e^{-i\omega_1 t} \quad 0 \quad e^{i\omega_1 t} \quad e^{i\omega_2 t} \quad \ldots] \tag{18}$$

and an amplitude coefficient vector:

$$\vec{c} = \left[\cdots \quad \frac{2^{-\sigma}}{2\cos(t\omega_2)} \quad \frac{1}{2} \quad 0 \quad \frac{1}{2} \quad \frac{2^{-\sigma}}{2\cos(t\omega_2)} \quad \cdots \right] \tag{19}$$

Hence, for real arguments, eq. (11) is the Fourier series for Euler product formula with complex Fourier coefficients:

$$f(p^\sigma) = \vec{c} \cdot \vec{e} = \sum_{-\infty}^{\infty} c_n e^{i\omega_n t} = \sum_{n=1}^{\infty} \left[c_n e^{i\omega_n t} + c_0 + c_{-n} e^{-i\omega_n t} \right] \qquad (20)$$

It should be paid attention that the function $f(p^\sigma)$ is a real-valued function. Accordingly, the coefficients for positive and negative angular frequency flow are conjugated with each other.

3. Physical Interpretations

The series of powers of natural numbers with real arguments, defined by eq. (11), is convergent, if the real argument is greater than one, $\sigma > 1$.

Physically, it can be imagined as two rotational particles with opposite rotational signs, namely, one particle is rotating clockwise and another one anticlockwise. If we view the two particles as a whole system, then the total rotational momentum (the sum of the rotational momentum of the two particles) will be zero. But the total rotational energy is $2f(p^\sigma)$, the greater the parameter, σ, is, the smaller the total energy will be.

Initially, the two particles are located at the coordinate origin, they are rotating in an over-damping field, this over-damping field finally causes the entangled oscillators to return to equilibrium without oscillating. Fig. 1 gives an example of $\sigma = 2$. Oscillators move slowly toward the equilibrium state.

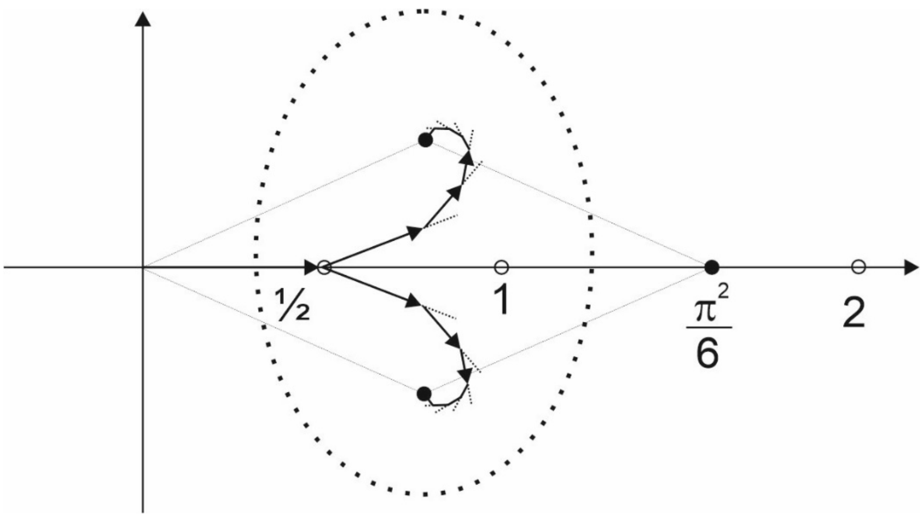

Fig. 1. Initially entangled two particles $(\sigma = 2)$ is viewed as a system: total rotational energy is $2f(p^2) = \frac{\pi^2}{6}$. After the decay the total vorticity=0

4. Riemann's functional equation holds at $Re(s) = \sigma = \frac{1}{2}$

One of Riemann's functional equations is

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\left(\frac{1}{2}-\frac{s}{2}\right)}\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\zeta(1-s) \quad (21)$$

Recalling the infinite series definition of the Riemann zeta function of eq. (1), this equation can be expressed explicitly as

$$\sum\left(\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\cdot n^{-\sigma}\right)e^{-i\cdot\omega_n t} = \sum\left(\pi^{-\left(\frac{1}{2}-\frac{s}{2}\right)}\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\cdot n^{\sigma-1}\right)e^{i\cdot\omega_n t} \quad (22)$$

In this case, the amplitude vectors for negative and positive angular frequencies are:

$$\overrightarrow{c_{-n}}(s) = \left[\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) \cdot n^{-\sigma}\right]; \quad \overrightarrow{c_n}(s) = \left[\pi^{-\left(\frac{1}{2}-\frac{s}{2}\right)}\Gamma\left(\frac{1}{2}-\frac{s}{2}\right) \cdot n^{\sigma-1}\right] \tag{23}$$

and the angular frequency vectors are:

$$\overrightarrow{e_{-n}} = \left[e^{-i\cdot\omega_n t}\right]; \quad \overrightarrow{e_n} = \left[e^{i\cdot\omega_n t}\right] \tag{24}$$

Hence, eq. (22) can be expressed more compactly as:

$$\sum_{n=1}^{\infty} \overrightarrow{c_{-n}} \cdot e^{-i\cdot\omega_n t} = \sum_{n=1}^{\infty} \overrightarrow{c_n} \cdot e^{i\cdot\omega_n t} \tag{25}$$

We define a complex function as:

$$\Phi(s) = \pi^{-s}\Gamma(s) \tag{26}$$

Hence, the amplitude vectors of eq. (23) can be rewritten as:

$$\overrightarrow{c_{-n}}(s) = \left[\Phi\left(\frac{s}{2}\right) \cdot n^{-\sigma}\right]; \quad \overrightarrow{c_n}(s) = \left[\Phi\left(\frac{1}{2}-\frac{s}{2}\right) \cdot n^{\sigma-1}\right] \tag{27}$$

Both sides of eq. (25) are infinite series, each term must be equal.

$$\Phi\left(\frac{s}{2}\right) \cdot n^{-\sigma} \cdot e^{-i\cdot\omega_n t} = \Phi\left(\frac{1}{2}-\frac{s}{2}\right) \cdot n^{\sigma-1} \cdot e^{i\cdot\omega_n t} \tag{28}$$

Namely:

$$\Phi\left(\frac{\sigma}{2}+i\frac{t}{2}\right) \cdot n^{-\sigma} \cdot e^{-i\cdot\omega_n t} = \Phi\left(\frac{1}{2}-\frac{\sigma}{2}-i\frac{t}{2}\right) \cdot n^{\sigma-1} \cdot e^{i\cdot\omega_n t} \tag{29}$$

Isolating the function of $\Phi\left(\frac{s}{2}\right)$:

$$\Phi\left(\frac{\sigma}{2}+i\frac{t}{2}\right)=\Phi\left(\frac{1}{2}-\frac{\sigma}{2}-i\frac{t}{2}\right)\cdot n^{2\sigma-1}\cdot e^{i\cdot 2\omega_n t} \tag{30}$$

The magnitudes of both sides must be equal. Therefore

$$\left|\Phi\left(\frac{\sigma}{2}+i\frac{t}{2}\right)\right|=\left|\Phi\left(\frac{1}{2}-\frac{\sigma}{2}-i\frac{t}{2}\right)\right|\cdot n^{2\sigma-1} \tag{31}$$

The amplitude of the RHS of eq. (31) multiplies a scaling factor of $n^{2\sigma-1}$. For amplitudes to match with the $n^{2\sigma-1}$ factor, we should consider special values of σ:

$$n^{2\sigma-1}=1 \tag{32}$$

Hence:

$$\sigma=\frac{1}{2} \tag{33}$$

That is, given $\sigma=\frac{1}{2}$, the amplitude conditions for both sides of the infinite series, eq. (25) are satisfied.

The arguments of the complex functions are related by $2\omega_n t$ modulo 2π:

$$arg\left[\Phi\left(\frac{\sigma}{2}+i\frac{t}{2}\right)\right]=arg\left|\Phi\left(\frac{1}{2}-\frac{\sigma}{2}-i\frac{t}{2}\right)\right|+2\omega_n t+2k\pi \tag{34}$$

That means, for the infinite series, eq. (25) to hold, the complex variable s must have its real part σ equal to $\sigma=\frac{1}{2}$. Thus, s can be written as:

$$s = \frac{1}{2} + it \tag{35}$$

Namely, the real part of s is fixed at 1/2, and the imaginary part t can vary.

Given these conditions, the Fourier coefficients can be written explicitly:

$$\overrightarrow{c_{-n}} = \left[\Phi\left(\frac{1}{4} + i\frac{t}{2}\right) \cdot n^{-\frac{1}{2}}\right]; \quad \overrightarrow{c_n} = \left[\Phi\left(\frac{1}{4} - i\frac{t}{2}\right) \cdot n^{-\frac{1}{2}}\right] \tag{36}$$

Thus, eq. (25) can be expressed explicitly as:

$$\sum_{n=1}^{\infty}\left[\Phi\left(\frac{1}{4} + i\frac{t}{2}\right) \cdot n^{-\frac{1}{2}}\right] \cdot e^{-i\cdot\omega_n t} = \sum_{n=1}^{\infty}\left[\Phi\left(\frac{1}{4} - i\frac{t}{2}\right) \cdot n^{-\frac{1}{2}}\right] \cdot e^{i\cdot\omega_n t} \tag{37}$$

From eq. (37) We know that each term

$$n^{-\frac{1}{2}} = \frac{e^{i\cdot\omega_n t}}{\Phi\left(\frac{1}{4} + i\frac{t}{2}\right)}; \quad n^{-\frac{1}{2}} = \frac{e^{-i\cdot\omega_n t}}{\Phi\left(\frac{1}{4} - i\frac{t}{2}\right)} \tag{38}$$

Adding both terms together:

$$2n^{-\frac{1}{2}} = \frac{e^{-i\cdot\omega_n t}}{\Phi\left(\frac{1}{4} + i\frac{t}{2}\right)} + \frac{e^{i\cdot\omega_n t}}{\Phi\left(\frac{1}{4} - i\frac{t}{2}\right)} \tag{39}$$

Thus, the Fourier expansion of the Euler product formula at $\sigma = \frac{1}{2}$ is

$$2f\left(p^{1/2}\right) = 2\zeta\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{e^{-i\cdot\omega_n t}}{\Phi\left(\frac{1}{4} + i\,\frac{t}{2}\right)} + \sum_{n=1}^{\infty} \frac{e^{i\cdot\omega_n t}}{\Phi\left(\frac{1}{4} - i\,\frac{t}{2}\right)} \quad (40)$$

This is a real-valued function, thus, this equation implies that the absolute values of the arguments of $\Phi\left(\frac{s}{2}\right)$ at $\sigma = \frac{1}{2}$ equal the basis vectors of $e^{i\cdot\omega_n t}$, but with opposite signs.

In other words, equation (40) represents the complex form of the Fourier expansion of the Euler product formula at $\sigma = \frac{1}{2}$. Or we can say that the Riemann zeta function at $\sigma = \frac{1}{2}$ can be expressed as a complex Fourier series. The complex coefficients of the negative and positive angular frequency flow are $\left[\Phi\left(\frac{1}{4} + i\,\frac{t}{2}\right)\right]^{-1}$ and $\left[\Phi\left(\frac{1}{4} - i\,\frac{t}{2}\right)\right]^{-1}$, respectively.

It has been proved by Siegel [1] that at $\sigma = \frac{1}{2}$, the Riemann functional equation (21) is a real-valued function:

$$\Phi\left(\frac{1}{4} + i\,\frac{t}{2}\right)\zeta\left(\frac{1}{2} + it\right) = \Phi\left(\frac{1}{4} - i\,\frac{t}{2}\right)\zeta\left(\frac{1}{2} - it\right) = real - valued \quad (41)$$

Hence, the amplitude vectors can be rewritten as:

$$\vec{c} = \Phi\left(\frac{1}{4} + i\,\frac{t}{2}\right) \cdot n^{-\sigma} \quad (42)$$

and its arguments have an opposite sign of $\zeta\left(\frac{1}{2} + it\right)$, namely have the following form:

$$arg(\vec{c}) = \omega_n t \quad (43)$$

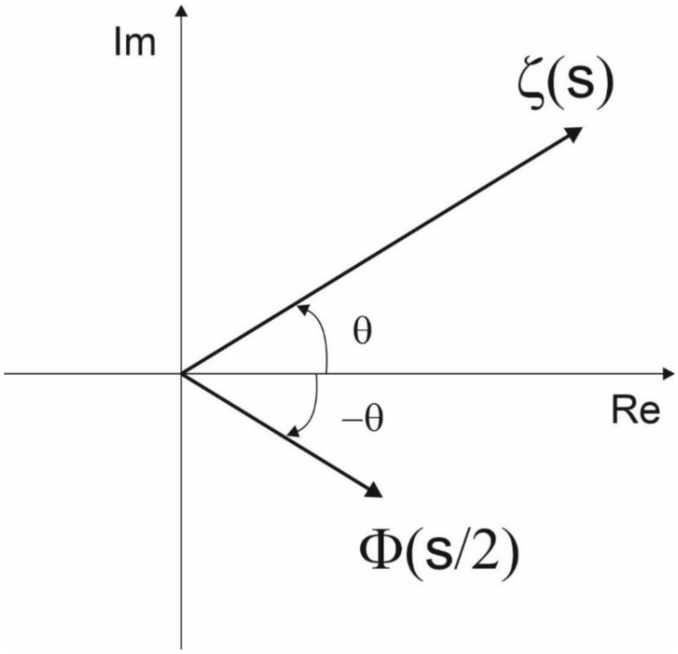

Fig. 2. The absolute values of the arguments of the functions $\Phi(s/2)$ and $\zeta(s)$ equal to each other, but with opposite signs.

Recalling the definition of Riemann's Xi function,

$$\xi(s) = \frac{1}{2}s(1-s)\left[\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)\right] \tag{44}$$

At $\sigma = \frac{1}{2}$, the complex numbers of s and (1-s) are conjugates, hence, in the complex plane, along the line $s = \frac{1}{2} + it$, the Riemann's Xi function of (42) is a real-valued function.

5. Holomorphic extensions of the Riemann zeta function

There are two versions of the functional equation:

$$\begin{cases} \zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) & Re(s) \le 1 \\ \zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s)\zeta(s) & Re(s) \ge 0 \end{cases} \tag{45}$$

Except for two points of s=0 and 1-s=0. Because the gamma function at s=0 has a pole.

Where the gamma function is defined:

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx \quad Re(s) > 0 \tag{46}$$

We define two complex functions:

$$\begin{cases} g(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \\ g(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \end{cases} \tag{47}$$

It can be proved that they behave as multiplicative inverses in complex plane, with one function being the inverse of the other for all points in their domain where both functions are non-zero:

$$g(s) \cdot g(1-s) = 1 \tag{48}$$

It was fully proved in Appendix C, for more details we can apply Appendix C.

Hence, two Riemann zeta functions, eq. (45), can be rewritten as

$$\begin{cases} \zeta(s) = g(s)\zeta(1-s) & Re(s) \le 1 \\ \zeta(1-s) = g(1-s)\zeta(s) & Re(s) \ge 0 \end{cases} \tag{49}$$

If we define the amplitude and argument of $g(s)$ to be r and θ, respectively, because of the complex multiplicative inversion behavior of g(s) and g(1-s) of the eq. (48), thus, the Riemann zeta function in the complex plane can be expressed as:

$$\begin{cases} \zeta(s) = \zeta(1-s) \cdot re^{i\theta_s} \\ \zeta(1-s) = \zeta(s) \cdot \frac{1}{r} e^{-i\theta_s} \end{cases} \tag{50}$$

where θ_s is the argument of $g(s)$ at the point of $s = \sigma + it$.

This implies that in the general case, the amplitudes of $\zeta(s)$ and $\zeta(1-s)$ are not equal to each other, rather, they behave as multiplicative inverses:

$$|\zeta(s)| \cdot |\zeta(1-s)| = 1 \tag{51}$$

If we plot $|\zeta(1-s)|$ on the y-axis and $|\zeta(s)|$ on the x-axis, the curve of the amplitudes will be a hyperbola with asymptotes along the axes. This amplitude hyperbola is symmetric with respect to the line y=x, this is at the location of $\sigma = \frac{1}{2}$.

Especially, in the complex plane, along the vertical line $\sigma = \frac{1}{2} + it$, the amplitudes of g(s) and g(1-s) are equal to each other, and both equal one:

$$\begin{cases} \left|g\left(\frac{1}{2} + it\right)\right| = 1 \\ \left|g\left(\frac{1}{2} - it\right)\right| = 1 \end{cases} \tag{52}$$

They are conjugates, thus, along the line $s = \frac{1}{2} + it$, the Riemann zeta functions, eq. (49), can be expressed as:

$$\begin{cases} \zeta\left(\frac{1}{2}+it\right) = \zeta\left(\frac{1}{2}-it\right)\cdot e^{i\theta_t} \\ \zeta\left(\frac{1}{2}-it\right) = \zeta\left(\frac{1}{2}+it\right)\cdot e^{-i\theta_t} \end{cases} \tag{53}$$

where θ_t is the argument of $g(s)$ at the point of $s = \frac{1}{2} + it$:

$$\theta_t = t \cdot \ln(2\pi) + tan^{-1}\left(tanh\left(\frac{\pi t}{2}\right)\right) + \beta \tag{54}$$

and where β is the argument of gamma function at the points of $\sigma = \frac{1}{2} - it$:

$$\beta = arg\left[\Gamma\left(\frac{1}{2} - it\right)\right] \tag{55}$$

Thus, the operating effect of $g\left(\frac{1}{2}+it\right)$ on the $\zeta\left(\frac{1}{2}-it\right)$ leads the $\zeta\left(\frac{1}{2}-it\right)$ to rotate an angle of θ_t clockwise, while $g\left(\frac{1}{2}-it\right)$ leads the $\zeta\left(\frac{1}{2}+it\right)$ to rotate an angle of θ_t anticlockwise, but the scaling factor of $g\left(\frac{1}{2}+it\right)$ and $g\left(\frac{1}{2}-it\right)$ keeps as constant, both factors are equal to one.

Moreover, if the argument $\theta_t = 0$, we have the following trivial equations:

$$\begin{cases} \zeta\left(\frac{1}{2}+it\right) = \zeta\left(\frac{1}{2}-it\right) \\ \zeta\left(\frac{1}{2}-it\right) = \zeta\left(\frac{1}{2}+it\right) \end{cases} \tag{56}$$

In this case, both equations take the same amplitude and argument simultaneously. The non-trivial zero points of the Riemann zeta function will be located at these points.

Hence, along the vertical line $s = \frac{1}{2} + it$, the amplitudes of $\zeta\left(\frac{1}{2} + it\right)$ and $\zeta\left(\frac{1}{2} - it\right)$ are the same for all cases, and the difference of their arguments is equal to $2\theta_t$, namely, the rotation radii of $\zeta\left(\frac{1}{2} + it\right)$ and $\zeta\left(\frac{1}{2} - it\right)$ equal each other. If one function is rotating and passing through the coordinate origin clockwise, another one must be rotating and passing through the origin anticlockwise, and furthermore, they will be located at the points of $\theta_t = 0$.

In the general case, the amplitudes of $\zeta(s)$ and $\zeta(1-s)$ are not equal to each other, because of their multiplicative inversion relationship, eq.(51). If the rotation radius of one function is bigger, another one must be smaller, hence, both functions cannot pass through the coordinate origin, otherwise, the multiplicative inversion relationship of eq. (51) cannot be held, unless the amplitude, r, of the function $g(s)$ is zero for all cases in the definition domain. But from the definition of the eq. (47), it cannot be equal to zero.

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Appendix A: sin(πs) and cos(πs)

Given a complex variable s in the complex plane:

$$s = \sigma + it \tag{A1}$$

The definition of complex sine function is

$$\sin(\pi s) = \sin(\pi\sigma + i\pi t) = \sin(\pi\sigma)\, cosh(\pi t) + i \cdot \cos(\sigma\pi)\, sinh(\pi t) \tag{A2}$$

With different values of σ, we can get its algebraic expressions, e.g., as following expressions:

Table A1: the most important expressions of sin(πs)	
$\sigma = 0$	$\sin(i\pi t) = i \cdot \sinh(\pi t)$
$\sigma = \frac{1}{4}$	$\sin\left(\frac{\pi}{4} + i\pi t\right) = \frac{1}{\sqrt{2}}[cosh(\pi t) + i \cdot sinh(\pi t)]$
$\sigma = \frac{1}{2}$	$\sin\left(\frac{\pi}{2} + i\pi t\right) = \cosh(\pi t)$
$\sigma = \frac{3}{4}$	$\sin\left(\frac{3\pi}{4} + i\pi t\right) = \frac{1}{\sqrt{2}}[cosh(\pi t) - i \cdot sinh(\pi t)]$
$\sigma = 1$	$\sin(\pi + i \cdot \pi t) = -i \cdot \sinh(\pi t)$

They are symmetric about $\sigma = \frac{1}{2}$.

Accordingly, in general, its Modulus (amplitude) is

$$|\sin(\pi s)| = \sqrt{[\sin(\pi\sigma)\, cosh(\pi t)]^2 + [\cos(\sigma\pi)\, sinh(\pi t)]^2} \tag{A3}$$

And its argument is

$$\theta = tan^{-1}[cot(\sigma\pi)tanh(\pi t)] \quad \text{(A4)}$$

The explicit algebraic expressions for the most important amplitudes between $0 \leq \sigma \leq 1$ are as follows:

Table A2: the most important amplitudes of sin(πs)	
$\sigma = 0$	$\lvert\sin(i \cdot \pi t)\rvert = \sinh(\pi t)$
$\sigma = \frac{1}{6}$	$\left\lvert\sin\left(\frac{\pi}{6} + i\pi t\right)\right\rvert = \frac{1}{2}\sqrt{\cosh^2(\pi t) + 3 \cdot \sinh^2(\pi t)}$
$\sigma = \frac{1}{4}$	$\left\lvert\sin\left(\frac{\pi}{4} + i\pi t\right)\right\rvert = \frac{1}{\sqrt{2}}\sqrt{\cosh(2\pi t)}$
$\sigma = \frac{1}{3}$	$\left\lvert\sin\left(\frac{\pi}{3} + i\pi t\right)\right\rvert = \frac{1}{2}\sqrt{3 \cdot \cosh^2(\pi t) + \sinh^2(\pi t)}$
$\sigma = \frac{1}{2}$	$\left\lvert\sin\left(\frac{\pi}{2} + i\pi t\right)\right\rvert = \cosh(\pi t)$
$\sigma = \frac{2}{3}$	$\left\lvert\sin\left(\frac{2\pi}{3} + i\pi t\right)\right\rvert = \frac{1}{2}\sqrt{3 \cdot \cosh^2(\pi t) + \sinh^2(\pi t)}$
$\sigma = \frac{3}{4}$	$\left\lvert\sin\left(\frac{3\pi}{4} + i\pi t\right)\right\rvert = \frac{1}{\sqrt{2}}\sqrt{\cosh(2\pi t)}$
$\sigma = \frac{5}{6}$	$\left\lvert\sin\left(\frac{5\pi}{6} + i\pi t\right)\right\rvert = \frac{1}{2}\sqrt{\cosh^2(\pi t) + 3 \cdot \sinh^2(\pi t)}$
$\sigma = 1$	$\lvert\sin(\pi + i \cdot \pi t)\rvert = \sinh(\pi t)$

The most important arguments between $0 \leq \sigma \leq 1$ are listed as follows:

Table A3: the most important arguments of sin(πs)	
$\sigma = 0$	$\theta = tan^{-1}[\infty] = \frac{\pi}{2}$
$\sigma = \frac{1}{6}$	$\theta = tan^{-1}\left[\sqrt{3} \cdot tanh(\pi t)\right]$
$\sigma = \frac{1}{4}$	$\theta = tan^{-1}[tanh(\pi t)]$
$\sigma = \frac{1}{3}$	$\theta = tan^{-1}\left[\frac{1}{\sqrt{3}} \cdot tanh(\pi t)\right]$
$\sigma = \frac{1}{2}$	$\theta = tan^{-1}[0] = 0$

$\sigma = \frac{2}{3}$	$\theta = tan^{-1}\left[-\frac{1}{\sqrt{3}} \cdot tanh(\pi t)\right]$
$\sigma = \frac{3}{4}$	$\theta = tan^{-1}[-tanh(\pi t)]$
$\sigma = \frac{5}{6}$	$\theta = tan^{-1}[-\sqrt{3} \cdot tanh(\pi t)]$
$\sigma = 1$	$\theta = tan^{-1}[-\infty] = -\frac{\pi}{2}$

It can be seen that both the amplitudes and arguments are symmetric about $\sigma = \frac{1}{2}$.

When the variable t approaches infinity, the asymptotes of the amplitudes are:

$$|\sin(\sigma\pi + i\pi t)| \approx \sinh(\pi t) \approx \cosh(\pi t) \qquad \text{(A5)}$$

and the asymptotes of the arguments is

$$\theta \approx tan^{-1}[cot(\sigma\pi)] = \frac{\pi}{2} - \sigma\pi \qquad \text{(A6)}$$

Thus, some asymptotes of the arguments between $0 \leq \sigma \leq 1$ are as follows:

Table A4: some asymptotes of the arguments of $sin(\pi s)$	
$\sigma = 0$	$\theta = tan^{-1}[\infty] = \frac{\pi}{2}$
$\sigma = \frac{1}{4}$	$\theta = tan^{-1}\left[cot\left(\frac{\pi}{4}\right)\right] \approx \frac{\pi}{4}$
$\sigma = \frac{1}{2}$	$\theta = tan^{-1}[0] = 0$
$\sigma = \frac{3}{4}$	$\theta = tan^{-1}\left[-cot\left(\frac{\pi}{4}\right)\right] \approx -\frac{\pi}{4}$
$\sigma = 1$	$\theta = tan^{-1}[-\infty] = -\frac{\pi}{2}$

The vector field of $sin(\pi s)$ is given in Fig. A1:

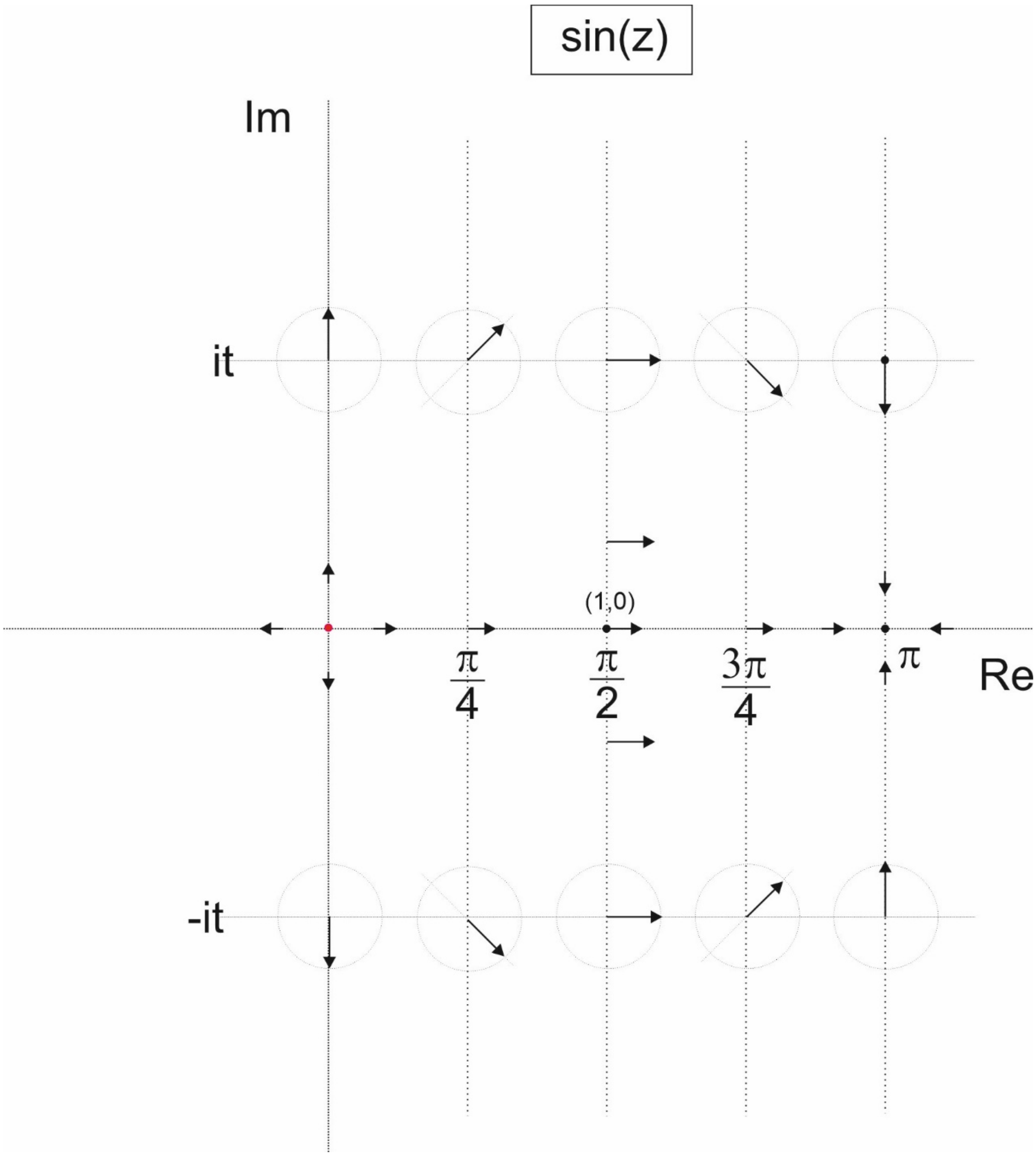

Fig. A1 the $sin(\pi s)$ vector field in the complex plane for $0 \leq \sigma \leq 1$

The definition of the complex cosine function is

$$\cos(\pi s) = \cos(\pi\sigma + i\pi t) = \cos(\pi\sigma)\, cosh(\pi t) - i \cdot \sin(\sigma\pi)\, sinh(\pi t) \qquad \text{(A7)}$$

Similar to the sine function, we can also get the most important algebraic expressions of $\cos(\pi s)$ for $0 \leq \sigma \leq 1$. Correspondingly, it is very easy to get the most important amplitudes and arguments. Here we give only its vector field in the complex plane as in Fig. A2.

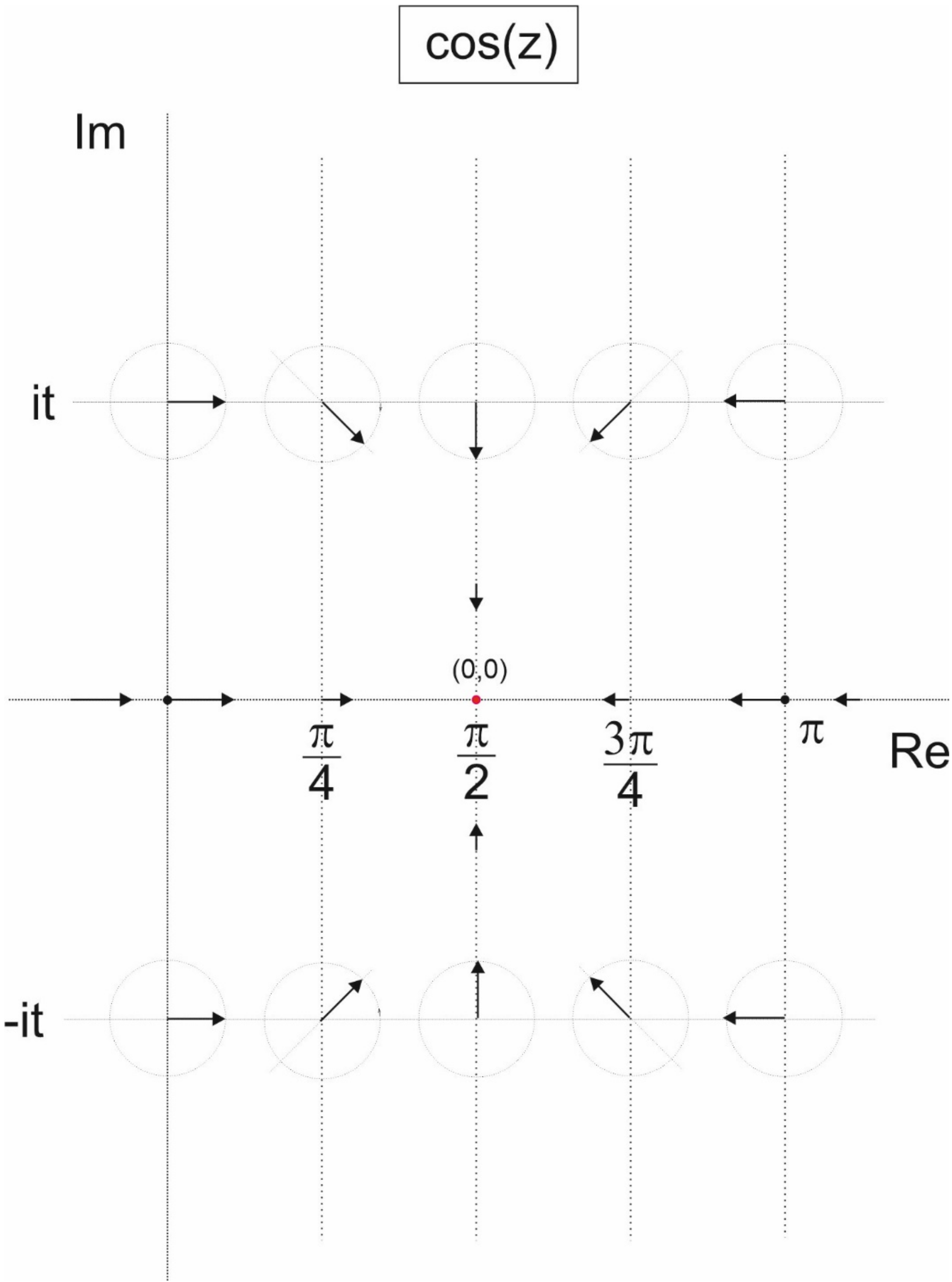

Fig. A2. The $cos(\pi s)$ vector field in the complex plane for $0 \leq \sigma \leq 1$

Comparing Fig. A1 with Fig. A2, it can be seen that the arguments of $sin(\pi s)$ and $cos(\pi s)$ have a shift of $\frac{\pi}{2}$:

$$cos(\pi s) = sin\left(\frac{\pi}{2} + \pi s\right) \tag{A8}$$

Appendix B: Gamma Function and its Vector Field

Given a complex variable $s = \sigma + it$ in the complex plane, the gamma function is related to sin(πs) by the reflection formula:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \tag{B1}$$

With the complex sin function definition in Appendix A, we have the general reflection formula in the region of $0 \leq \sigma \leq 1$:

$$\Gamma(\sigma + it)\Gamma(1 - \sigma - it) = \frac{\pi}{\sin(\pi\sigma)\, cosh(\pi t) + i \cdot \cos(\pi\sigma)\sinh(\pi t)} \tag{B2}$$

With different values of σ, we can get some important algebraic expressions, e.g., as follows:

Table B1: some important algebraic expressions	
$\sigma = 0$	$\Gamma(it)\Gamma(1-it) = -i \cdot \frac{\pi}{\sinh(\pi t)}$
$\sigma = \frac{1}{4}$	$\Gamma\left(\frac{1}{4} + it\right)\Gamma\left(\frac{3}{4} - it\right) = \frac{\sqrt{2}\cdot\pi}{cosh(\pi t) + i \cdot \sinh(\pi t)}$
$\sigma = \frac{1}{2}$	$\Gamma\left(\frac{1}{2} + it\right)\Gamma\left(\frac{1}{2} - it\right) = \frac{\pi}{cosh(\pi t)}$
$\sigma = \frac{3}{4}$	$\Gamma\left(\frac{3}{4} + it\right)\Gamma\left(\frac{1}{4} - it\right) = \frac{\sqrt{2}\cdot\pi}{cosh(\pi t) - i \cdot \sinh(\pi t)}$
$\sigma = 1$	$\Gamma(1+it)\Gamma(-it) = i \cdot \frac{\pi}{\sinh(\pi t)}$

One of the properties of the Gamma function is its recurrence relation, which relates $\Gamma(1+s)$ to $\Gamma(s)$. The recurrence relation is:

$$\mathrm{s} \cdot \Gamma(s) = \Gamma(1+s) \tag{B3}$$

On the imaginary axis, $\sigma = 0$, substituting these values into the recurrence relation, we have

$$(\mathrm{it})\Gamma(it) = \Gamma(1+it); \qquad \Gamma(it) = \frac{1}{it}\cdot\Gamma(1+it) \tag{B4}$$

Substituting this equation into the reflection formula for the case of $\sigma = 1$, we can get:

$(\mathrm{it})\Gamma(it)\Gamma(-it) = i\cdot\frac{\pi}{\sinh(\pi t)}$	(B5)

Hence, the reflection formula on the imaginary axis is:

$$\Gamma(it)\Gamma(-it) = \frac{1}{t}\cdot\frac{\pi}{\sinh(\pi t)} \tag{B6}$$

Substituting (B4) into the reflection formula for the case of $\sigma = 0$, we have the reflection formula along the line $\sigma = 1$:

$$\Gamma(1+it)\Gamma(1-it) = t\cdot\frac{\pi}{\sinh(\pi t)} \tag{B7}$$

Given the value of $\sigma = \frac{1}{2}$, we have the reflection formula along the line $\sigma = \frac{1}{2}$, as is listed in Table B1.

$$\Gamma\left(\frac{1}{2}+it\right)\Gamma\left(\frac{1}{2}-it\right) = \frac{\pi}{cosh(\pi t)} \tag{B8}$$

Equations (B6), (B7), and (B8) are the conjugate pairs, respectively, thus we can get some import moduli of the gamma function as follows:

The modulus of the gamma function on the imaginary axis:

$$|\Gamma(it)| = \frac{1}{\sqrt{t}} \cdot \frac{\sqrt{\pi}}{\sqrt{\sinh(\pi t)}} \tag{B9}$$

The modulus of the gamma function along the vertical line of $\sigma = 1$:

$$|\Gamma(1 + it)| = \sqrt{t} \cdot \frac{\sqrt{\pi}}{\sqrt{\sinh(\pi t)}} \tag{B10}$$

and the modulus of the gamma function along the vertical line of $\sigma = \frac{1}{2}$:

$$\left|\Gamma\left(\frac{1}{2} + it\right)\right| = \frac{\sqrt{\pi}}{\sqrt{cosh(\pi t)}} \tag{B11}$$

Obviously, when t=0, it degenerates to the value on the real axis:

$$\left|\Gamma\left(\frac{1}{2}\right)\right| = \sqrt{\pi} \tag{B12}$$

Hence, we have the following ratios of the gamma function moduli:

$$|\Gamma(it)|/\left|\Gamma\left(\frac{1}{2} + it\right)\right| = \frac{1}{\sqrt{t}} \cdot \sqrt{coth(\pi t)} \tag{B13}$$

and

$$|\Gamma(1 + it)|/\left|\Gamma\left(\frac{1}{2} + it\right)\right| = \sqrt{t} \cdot \sqrt{coth(\pi t)} \tag{B14}$$

Assuming the modulus of the gamma function changes continuously in the complex plane, then it can be deduced that along the horizontal line of $\sigma = const$, in the left side region of $\sigma < \frac{1}{2}$, the modulus $|\Gamma(\sigma + it)|$ is smaller than the value of $\left|\Gamma\left(\frac{1}{2} + it\right)\right|$, while in the right side region of $\sigma > \frac{1}{2}$, the modulus $|\Gamma(\sigma + it)|$ is greater than the value of $\left|\Gamma\left(\frac{1}{2} + it\right)\right|$.

Furthermore, it is observed from the reflection formula of (B1) and (B2), the argument of the expression $\Gamma(s)\Gamma(1-s)$ has a reverse relationship of the argument of $\sin(\pi s)$:

$$arg[\Gamma(s)\Gamma(1-s)] = tan^{-1}[-cot(\pi\sigma)tanh(\pi t)] \qquad \text{(B15)}$$

For large t, the asymptotes of the argument are

$$arg[\Gamma(s)] + rg[\Gamma(1-s)] \approx tan^{-1}[-cot(\pi\sigma)] = \sigma \cdot \pi - \frac{\pi}{2} \qquad \text{(B16)}$$

When $\sigma = 0$, $\Gamma(it)$ and $\Gamma(1-it)$ have a negative complementary angle relationship:

$$arg[\Gamma(it)] + arg[\Gamma(1-it)] \approx -\frac{\pi}{2} \qquad \text{(B17)}$$

when $\sigma = 1$, $\Gamma(-it)$ and $\Gamma(1+it)$ have a positive complementary angle relationship:

$$arg[\Gamma(1+it)] + arg[\Gamma(-it)] \approx \frac{\pi}{2} \qquad \text{(B18)}$$

Obviously, when $\sigma = \frac{1}{2}$, $\Gamma\left(\frac{1}{2} + it\right)$ and $\Gamma\left(\frac{1}{2} - it\right)$ are conjugates, the sum of their arguments is zero:

$$arg\left[\Gamma\left(\frac{1}{2}+it\right)\right]+arg\left[\Gamma\left(\frac{1}{2}-it\right)\right]=0 \tag{B19}$$

If the argument of $\Gamma\left(\frac{1}{2}-it\right)$ is equal to β, then the argument of $\Gamma\left(\frac{1}{2}+it\right)$ must be equal to $-\beta$. Fig. B1 gives a representation vector field for the gamma function in the region of $0 \le \sigma \le 1$.

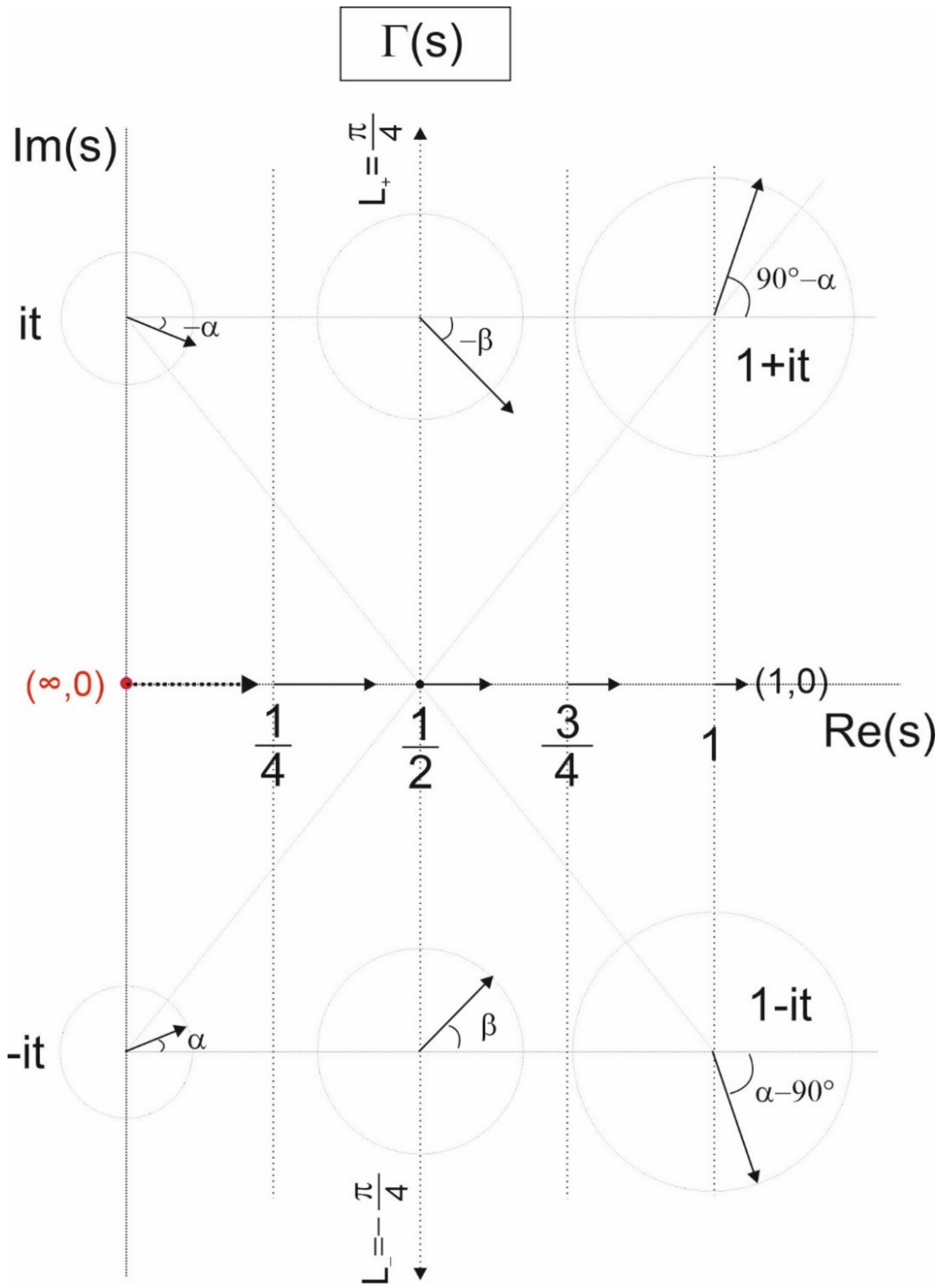

Fig. B1 Gamma function vector field in the region $0 \le \sigma \le 1$.

Appendix C: Complex Multiplicative Inverse of g(s) and g(1-s)

The complex scaling and rotation functions for the Riemann zeta function are defined by eq. (47):

$$\begin{cases} g(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \\ g(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \end{cases} \tag{C1}$$

We will show multiplicative inverses behavior between g(s) and g(1-s), namely, the identity of eq. (48):

$$g(s) \cdot g(1-s) = 1 \tag{C2}$$

The product of both functions is

$$g(s) \cdot g(1-s) = \frac{1}{\pi} \cdot \left[2 \cdot \sin\left(\frac{\pi s}{2}\right) \cos\left(\frac{\pi s}{2}\right)\right] [\Gamma(s)\Gamma(1-s)] \tag{C3}$$

Using the double-angle formula for the sin function and the gamma reflection formula to simplify it:

$$g(s) \cdot g(1-s) = \frac{1}{\pi} \cdot [\sin(\pi s)] \cdot \left[\frac{\pi}{\sin(\pi s)}\right] = 1 \tag{C4}$$

Thus, for any complex variable of $s = \sigma + it$, this multiplicative inverse behavior always holds.

Hence, given a complex variable of $s = \sigma + it$, we have

$$\begin{cases} g(\sigma+it) = \dfrac{(2\pi)^{\sigma}}{\pi}(2\pi)^{it}\sin\left(\dfrac{\pi\sigma}{2}+i\dfrac{\pi t}{2}\right)\Gamma(1-\sigma-it) \\ g(1-\sigma-it) = \dfrac{2}{(2\pi)^{\sigma}}(2\pi)^{-it}\cos\left(\dfrac{\pi\sigma}{2}+i\dfrac{\pi t}{2}\right)\Gamma(\sigma+it) \end{cases} \tag{C5}$$

If we define the modulus and argument to be r and θ for $g(\sigma+it)$, because of the multiplicative inverse behavior, the modulus and argument of $g(1-\sigma-it)$ must be $1/r$ and $-\theta$, respectively.

Substituting these into the Riemann zeta function, we have

$$\begin{cases} \zeta(s) = \zeta(1-s)\cdot\left[re^{i\theta}\right] \\ \zeta(1-s) = \zeta(s)\cdot\left[\dfrac{1}{r}e^{-i\theta}\right] \end{cases} \tag{C6}$$

It can be shown that when $\sigma=\frac{1}{2}$, the moduli of $g(s)$ and $g(1-s)$ are equal to 1.

Given $\sigma=\frac{1}{2}$, the eq. (C1) become:

$$\begin{cases} g\left(\dfrac{1}{2}+it\right) = \dfrac{\sqrt{2}}{\sqrt{\pi}}\cdot(2\pi)^{it}\sin\left(\dfrac{\pi}{4}+i\dfrac{\pi t}{2}\right)\Gamma\left(\dfrac{1}{2}-it\right) \\ g\left(\dfrac{1}{2}-it\right) = \dfrac{\sqrt{2}}{\sqrt{\pi}}\cdot(2\pi)^{-it}\cos\left(\dfrac{\pi}{4}+i\dfrac{\pi t}{2}\right)\Gamma\left(\dfrac{1}{2}+it\right) \end{cases} \tag{C7}$$

Using the modulus expression for $\sin\left(\frac{\pi}{4}+i\frac{\pi t}{2}\right)$ in table A2 and the modulus expression for $\Gamma\left(\frac{1}{2}-it\right)$ of the eq. (B8), we can get the moduli for functions $g\left(\frac{1}{2}+it\right)$ and $g\left(\frac{1}{2}-it\right)$:

$$\begin{cases} \left|g\left(\frac{1}{2}+it\right)\right| = \frac{1}{\sqrt{\pi}} \cdot \sqrt{cosh(\pi t)} \cdot \frac{\sqrt{\pi}}{\sqrt{cosh(\pi t)}} = 1 \\ \left|g\left(\frac{1}{2}-it\right)\right| = \frac{1}{\sqrt{\pi}} \cdot \sqrt{cosh(\pi t)} \cdot \frac{\sqrt{\pi}}{\sqrt{cosh(\pi t)}} = 1 \end{cases} \tag{C8}$$

Here, we have applied the following identity for hyperbolic functions:

$$\left|cosh\left(\frac{\pi t}{2}\right) \pm i \cdot sinh\left(\frac{\pi t}{2}\right)\right| = \sqrt{cosh^2\left(\frac{\pi t}{2}\right) + sinh^2\left(\frac{\pi t}{2}\right)} = \sqrt{cosh(\pi t)} \tag{C9}$$

These imply that when $\sigma = \frac{1}{2}$, the scaling factor g(s) and g(1-s) is always 1, independent of the variable of t, which implies also a reflection symmetry of the moduli of zeta function about $\sigma = \frac{1}{2}$, namely, the moduli in the regions of $\sigma < \frac{1}{2}$ and $\sigma > \frac{1}{2}$ have a multiplicative inverse complementary relationship.

Substituting these into the eq. (C6), finally we get the equations of (53) for $\sigma = \frac{1}{2}$. The scaling factor for the modulus of the zeta function keeps 1, while g(s) function lets the zeta function rotate continuously, the conjugate pair of the zeta functions have opposite rotational signs, but with the same moduli.

Appendix D: Other Variations of Zeta Function

Given any complex function, it is always true:

$$\begin{cases} \zeta(s) = \zeta(s) \\ \zeta(1-s) = \zeta(1-s) \end{cases} \tag{D1}$$

Multiplying and divided by $\sin(\pi s)$:

$$\begin{cases} \zeta(s) = \dfrac{\sin(\pi s)}{\sin(\pi s)} \cdot \zeta(s) \\ \zeta(1-s) = \dfrac{\sin(\pi s)}{\sin(\pi s)} \cdot \zeta(1-s) \end{cases} \tag{D2}$$

where, $s \neq 0, 1$.

Using the gamma reflection formula:

$$\begin{cases} \zeta(s) = \dfrac{1}{\pi} \cdot [\Gamma(s)\Gamma(1-s)] \cdot \sin(\pi s) \ \cdot \zeta(s) \\ \zeta(1-s) = \dfrac{1}{\pi} \cdot [\Gamma(s)\Gamma(1-s)] \cdot \sin(\pi s) \cdot \zeta(1-s) \end{cases} \tag{D3}$$

Using the double-angle identity for sin function:

$$\begin{cases} \zeta(s) = \dfrac{2}{\pi} \cdot \Gamma(s)\Gamma(1-s) \left[\sin\left(\dfrac{\pi s}{2}\right) \cos\left(\dfrac{\pi s}{2}\right)\right] \zeta(s) \\ \zeta(1-s) = \dfrac{2}{\pi} \cdot \Gamma(s)\Gamma(1-s) \left[\sin\left(\dfrac{\pi s}{2}\right) \cos\left(\dfrac{\pi s}{2}\right)\right] \zeta(1-s) \end{cases} \tag{D4}$$

Rearranging:

$$\begin{cases} \zeta(s) = \left[\frac{\sqrt{2}}{\sqrt{\pi}} \cdot \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\right]\left[\frac{\sqrt{2}}{\sqrt{\pi}} \cdot \cos\left(\frac{\pi s}{2}\right)\Gamma(s)\,\zeta(s)\right] \\ \zeta(1-s) = \left[\frac{\sqrt{2}}{\sqrt{\pi}} \cdot \cos\left(\frac{\pi s}{2}\right)\Gamma(s)\right]\left[\frac{\sqrt{2}}{\sqrt{\pi}} \cdot \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)\right] \end{cases} \tag{D5}$$

We define the following functional equations:

$$\begin{cases} \zeta(1-s) = \left[\frac{\sqrt{2}}{\sqrt{\pi}} \cdot \cos\left(\frac{\pi s}{2}\right)\Gamma(s)\,\zeta(s)\right] \\ \zeta(s) = \left[\frac{\sqrt{2}}{\sqrt{\pi}} \cdot \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)\right] \end{cases} \tag{D6}$$

Hence, we have

$$\begin{cases} \zeta(s) = \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \sin\left(\frac{\pi s}{2}\right) \cdot [\Gamma(1-s)\zeta(1-s)] \\ \zeta(1-s) = \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \cos\left(\frac{\pi s}{2}\right) \cdot [\Gamma(s)\zeta(s)] \end{cases} \tag{D7}$$

Recalling the integral definition of the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx \tag{D8}$$

Thus, we have the following equations:

$$\begin{cases} \Gamma(s)\zeta(s) = \int\limits_0^\infty \frac{x^{s-1}}{e^x - 1} dx \\ \Gamma(1-s)\zeta(1-s) = \int\limits_0^\infty \frac{x^{-s}}{e^x - 1} dx \end{cases} \tag{D9}$$

Substituting eq. (D9) into (D7), we can get another integral expression for Zeta function:

$$\begin{cases} \zeta(s) = \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \sin\left(\frac{\pi s}{2}\right) \cdot \int\limits_0^\infty \frac{x^{-s}}{e^x - 1} dx \\ \zeta(1-s) = \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \cos\left(\frac{\pi s}{2}\right) \cdot \int\limits_0^\infty \frac{x^{s-1}}{e^x - 1} dx \end{cases} \tag{D10}$$

When $s = \frac{1}{2} + it$, both equations become:

$$\begin{cases} \zeta\left(\frac{1}{2} + it\right) = \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \sin\left(\frac{\pi}{4} + i\frac{\pi t}{2}\right) \cdot \int\limits_0^\infty \frac{x^{-it}}{\sqrt{x}(e^x - 1)} dx \\ \zeta\left(\frac{1}{2} - it\right) = \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \cos\left(\frac{\pi}{4} + i\frac{\pi t}{2}\right) \cdot \int\limits_0^\infty \frac{x^{it}}{\sqrt{x}(e^x - 1)} dx \end{cases} \tag{D11}$$

Using the identity:

$$\cos\left(\frac{\pi}{4} + i\frac{\pi t}{2}\right) = \sin\left(\frac{\pi}{4} - i\frac{\pi t}{2}\right) \tag{D12}$$

The equations (D11) can be rewritten as:

$$\begin{cases} \zeta\left(\frac{1}{2}+it\right) = \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \sin\left(\frac{\pi}{4}+i\frac{\pi t}{2}\right) \cdot \int_0^\infty \frac{x^{-it}}{\sqrt{x}(e^x-1)}dx \\ \zeta\left(\frac{1}{2}-it\right) = \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \sin\left(\frac{\pi}{4}-i\frac{\pi t}{2}\right) \cdot \int_0^\infty \frac{x^{it}}{\sqrt{x}(e^x-1)}dx \end{cases} \tag{D13}$$

For large t, the asymptotes behavior of the both equations are:

$$\begin{cases} \zeta\left(\frac{1}{2}+it\right) \approx \frac{1}{\sqrt{\pi}} cosh\left(\frac{\pi t}{2}\right) \cdot (1+i) \cdot \int_0^\infty \frac{x^{-it}}{\sqrt{x}(e^x-1)}dx \\ \zeta\left(\frac{1}{2}-it\right) \approx \frac{1}{\sqrt{\pi}} cosh\left(\frac{\pi t}{2}\right) \cdot (1-i) \cdot \int_0^\infty \frac{x^{it}}{\sqrt{x}(e^x-1)}dx \end{cases} \tag{D14}$$

The complex numbers $(1+i)$ and $(1-i)$ can be written in polar form:

$$\begin{cases} \zeta\left(\frac{1}{2}+it\right) \approx \left[\frac{\sqrt{2}}{\sqrt{\pi}} cosh\left(\frac{\pi t}{2}\right)\right] \cdot e^{i\frac{\pi}{4}} \cdot \int_0^\infty \frac{x^{-it}}{\sqrt{x}(e^x-1)}dx \\ \zeta\left(\frac{1}{2}-it\right) \approx \left[\frac{\sqrt{2}}{\sqrt{\pi}} cosh\left(\frac{\pi t}{2}\right)\right] \cdot e^{-i\frac{\pi}{4}} \cdot \int_0^\infty \frac{x^{it}}{\sqrt{x}(e^x-1)}dx \end{cases} \tag{D15}$$

It is recognized that the integrals in both equations are complex functions.

Multiplying the rotation factors $e^{i\frac{\pi}{4}}$ and $e^{-i\frac{\pi}{4}}$ will change its angles. The angles obtained by successive multiplications will eventually return to their starting point after 8 multiplications, forming a cyclic group of order 8.

These rotation operations, having magnitude 1, lie on the unit circle in the complex plane and correspond to the 8th roots of unity.

Fig. D1 visualizes the unit circle in the complex plane, multiplication of 1+i (normalized) corresponds to one of the 8 equally spaced points on the circle, similarly, multiplication of 1-i corresponds also to one of the 8 equally spaced points on the circle, but anticlockwise.

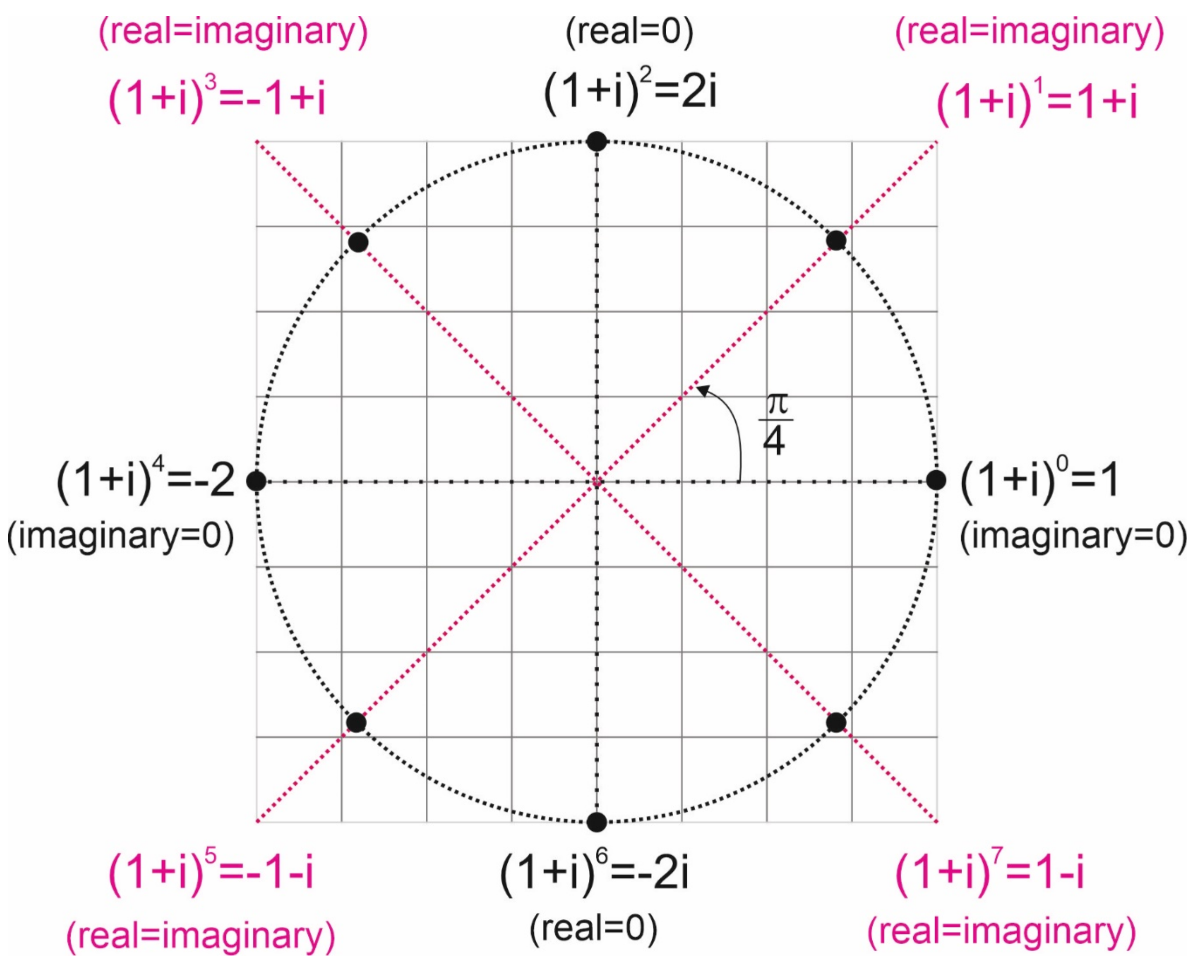

Fig. D1. C_8 cyclic group when multiplications of (1+i)

It should be noted, in the equations (D4), if we factorize the term 2 and π in the following manner:

$$\begin{cases} 2 = 2^{s-s+1} = 2^s \cdot 2^{1-s} \\ \pi^{-1} = \pi^{-1+s-s} = \pi^{-s} \cdot \pi^{s-1} \end{cases} \tag{D16}$$

Re-factorizing and re-defining the functions of (D6), we can also get the original Riemann zeta functional definition of eq. (45).