Research Article

Helicity Effects in Component-Wise Dimensionally Reduced Flows with a One-Dimensional Velocity Component

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The absolute equilibrium analysis of the nonlinear dynamical system from the Galerkin-truncated component-wise dimensionally reduced flows with a one-dimensional velocity component indicates the hybrid fastening, directionality and polarization effects of helicity, with implications of anisotropic turbulence noise problems.

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I. Introduction and preparation

In realistic problems, we often encounter anisotropic flows, requiring judgments or control over related dynamic processes such as mode transitions and energy propagation. However, even for energy distribution in homogeneous isotropic turbulence, a systematic non-equilibrium statistical analysis remains lacking.

T.-D. Lee (1952) [1] and R. Kraichnan (1955) [2] conducted pioneering studies on "absolute statistical equilibrium" for incompressible and compressible three-dimensional turbulence, respectively, yielding profound insights. Ref. [3] updated Kraichnan's work [2] with helical decomposition and helicity invariance, proposing that helicity modifies statistics via a "tightening" or "fastening" effect that reduces flow compressibility, later extended to plasma fluid models [4]. This helicity-induced tightening has been partially validated through energy spectrum analysis in both low- [5] and high-Reynolds-number numerical turbulence [6], motivating further theoretical exploration — including in the newly proposed real Schur flows (RSFs) [7],[8][1].

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In general, terms such as "anisotropy," "polarization," "directivity", and "directionality" are closely related yet ambiguously bounded, with meanings varying across disciplines and contexts. We neither intend nor can unify or standardize the nomenclature but instead explicitly clarify our choice:

- "Anisotropy" refers broadly to directional differences in the conventional sense;
- "Polarization" indicates that directional indicators (e.g., wave vector components rather than wavenumber) explicitly appear in a physical quantity (similar to "directivity" in acoustics and close to the "local polarization" notion in some literatures [9][2]), but not in a dimensional reduction manner below;
- "Directionality" specifically denotes cases (as shown later in Eq. 19 and others in Sec. IIB) where Kronecker delta functions explicitly appear in the expression of a physical function to indicate dimensional reduction, which is close to the "global polarization" in some literatures [9].

An anisotropic field inherently implies infinite quantitative possibilities, but can be categorized into two types. One type can be described by adjusting the aspect ratios of the periodic box in the above calculations; for cases other than $1:1:1$, it means that the physical scales of all variables consistently exhibit the same anisotropic characteristics — mathematically, this only requires renormalizing the wave vectors in different directions in the analysis above and does not pose a new problem. The other type is when different physical variables possess distinctly different anisotropic scale characteristics, an extreme ideal model of which is the componentwise dimensionally reduced flow (CWDRF) we will discuss here.

This study aims to provide heuristic theoretical foundations for related issues through absolute statistical equilibrium analysis of some types of CWDRFs. We first briefly review Kraichnan's approach to absolute statistical equilibrium analysis in 3D compressible isotropic turbulence [2] to contextualize subsequent discussions.

A. Absolute statistical equilibrium analysis of compressible turbulence

Let $\mathbf{u}$, ρ, and c represent the flow velocity, density, and speed of sound, respectively, with the pressure p given by the adiabatic barotropic relation

$$p = c^2\rho,\ \rho = \rho_0 e^{\zeta}. \tag{1}$$

By appropriate scaling and choice of units, the background ρ_0 can be taken as unity $(= 1)$, although we may keep them explicit when needed to indicate physical meanings. The inviscid governing equations

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are

$$\partial_t \zeta + \zeta_{,\sigma} u_\sigma + u_{\sigma,\sigma} = 0, \tag{2}$$
$$\partial_t u_\lambda + u_\sigma u_{\lambda,\sigma} + c^2 \zeta_{,\lambda} = 0, \tag{3}$$

where $(\bullet)_{,\gamma} = \partial(\bullet)/\partial x^\gamma$. Consider a periodic box with equal sides, of volume $V = (2\pi)^3$. For the Fourier expansion coefficient $\hat{v}(\mathbf{k})$ of any variable $v(\mathbf{r})$, the discrete wave vector $\mathbf{k}$ may sometimes be written as a subscript for convenience (e.g., $\hat{v}_{\mathbf{k}}$), and may even be omitted when the meaning is clear from context. For instance, $\mathbf{u}(\mathbf{r}) = \sum_{\mathbf{k}} \hat{\mathbf{u}}(\mathbf{k}) \exp\{\hat{i}\mathbf{k}\cdot\mathbf{r}\}$, where $\hat{i}^2 = -1$. Applying a Galerkin truncation (e.g., setting modes with $k = |\mathbf{k}| > K$ to zero), Kraichnan [2] considered the real and imaginary parts of $\hat{v}$ as constituting the phase space of the system. It is readily seen that for inviscid flow the phase flow is incompressible, i.e., it satisfies Liouville's theorem. For small disturbances, the mean energy per unit mass is

$$\mathcal{E} = \frac{\langle u^2 + c^2\zeta^2\rangle_{123}}{2} = \frac{\sum_{\mathbf{k}} [\hat{u}_\lambda(\mathbf{k})\hat{u}^*_\lambda(\mathbf{k}) + c^2|\hat{\zeta}(\mathbf{k})|^2]}{2}, \tag{4}$$

where $\langle\bullet\rangle_{123} := \frac{1}{(2\pi)^3}\int_V \bullet\, dV$, and the potential energy fluctuation is taken to second-order approximation as

$$\int_{\rho_0}^{\rho} \frac{p - p_0}{\rho^2} d\rho \approx \frac{c^2\zeta^2}{2}. \tag{5}$$

Kraichnan assumed that the system would approach a state of absolute statistical equilibrium (as Lee [1] did — *c.f.* Ref. [10] for more on this), starting from the canonical ensemble distribution $\sim \exp\{-\alpha\mathcal{E}\}$ (α being a parameter related to "temperature"), and derived an equipartition of energy, thereby analyzing the possible different dissipation rates of turbulence with or without noise.

Superficially, because the concept of statistical equilibrium involves infinite time, any approximation such as that for the potential energy above might seem unjustifiable, as it could eventually become uncontrolled. However, what we actually require is the partial thermalization or the tendency toward thermalization of the turbulent system within certain finite spatiotemporal scales. Within these physically relevant finite intervals, the adopted approximation may be reasonable. Moreover, although real turbulence involves dissipation and various other factors and never reaches absolute statistical equilibrium, a qualitative trend analysis within dominating nonlinear regimes is not without justification: choosing the appropriate quadratic constraints is more from the physical relevance rather than formal mathematical rigor.

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At that time, it was not known that this ideal system also conserves helicity [11]. After learning of it, Kraichnan applied it to the absolute statistical equilibrium analysis of incompressible isotropic turbulence [12]:

$$\mathcal{H} = \sum_{\mathbf{k}} \hat{i}\mathbf{k} \times \hat{\mathbf{u}}_{\mathbf{k}} \cdot \hat{\mathbf{u}}_{\mathbf{k}}^{*}/2. \tag{6}$$

The issue of helicity in compressible flow has since been updated and supplemented [3] (a brief review and comparison will be given in Section IIA below). However, such an analysis does not directly address the anisotropy itself in anisotropic problems.

B. CWDRFs

Consider the matrix representation of the velocity gradient tensor $\nabla\mathbf{u}$ in three-dimensional space:

$$G = \begin{pmatrix} u_{1,1} & u_{2,1} & \cancel{u_{3,1}} \\ u_{1,2} & u_{2,2} & \cancel{u_{3,2}} \\ \xcancel{u_{1,3}} & \xcancel{u_{2,3}} & u_{3,3} \end{pmatrix}, \tag{7}$$

where $u_{i,j} := \partial_{x_j} u_i$. As indicated by the multiple and single slashes in the matrix, setting the two elements below or above the block diagonal to zero yields the real Schur form (referred to as the "223" and "331" forms/types, respectively), which is covered in every standard matrix theory textbook. Any real matrix can be transformed into either of these forms via an orthogonal coordinate transformation (a "generic" property). If, consistently in space and time,

$$u_{1,3} \equiv 0 \equiv u_{2,3}, \tag{8}$$

then we have a coupling between two components (u_1 and u_2) that are two-dimensional and one component (u_3) that is three-dimensional (2C2Dcw1C3D or 2D2D3D). The corresponding flow can be briefly called a 223 real Schur flow (223RSF). Similarly, "3D3D1D" and the 331RSF correspond to

$$u_{3,1} \equiv 0 \equiv u_{3,2}. \tag{9}$$

For convenience, we will also use self-evident terms like "223 matrix" and "331 form/type" etc.

CWDRFs also come in other forms. For example, besides 2D2D3D and 3D3D1D [whose intersection satisfying both (8) and (9) is 2D2D1D], there are flows corresponding to upper or lower triangular matrices of G in Eq. (7), adding to Eqs. (8) and (8), respectively,

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$$u_{1,2} \equiv 0 \tag{10}$$

and

$$0 \equiv u_{2,1}, \tag{11}$$

and named 1D2D3D and 3D2D1D according to similar rules: they are also called “lone Schur flows (LSF)” because their matrix eigenvalues do not come in complex conjugate pairs [13]3). However, the generic nature of the real Schur form suggests that the 223RSFs and 331RSFs considered in Ref. [7][8] may have a fundamental role (a naively wild analogy is associated to the local inertial frames and the global flat space-time in special relativity, and see the attempt in Ref. [4] to “reduce” the helicity effects to RSFs). Moreover, the real Schur form of the velocity gradient is preserved by the self-advection nonlinear term in hydrodynamic-type equations (so RSFs of the Burgers equation trivially form an invariant submanifold of its solutions!). Different authors had used the 223-type Schur matrix of G for various purposes in studies of hydrodynamics (c.f., the comprehensive bibliographies in Refs. [14],[15]4), but none had considered the corresponding 331 real Schur form (let alone the dynamical RSFs) before Ref. [7][8], to the best of our knowledge.

Ref. [7][8] mentions that active fluids—nonequilibrium fluids composed of self-propelled particles, such as bacterial suspensions or artificial microswimmer solutions, where the particles consume energy to generate motion, leading to unique collective behaviors like spontaneous flow, vortices, and even turbulence—might realize them; and, removing the incompressibility condition in the Taylor-Proudman theorem for classical rapidly rotating flows could lead to 223RSFs with the additional property of horizontal incompressibility [5]; furthermore, following Theorem 3 on the topology of 331RSFs to be stated later, we will remark on 331RSF’s possible connections with flows containing thin layer(s) of strong stratification(s), such as Venus’s temperature inversion layers. Although a mathematically rigorous proof of the specific conditions that can lead to RSFs remains to be completed, we can achieve them by real-time removal of the variations of appropriate velocity components in specific directions in classical hydrodynamic equations and by constructing self-consistent dynamical models, analyze their properties, and gain physical insights [5][7][8].

Taking the ideal CWDRF equations in a periodic box constructed from Eqs. (1,2,3) in [5] as an example, it has been verified (including relevant numerical tests inside and outside [5]) that the conditions and form of Kraichnan’s absolute statistical equilibrium analysis from Sec. I.1 (such as Liouville’s theorem, helicity conservation — Appendices A and B, and the arguments for approximating the pressure work/potential energy fluctuation part) also formally apply, say, to the following 331RSF equation,

$$\partial_t \zeta = \langle \varrho \rangle_{123} - \langle \varrho \rangle_3 - \langle \varrho \rangle_{12}, \tag{12a}$$
$$\partial_t \mathbf{u}_h + \mathbf{u} \cdot \nabla \mathbf{u}_h = -c^2 \nabla_h \zeta, \tag{12b}$$
$$\partial_t u_3 + u_3 \cdot \nabla u_3 = -c^2 \nabla \zeta, \tag{12c}$$

and the 1D2D3D LSF equation,

$$\partial_t \zeta = 2\langle \varrho \rangle_{123} - \langle \varrho \rangle_{23} - \langle \varrho \rangle_{13} - \langle \varrho \rangle_{12}, \tag{13a}$$
$$\partial_t u_1 + u_1 u_{1,1} = -c^2 \zeta_{,1}, \tag{13b}$$
$$\partial_t u_2 + u_1 u_{2,1} + u_2 u_{2,2} = -c^2 \zeta_{,2}, \tag{13c}$$
$$\partial_t u_3 + \mathbf{u} \cdot \nabla u_3 = -c^2 \zeta_{,3}, \tag{13d}$$

where $\varrho := u_\sigma \zeta_{,\sigma} + u_{\lambda,\lambda}$, and the (multiple) subscript J in $\langle \bullet \rangle_J$ denotes averaging over the corresponding spatial coordinates. The form of Eqs. (12a) and (13a) is designed to preserve the decompositions $\zeta = \mathcal{Z}_h^{RSF}(\mathbf{x}_h, t) + \mathcal{Z}_3^{RSF}(x_3, t)$ [5] and $\zeta = \mathcal{Z}_1^{LSF}(\mathbf{x}_h, t) + \mathcal{Z}_2^{LSF}(x_2, t) + \mathcal{Z}_3^{LSF}(x_3, t)$ [13], respectively, thereby maintaining the corresponding CWDRF forms of $\nabla \mathbf{u}$. For the 2D2D1D CWDRF simultaneously satisfying conditions (8,9), one can directly verify that Eq. (2) naturally takes the form (12a); i.e., the Euler equations have an invariant submanifold of 2D2D1D flows. However, the Euler equations do not necessarily preserve $\varrho = \langle \varrho \rangle_{12} + \langle \varrho \rangle_3 - \langle \varrho \rangle_{123}$; that is, the entire set of RSFs does not form an invariant submanifold (the relationship between the two types of real Schur CWDRFs and classical general flows, especially possible nonequilibrium equivalent ensembles, is a topic for further studies.)

It is intuitively clear that 1D2D3D and 3D2D1D LSFs are the same, by simply swapping the x_1 and x_3 coordinates, and that 223RSF and 331RSF are different for distinct componentwise dimension reductions, which actually can be formulated as a no-go theorem and proved by straightforward matrix computation [16].

It should be noted that the above CWDRFs are, first and foremost, mathematically self-consistent "toy models" with special structures, constructed purely rationally within a specific framework. How they are compatible with other physics or systems, or how they can be realized in reality, is a separate issue.

Next, in Sec. IIA, we revisit the earlier update [3] to Kraichnan's work introduced in Sec. I.1, expressing it differently to give a more refined description of the polarization effect of helicity on correlation functions. With this warm-up and groundwork, we then carry out the absolute equilibrium analyssi of 331RSFs and LSFs in Sec. IIB, demonstrating the fastening, polarization and directionality hybrid effects of helicity. Finally, Sec. III provides a summary and outlook associated to the obvious implications for flow acoustics.

II. Analysis

It is helpful to set up the general foundational mathematical and physical background for our analysis. For the intuitively reasonably clear and technically involved results, we leave the details of the proofs to another communication for more general purposes [16].

First of all, although the velocity gradient forms of the 331- and 223-RSFs both consistently have two fixed elements set to zero, the values of the other matrix elements vary in space and time. Thus, intuitively, the two flows are "different." However, a rigorous mathematical statement requires the following:

Theorem 1. *There does not exist an orthogonal (real Schur) coordinate transformation that can consistently transform all 223 real Schur matrices (and correspondingly, 223RSFs) into the 331 form (and correspondingly, 331RSFs).*

The above theorem can be proven by direct computation and comparison of matrix element values before and after coordinate transformation. Moreover, the computation will simultaneously reveal, consistent with intuition, that for the purely upper or lower triangular matrices mentioned earlier:

Theorem 2. *There exists a unified orthogonal coordinate transformation that can convert all upper triangular matrices (correspondingly, 3D2D1D LSFs) into lower triangular form (correspondingly, 1D2D3D LSFs).*

Remark 1. *The simplest example is the anti-diagonal permutation matrix, which swaps the first and third coordinate axes. Therefore, we need and will only carry out the analysis of 1D2D3D or 3D2D1D LSFs {m/13/}, with the results applying to the alternative by simply swapping the coordinate indexes.*

The "difference" between 331RSFs and 223RSFs is also reflected in the unique flow topology arising from the one-dimensional nature of $u_3(x_3)$ (we consider classical solutions rather than possible generalized or weak solutions, so properties like uniqueness of streamlines and non-intersection hold). For instance, unlike 223RSFs, we have the following:

Theorem 3. *In a 331RSF, closed streamlines can only form in the equilibrium plane of u_3, i.e., the x_1-x_2 plane where $u_3(x_3^*) = 0$.*

Remark 2. *This theorem indicates the possible physical relevance between 331RSFs and strongly stratified flows in Nature, with the u_3-equilibrium plane(s) bearing some resemblance to the ideally infinite-thin temperature inversion layers (of the Venus' atmosphere, say), the Earth's stratospeheres, and, ocean pycno-, thermo- and*

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halo-clines which inhibit the vertical convections. Of course, closer dynamical examination beyond topological patterns is needed for further clarification.

The proof of Theorem 3 relies on the monotonicity of one-dimensional autonomous systems in the basics of ordinary differential equations [17]. The same argument has important implications for other related CWDRFs. For example, it can further be proved that LSF is in general vortical but never swirling [13]:

Theorem 4. *LSF does not admit closed streamlines.*

Detailed proofs of the above claims and analyses of other mathematical properties are beyond the scope of this note and are provided in a separate communication [16] where the following definition is also offered to justify our usage of "swirl" different to "vortex" in the above:

Definition 1. *"Vortex (structure)" is some distribution of vorticity $\omega = \nabla \times \mathbf{u}$. As for the choice of the distributions, it should be dependent on the purpose. Any other "structures" not defined by vorticity should not be called "vortex (structure)". And, a closed streamline defines the "swirl".*

The following is the reason why we will not apply the helical decomposition technique as in Ref. [3].

Definition 2. *A vector field $\mathbf{v}$ is 'helical' if the helicity does not vanish and is 'purely helical' ('unichiral') if each of the Fourier component $\hat{\mathbf{v}}(\mathbf{k}) = \int\int\int \mathbf{v}\exp\{-\hat{\imath}\mathbf{x}\cdot\mathbf{v}\}d^3\mathbf{x}$ is a 'helical mode', the latter meaning that $\hat{\imath}\mathbf{k}\times\hat{\mathbf{v}} = \pm k\hat{\mathbf{v}}$ with $\hat{\imath}^2 = -1$. The $+$ and $-$ signs are usually assigned respectively to right- and left-handnesses. $\mathbf{v}$ is 'maximally helical' if the $+$ or $-$ sign applies uniformly for every $\mathbf{k}$ ('homochiral'), and 'Beltramian' if $\exists\kappa, \kappa^2 > 0, \nabla\times\mathbf{v} = \kappa\mathbf{v}$.*

3D3D1D field, like the 2D2D3D one, can be helical, and here is an explicit 1D2D3D example (swapping the x_1 and x_3 axes, we have the corresponding 3D2D1D, a further reduction of 3D3D1D field):

$$\begin{pmatrix} b_1\cos(kx_1) \\ b_2\sin(kx_1) + b_3\cos(kx_2) \\ b_4\cos(kx_1) + b_5\sin(kx_2) + b_6\sin(kx_3) \end{pmatrix} \tag{14}$$

where $b_\bullet$ are spatially uniform real coefficients and $b_2b_4 \neq 0$; but, on the degree of chirality of the RSFs, we can establish

Theorem 5. *A 3D3D1D or 2D2D3D $\mathbf{u}$ with $u_{3,3} \neq 0$ can not be purely helical.*

Proof. $u_{3,1} = u_{3,2} \equiv 0$ for 3D3D1D $\mathbf{u}$ means $\hat{u}_3(\mathbf{k}) = 0 \forall k_1^2 + k_2^2 \neq 0$. The purely-helical Fourier mode $\hat{\mathbf{u}}$, satisfies

$$\hat{\imath}(k_1\hat{u}_2 - k_2\hat{u}_1) = \pm k\hat{u}_3 \tag{15}$$

by definition 2, so $u_3(\mathbf{k})=0$ for $k_1=k_2=0\neq k_3$, contradicting $u_{3,3}\neq 0$; Eq. (15) similarly leads to contradiction in the 2D2D3D case. □

This theorem unifies that of the 2D2D3D field in Ref. [5].

A. Polarization information in isotropic compressible helical turbulence

Following Kraichnan [2][3], we consider the canonical ensemble distribution $\sim \exp\{-C\}$, but now the constants of motion are taken as $C=\alpha\mathcal{E}+\beta\mathcal{H}$ with the multiplier β, and the expressions for $\mathcal{E}$ and $\mathcal{H}$ directly given by Eqs. (4) and (6), without using the helical decomposition as in Ref. [3]. Further separating the real and imaginary parts of the Fourier coefficients, e.g., $\hat{u}_i=R_i+\hat{i}I_i$ $(i=1,2,3)$, we have the matrix for the quadratic form of C in terms of $(R_1,R_2,R_3,I_1,I_2,I_3)$,

$$\begin{pmatrix} \alpha & 0 & 0 & 0 & \beta k_3 & -\beta k_2 \\ 0 & \alpha & 0 & -\beta k_3 & 0 & \beta k_1 \\ 0 & 0 & \alpha & \beta k_2 & -\beta k_1 & 0 \\ 0 & -\beta k_3 & \beta k_2 & \alpha & 0 & 0 \\ \beta k_3 & 0 & -\beta k_1 & 0 & \alpha & 0 \\ -\beta k_2 & \beta k_1 & 0 & 0 & 0 & \alpha \end{pmatrix} \tag{16}$$

from which we obtain the correlation matrix for the variable tuple $(R_1,R_2,R_3,I_1,I_2,I_3)$:

$$\begin{pmatrix} \frac{\alpha^2-\beta^2k_1^2}{\alpha^3-\alpha\beta^2k^2} & \frac{-\beta^2k_1k_2}{\alpha^3-\alpha\beta^2k^2} & \frac{-\beta^2k_1k_3}{\alpha^3-\alpha\beta^2k^2} & 0 & \frac{-\beta k_3}{\alpha^2-\beta^2k^2} & \frac{\beta k_2}{\alpha^2-\beta^2k^2} \\ \frac{-\beta^2k_1k_2}{\alpha^3-\alpha\beta^2k^2} & \frac{\alpha^2-\beta^2k_2^2}{\alpha^3-\alpha\beta^2k^2} & \frac{-\beta^2k_2k_3}{\alpha^3-\alpha\beta^2k^2} & \frac{\beta k_3}{\alpha^2-\beta^2k^2} & 0 & \frac{-\beta k_1}{\alpha^2-\beta^2k^2} \\ \frac{-\beta^2k_1k_3}{\alpha^3-\alpha\beta^2k^2} & \frac{-\beta^2k_2k_3}{\alpha^3-\alpha\beta^2k^2} & \frac{\alpha^2-\beta^2k_3^2}{\alpha^3-\alpha\beta^2k^2} & -\frac{\beta k_2}{\alpha^2-\beta^2k^2} & \frac{\beta k_1}{\alpha^2-\beta^2k^2} & 0 \\ 0 & \frac{\beta k_3}{\alpha^2-\beta^2k^2} & \frac{-\beta k_2}{\alpha^2-\beta^2k^2} & \frac{\alpha^2-\beta^2k_1^2}{\alpha^3-\alpha\beta^2k^2} & \frac{-\beta^2k_1k_2}{\alpha^3-\alpha\beta^2k^2} & \frac{-\beta^2k_1k_3}{\alpha^3-\alpha\beta^2k^2} \\ \frac{-\beta k_3}{\alpha^2-\beta^2k^2} & 0 & \frac{\beta k_1}{\alpha^2-\beta^2k^2} & \frac{-\beta^2k_1k_2}{\alpha^3-\alpha\beta^2k^2} & \frac{\alpha^2-\beta^2k_2^2}{\alpha^3-\alpha\beta^2k^2} & \frac{-\beta^2k_2k_3}{\alpha^3-\alpha\beta^2k^2} \\ \frac{\beta k_2}{\alpha^2-\beta^2k^2} & -\frac{\beta k_1}{\alpha^2-\beta^2k^2} & 0 & -\frac{\beta^2k_1k_3}{\alpha^3-\alpha\beta^2k^2} & \frac{-\beta^2k_2k_3}{\alpha^3-\alpha\beta^2k^2} & \frac{\alpha^2-\beta^2k_3^2}{\alpha^3-\alpha\beta^2k^2} \end{pmatrix}, \tag{17}$$

where $k=|\mathbf{k}|$. Here, the six degrees of freedom of the real and imaginary parts of $\hat{\mathbf{u}}$ include

$$\text{the compressional mode } \hat{u}^{\parallel} \text{ and the vortical modes } \hat{\mathbf{u}}^{\perp}=\hat{u}^{+}\hat{\mathbf{h}}^{+}+\hat{u}^{-}\hat{\mathbf{h}}^{-}, \tag{18}$$

which in the helical representation of Ref. [3] consist of two real and imaginary degrees of freedom in the $\mathbf{k}$ direction for the former, and two pairs of real and imaginary degrees of freedom for the left- and right-handed helical eigenmodes ($\hat{\mathbf{h}}^{\pm}$). The absolute statistical equilibrium properties of $\hat{u}^{\parallel}$ are exactly the same as those of $\hat{\zeta}$ (see Section C below), but their imprints in real turbulence can differ significantly due

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to varying influences from other factors (such as dissipation) — this has been discussed in depth in previous works [2][3] and will not be repeated here.

If we are only concerned with the difference between helical ($\beta \neq 0$) and non-helical ($\beta = 0$) cases in the total energy spectrum of fluctuations, a simple summation over components yields results consistent with Ref. [3]. Moreover, the correlation function between $\hat{\zeta}$ and $\hat{\mathbf{u}}$ is zero (independent). And, $\hat{\zeta}$'s autocorrelation function takes the same form $1/\alpha$ with and without helicity. Its relative value indeed decreases in the helical case ("helicity tightening/fastening the flow"): this does not reflect the polarization effect of interest here, hence it is not listed above. Below, we discuss the polarization issue:

First, the most striking feature is that helicity introduces rich cross-correlation information beyond $\langle R_i I_i \rangle$ ($\langle \bullet \rangle$ denotes statistical average): the fact that $\langle R_i I_i \rangle$ does not reflect any helicity effect is because the right-hand side of Eq. (6) actually equals $\mathbf{k} \cdot (\mathbf{R} \times \mathbf{I})$. We see that helicity also leads to polarization (distortion of the correlation function with direction on the same wavevector shell of constant k). Such information is so diverse in the correlation functions of real and imaginary parts, and its expected imprints in dissipative turbulence will also be rich. If we further decompose it as in Ref. [3], we find that the compressional mode, like the density mode $\hat{\zeta}$, does not carry polarization effects; that is, the latter is encoded solely by the vortical modes.

It should be noted that avoiding the helical decomposition (18) is not only for easier comparison with the RSF results below — the latter lacks nontrivial pure helical modes — but more essentially, it allows the helicity polarization effect to be manifested. Using the helical decomposition (18) indeed brings computational convenience and cleanly reveals the "fastening" effect; however, conversely, the helicity polarization effect is entirely hidden in the local helical coordinates $\hat{\mathbf{h}}_{\mathbf{k}}^{\pm}$ in wavevector space, so polarization appears as differences between $\hat{u}^+$ and $\hat{u}^-$ [3], rather than being reflected in the statistics of the coefficients $\hat{u}_{\mathbf{k}}^{\pm}$ themselves with respect to the direction of the wavevector. Therefore, the two representations complement each other computationally, and the current correlation matrix carries higher information content.

The polarization effect of helicity in isotropic turbulence is not only completely neutralized in "three-dimensional energy spectra" such as $\langle R^2(\mathbf{k}) = \sum_i R_i^2(\mathbf{k}) \rangle$, but also, due to the $1:1:1$ spatial scale mentioned in the introduction, neutralized in one-dimensional cross-correlation spectra such as $\langle \sum_{|\mathbf{k}|=k} R_1(\mathbf{k}) R_2(\mathbf{k}) \rangle$ (and consequently in physical space statistics like $\langle u_i u_j \rangle$) and thus not manifested.

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This highlights the significance of three-dimensional correlation spectra like $\langle R_i^2(\mathbf{k})\rangle$ and $\langle R_1(\mathbf{k})I_2(\mathbf{k})\rangle$ in reflecting helicity polarization effects while overall isotropy is maintained.

B. Helicity fastening, directionality and polarization in CWDRF turbulence

Equation (9) for 331RSF is expressed in Fourier space as $\hat{u}_3 k_1 \equiv 0 \equiv \hat{u}_3 k_2$, which implies

$$\hat{u}_3(1-\delta_{0,k_1}\delta_{0,k_2}) \equiv 0 \tag{19}$$

from which we have the matrix for the quadratic form of C in terms of the variables of $(R_1, R_2, R_3, I_1, I_2, I_3)$,

$$\begin{pmatrix} \alpha & 0 & 0 & 0 & \beta k_3 & 0 \\ 0 & \alpha & 0 & -\beta k_3 & 0 & 0 \\ 0 & 0 & \alpha\delta_{0,k_1}\delta_{0,k_2} & 0 & 0 & 0 \\ 0 & -\beta k_3 & 0 & \alpha & 0 & 0 \\ \beta k_3 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha\delta_{0,k_1}\delta_{0,k_2} \end{pmatrix}, \tag{20}$$

and the correlation matrix

$$\begin{pmatrix} \frac{\alpha}{\alpha^2-\beta^2k_3^2} & 0 & 0 & 0 & \frac{-\beta k_3}{\alpha^2-\beta^2k_3^2} & 0 \\ 0 & \frac{\alpha}{\alpha^2-\beta^2k_3^2} & 0 & \frac{\beta k_3}{\alpha^2-\beta^2k_3^2} & 0 & 0 \\ 0 & 0 & \frac{\delta_{0,k_1}\delta_{0,k_2}}{\alpha} & 0 & 0 & 0 \\ 0 & \frac{\beta k_3}{\alpha^2-\beta^2k_3^2} & 0 & \frac{\alpha}{\alpha^2-\beta^2k_3^2} & 0 & 0 \\ \frac{-\beta k_3}{\alpha^2-\beta^2k_3^2} & 0 & 0 & 0 & \frac{\alpha}{\alpha^2-\beta^2k_3^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\delta_{0,k_1}\delta_{0,k_2}}{\alpha} \end{pmatrix}. \tag{21}$$

It should be remarked that in the correlation function(s) vanishing with the Kronecker delta(s), such as the $\langle R_3 R_3\rangle = \frac{\delta_{0,k_2}\delta_{0,k_1}}{\alpha}$ in Eq. (21), the situation is actually that the corresponding (random) variable(s) being simply absent, with the matrix (20) for the quadratic form C becoming singular along with their corresponding Kronecker deltas. In such a case, we must remove the corresponding rows and columns to compute the correlations of the remaining variables via a 4×4 matrix inversion, and, to formally unify the presentation of results for both 6×6 and 4×4 inversions in a 6×6 matrix, we have introduced Kronecker deltas to ensure that the (auto)correlations vanish for the corresponding absent variables.

Next, for the 3D2D1D LSF (after only further dimensional reduction of $u_{2,1}\equiv 0$ in 331RSF) which, rather than the 1D2D3D LSF, is chosen to also have u_3 being 1D as in 331RSF, for easier comparison of the results, we have $u_{3,1} = u_{3,2} \equiv 0 \equiv u_{2,1}$, i.e.,

$$\hat{u}_3(1-\delta_{0,k_1}\delta_{0,k_2}) \equiv 0 \equiv \hat{u}_2(1-\delta_{0,k_1}), \tag{22}$$

and then the matrix for the quadratic form of C in terms of the variables of $(R_1, R_2, R_3, I_1, I_2, I_3)$,

$$\begin{pmatrix} \alpha & 0 & 0 & 0 & \beta k_3 \delta_{0,k_1} & 0 \\ 0 & \alpha\delta_{0,k_1} & 0 & -\beta k_3 \delta_{0,k_1} & 0 & 0 \\ 0 & 0 & \alpha\delta_{0,k_1}\delta_{0,k_2} & 0 & 0 & 0 \\ 0 & -\beta k_3 \delta_{0,k_1} & 0 & \alpha & 0 & 0 \\ \beta k_3 \delta_{0,k_1} & 0 & 0 & 0 & \alpha\delta_{0,k_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha\delta_{0,k_1}\delta_{0,k_2} \end{pmatrix}. \tag{23}$$

Thus, with similar treatments and calculations as in the above 331RSF case, we obtain the following 3D2D1D LSF correlation matrix:

$$\begin{pmatrix} \frac{\alpha}{\alpha^2-(\beta k_3)^2\delta_{0,k_1}} & 0 & 0 & 0 & \frac{-\beta k_3\delta_{0,k_1}}{\alpha^2-(\beta k_3)^2} & 0 \\ 0 & \frac{\alpha\delta_{0,k_1}}{\alpha^2-(\beta k_3)^2} & 0 & \frac{\beta k_3\delta_{0,k_1}}{\alpha^2-(\beta k_3)^2} & 0 & 0 \\ 0 & 0 & \frac{\delta_{0,k_1}\delta_{0,k_2}}{\alpha} & 0 & 0 & 0 \\ 0 & \frac{\beta k_3\delta_{0,k_1}}{\alpha^2-(\beta k_3)^2} & 0 & \frac{\alpha}{\alpha^2-(\beta k_3)^2\delta_{0,k_1}} & 0 & 0 \\ \frac{-\beta k_3\delta_{0,k_1}}{\alpha^2-(\beta k_3)^2} & 0 & 0 & 0 & \frac{\alpha\delta_{0,k_1}}{\alpha^2-(\beta k_3)^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\delta_{0,k_1}\delta_{0,k_2}}{\alpha} \end{pmatrix}. \tag{24}$$

The Kronecker delta functions $\delta_{0,\bullet}$ show the directionality, and any direct presence of βk_1 and βk_2 signals the polarization in the helical case ($\beta \neq 0$). For 3D2D1D LSF, the matrix for the quadratic form C contains more δ_{0,k_1} multipliers due to the additional dimensional reduction, compared to the 331RSF case, which leads accordingly to the more δ_{0,k_1} multipliers, denoting the corresponding directionality, in the correlation matrix of the former. It is interesting to note that $\langle R_1^2 \rangle = \langle I_1^2 \rangle = \frac{\alpha}{\alpha^2-(\beta k_3)^2\delta_{0,k_1}}$ also contains the factor δ_{0,k_1}, although $\hat{u}_1$ has no dimensional reduction: this can be associated to the remarkable topology stated in Theorem 4 for the complete absence of swirl (closed streamlines) of vortical LSF due to the further dimensional reduction of u_2 denoted by δ_{0,k_1}, while 331RSF can have swirls in the x_1-x_2 equilibrium plane.

The 223RSF results are of similar fashion but, without fixing a component to be of 1D state, will involve explicitly βk_h with $k_h^2 = k_1^2 + k_2^2$ for very different anisotropic characteristics, so, to avoid the repeat of similar notions with quantitative differences but without the similarity associated to the presence of 1D component, they are left for a separate study together with other more practical issues closely related to the specific flow acoustics.

C. Further discussions

We can, of course, further compute other physical quantities of interest from the previous results. For example, for the parallel mode ${}^{|}u_i = {}^{|}R_i + I_i := \hat{\boldsymbol{u}} \cdot \boldsymbol{k} k_i / k^2$, from the isotropic case we obtain

$$\left\langle {}^{|}R_i^2 \right\rangle = \left\langle \left(\frac{\sum_{j=1}^{3} R_j k_j}{k^2} \right)^2 k_i^2 \right\rangle = \frac{k_i^2}{\alpha k^2} = \left\langle {}^{|}I_i^2 \right\rangle, \tag{25}$$

thus, as in Ref. [3], no polarization in

$$\langle {}^{|}R^2 \rangle := \sum_{i=1}^{3} \langle {}^{|}R_i^2 \rangle = 1$$

but the "fastening" effect is indeed reflected in the relative enhancement of the vortical mode "energy" $\left\langle {}^{\perp}R^2 \right\rangle := \left\langle \sum_{i=1}^{3} R_i^2 - {}^{|}R^2 \right\rangle = \frac{2\alpha}{\alpha^2 - \beta^2 k^2}$.

And, from the 331RSF result, we obtain

$$\left\langle {}^{|}R^2 \right\rangle = \left(\frac{\alpha k_1^2}{\alpha^2 - \beta^2 k_3^2} + \frac{\alpha k_2^2}{\alpha^2 - \beta^2 k_3^2} + \frac{\delta_{0,k_1} \delta_{0,k_2} k_3^2}{\alpha} \right) \frac{k_i^2}{k^4} = \left\langle {}^{|}I_i^2 \right\rangle; \tag{27}$$

while for the LSF case,

$$\left\langle {}^{|}R_i^2 \right\rangle = \left(\frac{k_1^2}{\alpha} + \frac{\delta_{0,k_1} \alpha k_2^2}{\alpha^2 - \beta^2 k_3^2} + \frac{\delta_{0,k_1} \delta_{0,k_2} k_3^2}{\alpha} \right) \frac{k_i^2}{k^4} = \left\langle {}^{|}I_i^2 \right\rangle. \tag{28}$$

Summation over $i = 1, 2, 3$ for $\left\langle {}^{|}R_i^2 \right\rangle$ only turns the factor k_i^2 / k^4 into $1/k^2$ without changing the polarization and directionality in either case. Their mixture greatly enriches the helicity-related information contained in CWDRFs. Such details are very interesting and need careful elucidation:

- The third term $\frac{\delta_{0,k_1} \delta_{0,k_2} k_3^2}{\alpha}$ common in both 331RSF and LSF is contributed by the 1D u_3 for only the compressive mode, leaving no counterpart for the vortical mode, and this term works only when $k_2 = 0 = k_1$ which makes the first and second terms vanish.
- If $k_2 \neq 0 = k_1$, both have the only same second term, subjected to the helicity effect signatured by the presence of β unless $k_3 = 0$.
- If $k_2 = 0 \neq k_1$, both have only the first terms, but are different in that 331RSF is subjected to the helicity effect and that LSF is not (for given k_1, the 3D2D1D-LSF $\left\langle {}^{|}R_i^2 \right\rangle$ is equipartitioned over k_3 but the 331RSF-$\left\langle {}^{|}R_i^2 \right\rangle$ increases with k_3 due to the explicit presence of the β-term): LSF presents less helicity effect.

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- $k_2 \neq 0 \neq k_1$, the 331RSF result contains both the first and second terms, both subjected the helicity effect, while the LSF result contains only the first term with no helicity effect: LSF presents less helicity effect.

We can also compute the vortical mode energy $\langle {}^{\perp}R^2 \rangle := \left\langle \sum_{i=1}^{3} R_i^2 - \mid R^2 \right\rangle$, obtaining

$$\langle {}^{\perp}R^2 \rangle = \frac{\alpha\left(1-k_1^2/k^2\right)}{\alpha^2-\beta^2 k_3^2} + \frac{\alpha\left(1-k_2^2/k^2\right)}{\alpha^2-\beta^2 k_3^2} + \frac{\delta_{0,k_1}\delta_{0,k_2}\left(1-k_3^2/k^2\right)}{\alpha} = \langle {}^{\perp}I^2 \rangle, \tag{29}$$

$$\langle {}^{\perp}R^2 \rangle = \frac{\left(1-k_1^2/k^2\right)}{\alpha} + \frac{\delta_{0,k_1}\alpha\left(1-k_2^2/k^2\right)}{\alpha^2-\beta^2 k_3^2} + \frac{\delta_{0,k_1}\delta_{0,k_2}\left(1-k_3^2/k^2\right)}{\alpha} = \langle {}^{\perp}I^2 \rangle, \tag{30}$$

respectively for 331RSF and 3D2D1D LSF, and analyze explicitly the diverse directionality and polarization, including the coexisting "tightening" effect, which however is already implicitly implied from the above itemized remarks and will not be further elaborated here.

III. Conclusion and outlook

We have analyzed the possible imprints of helicity in isotropic and anisotropic (particularly, componentwise dimension-reduced) turbulence from the perspective of absolute statistical equilibrium. In previous isotropic statistical analyses [3], due to the use of local (complex) helical "coordinates" $\hat{\mathbf{h}}_{\mathbf{k}}^{\pm}$ in wavevector space, the directional aspects of the polarization effect were entirely absorbed by the latter, leaving only the "tightening effect" in the coefficients $\hat{u}^{\pm}$ without explicit wavevector direction information. This paper "restores" these directional details, by adopting the conventional global wavevector coordinates, and uses them as a reference for comparison with 331RSF and 3D2D1D LSF results.

Comparing Eqs. (21) and (24) with (17), the apparently vanishing cross-correlations (even with $\beta \neq 0$) due to dimensional reduction implies, first of all, less helicity effect, and, as already pointed out in Sec. IIC, further dimensional reduction from 331RSF to 3D2D1D LSF leads to even less helicity effect. But, on the other hand, dimensional reduction also makes the helicity effects richer, in the sense of mixing the fastening effect with the polarization and directionality effects.

Other CWDRFs such as the 223RSF should similarly, yet with specific differences, exhibit rich mixtures of helicity effects including tightening, directionality, and polarization. Considering the coupling of force, heat, sound, etc., these results have broad potential value. Particularly from the perspective of flow acoustics, they provide an important foundation and inspiration for flow-control design and inverse problems. Of course, all theoretical assumptions and results await verification by numerical and physical measurements and observations (which is also why, at this still not fully mature stage, we do not exhaustively and systematically compare all possibilities).

As with the numerical experiments and verification in Refs. [5][6] regarding the theoretical conjecture of the "helicity fastening effect" inspired by absolute statistical equilibrium analysis in [3], the imprints of all these properties in real scenarios require, and can undergo further systematic numerical and experimental testing. Integrating with other disciplines, it is promising to develop specific techniques motivated by our results, related to actively controlling the helicity of flow fields around aircraft, drones, etc., to "steer" noise energy toward specific directions (e.g., away from residential areas or avoiding sensitive sensors), modulating and filtering self-noise interference, or optimizing shapes to "sculpt" wakes, as well as applying to inverse problems such as inferring the speed, size, and even shape of moving bodies, and, further more, optimizing hydrophones (associated to "vector acoustics" [18]), array configurations, and signal processing.

On the other hand, since thermalization effects of interactions in non-equilibrium systems are often complex, involving competition and collaboration with other dynamical elements and specific spatiotemporal scales and environments, a certain degree of caution and acuity in appropriately adjusting suitable theories for different regimes is necessary. In CWDRFs, in some cases, different physical components may have very different spatial interaction scales, respond to a particular factor at drastically different speeds, thus different degrees of (partial) thermalization.

Regarding the non-equilibrium statistical dynamical ensemble theoretical model for incompressible flow with energy-helicity dual constraints also addressed in Ref. [3], it can, in principle, be extended to the CWDRFs considered here for theoretical and numerical analysis. However, this requires dedicated discussion in a separate study. In summary, this paper takes the first step towards comprehensively breaking through the framework and content of Ref. [3], achieving a deeper, more detailed, and realistic analysis. This opens up a broader theoretical and applied space, where systematic numerical and physical experimental research holds great promise.

In the end, we take a step back to the fundamentals of theoretical fluid mechanics. At the very basic level, by studying CWDRF statistics, we are led to the discovery of helicity invariance in a broader variety of models or the extension by removing its conventional association with the traditional (local) mass conservation law (Appendix A). The detailed conservation in each interacting triad of the corresponding hydrodynamic-type systems or its Galerkin-truncation version, as shown in Appendix B, is not only the basis of the statistical mechanics but also a signature of its unique importance and wide impact in various aspects of flow physics.

Appendix A. Helicity invariance of inviscid flows in a material volume

The conservation of helicity, defined as $\mathcal{H} = \int \mathbf{u} \cdot \boldsymbol{\omega} dV$ with $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, plays a fundamental role in fluid dynamics. Moffatt [11] proved helicity invariance for ideal barotropic flows, using the mass conservation (continuity equation) in the form $D(\rho dV)/Dt = 0$ (see also Khesin & Chekanov's [19] proof with an elegant and powerful treatment for more general purposes). This note demonstrates that the helicity conservation does not actually rely on the continuity equation, the latter being replaced by the Reynolds transport theorem for a material volume $V(t)$ in our new proof following the fashion of that by Moffatt [who tracked the material volume]. Consequently, helicity is conserved in a broader class of hydrodynamic-type equations, including the pressureless ideal Burgers equation and the CWDRFs where the continuity equation is absent or modified.

Consider a general velocity field $\mathbf{u}(\mathbf{x}, t)$ evolving according to

$$\frac{D\mathbf{u}}{Dt} \equiv \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{F}, \tag{A1}$$

where $\mathbf{F}$ is a force per unit mass. The vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ satisfies

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - \boldsymbol{\omega}(\nabla \cdot \mathbf{u}) + \nabla \times \mathbf{F}. \tag{A2}$$

The helicity density $h = \mathbf{u} \cdot \boldsymbol{\omega}$ evolves as

$$\frac{Dh}{Dt} = \boldsymbol{\omega} \cdot \mathbf{F} + \mathbf{u} \cdot \nabla \times \mathbf{F} - h(\nabla \cdot \mathbf{u}) + \mathbf{u} \cdot [(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}]. \tag{A3}$$

Using the Reynolds transport theorem for a material volume $V(t)$, the total helicity $\mathcal{H} = \int_{V(t)} h dV$ changes as

$$\frac{d\mathcal{H}}{dt} = \int_{V(t)} \left(\frac{Dh}{Dt} + h(\nabla \cdot \mathbf{u}) \right) dV = \int_{V(t)} (\boldsymbol{\omega} \cdot \mathbf{F} + \mathbf{u} \cdot \nabla \times \mathbf{F} + \mathbf{u} \cdot [(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}]) \, dV. \tag{A4}$$

Notice that the $h(\nabla \cdot \mathbf{u})$ terms cancel, irrespective of the continuity equation. If $\mathbf{F} = -\nabla\Phi$ for some scalar potential Φ, then $\nabla \times \mathbf{F} = 0$ and $\boldsymbol{\omega} \cdot \mathbf{F} = -\boldsymbol{\omega} \cdot \nabla\Phi = -\nabla \cdot (\Phi\boldsymbol{\omega})$, with $\nabla \cdot \boldsymbol{\omega} = 0$; the latter also leads to $\mathbf{u} \cdot [(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}] = \boldsymbol{\omega} \cdot \nabla u^2/2 = \nabla \cdot (\boldsymbol{\omega} u^2)/2$. Equation (A4) becomes

$$\frac{d\mathcal{H}}{dt} = \int_{\partial V(t)} (u^2/2 - \Phi)\boldsymbol{\omega}\cdot\mathbf{n}dS. \tag{A5}$$

Thus, $d\mathcal{H}/dt = 0$ if $\boldsymbol{\omega}\cdot\mathbf{n} = 0$ on the boundary (e.g., periodic domain or decay at infinity). The mass conservation has not been used.

For ideal barotropic flow with conservative body forces, the equation of motion can be written as

$$\frac{D\mathbf{u}}{Dt} = -\nabla(H + \Phi), \tag{A6}$$

where $H = \int dp/\rho$ and Φ is the potential of the conservative body forces. This force is a gradient, so the helicity evolution equation (A4) becomes

$$\frac{d\mathcal{H}}{dt} = \int_{V(t)} (-\boldsymbol{\omega}\cdot\nabla(H+\Phi))\, dV = -\int_{\partial V(t)} (H+\Phi)\boldsymbol{\omega}\cdot\mathbf{n}dS, \tag{A7}$$

where we have used $\nabla\cdot\boldsymbol{\omega} = 0$. Thus, $d\mathcal{H}/dt = 0$ under boundary conditions $\boldsymbol{\omega}\cdot\mathbf{n} = 0$. For flows in a periodic box (without boundary), the moving volume is the whole torus and the proof applies. Note that this derivation does not require the continuity equation. Moffatt's original proof used the mass conservation in the form $D(\rho dV)/Dt = 0$ to simplify the calculation, but as shown here, it is not necessary.

For the three-dimensional Burgers equation (pressureless, inviscid, without external forces) and the CWDRF equations studied here, it is only a matter of specialization or reduction.

The invariance of helicity in various ideal (compressible) flows irrespective of the mass equation can also be proved directly with the Eulerian description, as shown elsewhere [16]. This fact supports the persistence of helicity (fastening) effects across diverse flow models.

Our proof implies that $\nabla\cdot\mathbf{u} = 0$ be unnecessary for the helicity conservation in incompressible flows (but Moffatt's not), which now leads us to re-examine Frisch's proof [20] which turns out to be extremely elegant (without using $\nabla\cdot\mathbf{u} = 0$!) and which applies also to the Burgers equation and to our CWDRFs. However, his writing is so concise that the use of $\nabla\cdot\mathbf{u} = 0$ in transforming the viscous term might cause the confusion that incompressibility be needed for the conservation law.

Appendix B. Helicity invariance of Galerkin-truncated inviscid flows

As indicated in the end of the main body of this article, the helicity invariance is of a very fundamental level of importance beyond the main statistical analysis of CWDRFs, so discussions below are not restricted to the mere purpose of proving Galerkin-truncated version in the latter.

Consider the three-dimensional Burgers equation in a periodic box,

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = 0,$$

and focus on a pair of triad-interaction wavevectors $\pm\mathbf{k}, \pm\mathbf{p}, \pm\mathbf{q}$, satisfying

$$\mathbf{k} + \mathbf{p} + \mathbf{q} = 0. \tag{B1}$$

By reality, the Fourier coefficient satisfies $\hat{\mathbf{u}}_{-\mathbf{s}} = \hat{\mathbf{u}}_{\mathbf{s}}^*$. The dynamical equations for these coefficients are

$$\partial_t \hat{\mathbf{u}}_{\mathbf{k}} = \hat{i}(\mathbf{p}\cdot\hat{\mathbf{u}}_{\mathbf{q}}^*)\hat{\mathbf{u}}_{\mathbf{p}}^* + \hat{i}(\mathbf{q}\cdot\hat{\mathbf{u}}_{\mathbf{p}}^*)\hat{\mathbf{u}}_{\mathbf{q}}^*, \tag{B2}$$
$$\partial_t \hat{\mathbf{u}}_{\mathbf{p}} = \hat{i}(\mathbf{q}\cdot\hat{\mathbf{u}}_{\mathbf{k}}^*)\hat{\mathbf{u}}_{\mathbf{q}}^* + \hat{i}(\mathbf{k}\cdot\hat{\mathbf{u}}_{\mathbf{q}}^*)\hat{\mathbf{u}}_{\mathbf{k}}^*, \tag{B3}$$
$$\partial_t \hat{\mathbf{u}}_{\mathbf{q}} = \hat{i}(\mathbf{k}\cdot\hat{\mathbf{u}}_{\mathbf{p}}^*)\hat{\mathbf{u}}_{\mathbf{k}}^* + \hat{i}(\mathbf{p}\cdot\hat{\mathbf{u}}_{\mathbf{k}}^*)\hat{\mathbf{u}}_{\mathbf{p}}^*, \tag{B4}$$

where $\hat{i}^2 = -1$. The total helicity (up to a constant volume factor) is defined as

$$\mathcal{H} = \hat{i}\mathbf{k}\cdot(\hat{\mathbf{u}}_{\mathbf{k}}\times\hat{\mathbf{u}}_{\mathbf{k}}^*) + \hat{i}\mathbf{p}\cdot(\hat{\mathbf{u}}_{\mathbf{p}}\times\hat{\mathbf{u}}_{\mathbf{p}}^*) + \hat{i}\mathbf{q}\cdot(\hat{\mathbf{u}}_{\mathbf{q}}\times\hat{\mathbf{u}}_{\mathbf{q}}^*).$$

Its time derivative is

$$\frac{d\mathcal{H}}{dt} = \sum_{\mathbf{s}\in\{\mathbf{k},\mathbf{p},\mathbf{q}\}} \hat{i}\mathbf{s}\cdot\left(\partial_t\hat{\mathbf{u}}_{\mathbf{s}}\times\hat{\mathbf{u}}_{\mathbf{s}}^* + \hat{\mathbf{u}}_{\mathbf{s}}\times\partial_t\hat{\mathbf{u}}_{\mathbf{s}}^*\right).$$

Substituting (B2)–(B4) and their complex conjugates, after straightforward algebra we obtain

$$\frac{d\mathcal{H}}{dt} = -2Re(S_H),$$

where

$$\begin{aligned} S_H = &= (\mathbf{p}\cdot\hat{\mathbf{u}}_{\mathbf{q}}^*)[\mathbf{k},\hat{\mathbf{u}}_{\mathbf{p}},\hat{\mathbf{u}}_{\mathbf{k}}] + (\mathbf{q}\cdot\hat{\mathbf{u}}_{\mathbf{p}}^*)[\mathbf{k},\hat{\mathbf{u}}_{\mathbf{q}},\hat{\mathbf{u}}_{\mathbf{k}}] \\ &+(\mathbf{q}\cdot\hat{\mathbf{u}}_{\mathbf{k}}^*)[\mathbf{p},\hat{\mathbf{u}}_{\mathbf{q}},\hat{\mathbf{u}}_{\mathbf{p}}] + (\mathbf{k}\cdot\hat{\mathbf{u}}_{\mathbf{q}}^*)[\mathbf{p},\hat{\mathbf{u}}_{\mathbf{k}},\hat{\mathbf{u}}_{\mathbf{p}}] \\ &+(\mathbf{k}\cdot\hat{\mathbf{u}}_{\mathbf{p}}^*)[\mathbf{q},\hat{\mathbf{u}}_{\mathbf{k}},\hat{\mathbf{u}}_{\mathbf{q}}] + (\mathbf{p}\cdot\hat{\mathbf{u}}_{\mathbf{k}}^*)[\mathbf{q},\hat{\mathbf{u}}_{\mathbf{p}},\hat{\mathbf{u}}_{\mathbf{q}}], \end{aligned} \tag{B5}$$

and $[\mathbf{a},\mathbf{b},\mathbf{c}] = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ denotes the scalar triple product. The permutative symmetry in the expression together with Eq. (B1) indicates that $S_H = 0$, which is indeed the case as shown in Appendix C.

Hence $\frac{d\mathcal{H}}{dt} = 0$, i.e., the helicity is conserved in each triad interaction of the 3D Burgers equation, which is followed immediately by the conclusion of helicity conservation for arbitrary Galerkin truncation.

Now consider the three-dimensional barotropic ideal fluid in a periodic box, governed by

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla\Phi,$$

where $\Phi = \int dp/\rho$ is the specific enthalpy H (and the potential of other conservative force/acceleration, as indicated by our choice of the notation).

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Projecting the momentum equation onto the Fourier mode $\mathbf{k}$ gives

$$\partial_t \hat{\mathbf{u}}_{\mathbf{k}} = -\hat{\imath}\mathbf{k}\hat{\Phi}_{\mathbf{k}} + \mathcal{N}_{\mathbf{k}},$$

where the nonlinear term $\mathcal{N}_{\mathbf{k}}$ is the same as for the Burgers case compared to which we only prove the conservation by the linear enthalpy term.

For each wavevector $\mathbf{s}$ we have

$$\hat{\imath}\mathbf{s} \cdot \Big[(-\hat{\imath}\mathbf{s}\hat{\Phi}_{\mathbf{s}}) \times \hat{\mathbf{u}}_{\mathbf{s}}^*\Big] = \mathbf{s} \cdot \Big(\mathbf{s}\hat{\Phi}_{\mathbf{s}} \times \hat{\mathbf{u}}_{\mathbf{s}}^*\Big) = \hat{\Phi}_{\mathbf{s}}\mathbf{s} \cdot (\mathbf{s} \times \hat{\mathbf{u}}_{\mathbf{s}}^*) = 0,$$

because $\mathbf{s} \times \hat{\mathbf{u}}_{\mathbf{s}}^*$ is perpendicular to $\mathbf{s}$. Similarly,

$$\hat{\imath}\mathbf{s} \cdot \Big[\hat{\mathbf{u}}_{\mathbf{s}} \times (\hat{\imath}\mathbf{s}\hat{\Phi}_{\mathbf{s}}^*)\Big] = -\hat{\Phi}_{\mathbf{s}}^*\mathbf{s} \cdot (\hat{\mathbf{u}}_{\mathbf{s}} \times \mathbf{s}) = 0.$$

Hence the pressure terms give no contribution to $\frac{d\mathcal{H}}{dt}$.

Appendix C. Detailed calculations showing $S_H = 0$

To show that S_H in Eq. (B5) vanishes, we introduce the shorthand

$$\mathbf{a} = \hat{\mathbf{u}}_{\mathbf{k}}, \mathbf{b} = \hat{\mathbf{u}}_{\mathbf{p}}, \mathbf{c} = \hat{\mathbf{u}}_{\mathbf{q}}, \mathbf{K} = \mathbf{k}, \mathbf{P} = \mathbf{p}.$$

Using the properties of the triple product and the resonance condition $\mathbf{K} + \mathbf{P} + \mathbf{q} = 0$, we rewrite S_H as

$$\begin{aligned} S_H = &= (\mathbf{P}\cdot\mathbf{c})[\mathbf{K},\mathbf{b},\mathbf{a}] + (\mathbf{q}\cdot\mathbf{b})[\mathbf{K},\mathbf{c},\mathbf{a}] + (\mathbf{q}\cdot\mathbf{a})[\mathbf{P},\mathbf{c},\mathbf{b}] \\ &+(\mathbf{K}\cdot\mathbf{c})[\mathbf{P},\mathbf{a},\mathbf{b}] + (\mathbf{K}\cdot\mathbf{b})[\mathbf{q},\mathbf{a},\mathbf{c}] + (\mathbf{P}\cdot\mathbf{a})[\mathbf{q},\mathbf{b},\mathbf{c}]. \end{aligned}$$

Expressing each triple product as a scalar–vector combination, e.g. $[\mathbf{K},\mathbf{b},\mathbf{a}] = \mathbf{K}\cdot(\mathbf{a}\times\mathbf{b})$, and using the antisymmetry of the cross product, we obtain

$$\begin{aligned} S_H = &= (\mathbf{P}\cdot\mathbf{c})\mathbf{K}\cdot(\mathbf{a}\times\mathbf{b}) - (\mathbf{q}\cdot\mathbf{b})\mathbf{K}\cdot(\mathbf{c}\times\mathbf{a}) + (\mathbf{q}\cdot\mathbf{a})\mathbf{P}\cdot(\mathbf{b}\times\mathbf{c}) \\ &-(\mathbf{K}\cdot\mathbf{c})\mathbf{P}\cdot(\mathbf{a}\times\mathbf{b}) - (\mathbf{K}\cdot\mathbf{b})\mathbf{q}\cdot(\mathbf{a}\times\mathbf{c}) - (\mathbf{P}\cdot\mathbf{a})\mathbf{q}\cdot(\mathbf{b}\times\mathbf{c}). \end{aligned}$$

Now we apply the vector identity $(\mathbf{X}\cdot\mathbf{Z})\mathbf{Y} - (\mathbf{Y}\cdot\mathbf{Z})\mathbf{X} = (\mathbf{X}\times\mathbf{Y})\times\mathbf{Z}$ to pair the terms. Define $\mathbf{W} = \mathbf{P}\times\mathbf{K}$. Using the resonance condition, $\mathbf{q} = -\mathbf{K} - \mathbf{P}$, one finds $\mathbf{K}\times\mathbf{q} = \mathbf{W}$ and $\mathbf{q}\times\mathbf{P} = \mathbf{W}$. Grouping the first with the fourth term, the second with the fifth, and the third with the sixth, we obtain

$$S_H = (\mathbf{W}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{b}) + (\mathbf{W}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{a}) + (\mathbf{W}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{c}).$$

Applying the identity $(\mathbf{U}\times\mathbf{V})\cdot(\mathbf{X}\times\mathbf{Y}) = (\mathbf{U}\cdot\mathbf{X})(\mathbf{V}\cdot\mathbf{Y}) - (\mathbf{U}\cdot\mathbf{Y})(\mathbf{V}\cdot\mathbf{X})$ to each of the three products gives

$$
\begin{aligned}
(\mathbf{W}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{b}) &= (\mathbf{W}\cdot\mathbf{a})(\mathbf{c}\cdot\mathbf{b}) - (\mathbf{W}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{a}),\\
(\mathbf{W}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{a}) &= (\mathbf{W}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{a}) - (\mathbf{W}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{c}),\\
(\mathbf{W}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{c}) &= (\mathbf{W}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{c}) - (\mathbf{W}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{b}).
\end{aligned}
$$

Summing these three expressions, all terms cancel pairwise, yielding $S_H = 0$.

Statements and Declarations

Data availability

The data that support the findings of this study are available from the corresponding author upon request.

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Footnotes

[1] This book provides a comprehensive discussion of different types of anisotropy in turbulent flows, including distinctions and connections between "directional anisotropy," "polarization anisotropy," "dimensional anisotropy," and "componental anisotropy."

[2] This work finds that Galerkin truncation can, in fact, admit traveling wave and quasi-periodic solutions, which can generically evolve into "longulent states" with soliton-like characteristics, rather than absolute statistical equilibrium. This means the nonlinear dispersion effect of Galerkin truncation depends on initial conditions: it may act as a persistent destabilizing mechanism leading to complete thermalization, or ultimately as a stabilizing mechanism maintaining the system in a (quasi-)ordered state, potentially related to turbulence intermittency. This does not contradict or challenge the present study but rather complements it, as both thermalization and intermittency mechanisms coexist in turbulence. Furthermore, even the development of "longulent states" requires certain instability or thermalization mechanisms, and imprints of the tendency toward absolute statistical equilibrium may still appear during the process. As for "laminar flows," they belong to a different framework from the turbulence we study and do not negate our analysis and understanding of turbulence.

[3] This author appeared to be also the first researcher originally applying general 223-type Schur matrix in fluid dynamics [C. J. Keylock, "Synthetic Velocity Gradient Tensors and the Identification of Statistically Significant Aspects of the Structure of Turbulence," Physical Review Fluids 2, 084607 (2017); C. J. Keylock, "The Schur Decomposition of the Velocity Gradient Tensor for Turbulent Flows," Journal of Fluid Mechanics 848, 876 – 905 (2018).]

[4] This author and collaborators seemed to be the first in using a very special form of the 223-type real Schur matrix in hydrodynamics [Z. Li, X. Zhang, and F. He, "Evaluation of vortex criteria by virtue of the quadruple decomposition of velocity gradient tensor," Acta Physica Sinica 63, 054704 – 054704 (2014)].

References

1. a, b*Lee TD (1952). "On Some Statistical Properties of Hydrodynamical and Magneto-Hydrodynamical Fields." Q Appl Math.* ***10****:69.*
2. a, b, c, d, e, f*Kraichnan RH (1955). "Statistical Theory of Turbulence for a Compressible Fluid." J Acoust Soc Am.* ***27****:438.*
3. a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p*Zhu JZ (2020). "Chiral Turbulence: Equilibrium and Non-Equilibrium Ensembles for Time-Reversible Systems." Sci Sin-Phys Mech Astron [Science China-Physics, Mechanics & Astronomy].* ***50****:040002; Zhu JZ (2016). "Isotropic Polarization of Compressible Flows." J Fluid Mech.* ***787****:440.*
4. a, b*Zhu JZ (2021). "Compressible Helical Turbulence: Fastened-Structure Geometry and Statistics." Phys Plasmas.* ***28****:032302.*
5. a, b, c, d, e, f, g, h*Zhu JZ (2021). "Real Schur Flow Computations, Helicity Fastening Effects and Bagua-Pattern Cyclones." Phys Fluids.* ***33****:107112.*
6. a, b*Yang Y, Zhu JZ (2022). "Direct Numerical Simulations of Compressible Helical Turbulence and Spectral Analysis of Energy Transfer." Phys Fluids.* ***34****:045113.*
7. a, b, c, d, e*Zhu JZ (2018). "Vorticity and Helicity Decompositions and Dynamics with Real Schur Form of the Velocity Gradient." Phys Fluids.* ***30****:031703.*
8. a, b, c, d, e*Zhu JZ (2021). "Thermodynamic and Vortic Structures of Real Schur Flows." J Math Phys.* ***62****:083101.*
9. a, b*Sagaut P, Cambon C (2008). Homogeneous Turbulence Dynamics. Cambridge: Cambridge University Press.*

 doi.org/10.32388/O4U7T7

10. ^ *Zhu JZ (2024). "Longons from the Nonlinear Dispersion of Galerkin Regularization." Preprint, https://arxiv.org/abs/2404.08583.*

11. a, b *Moffatt K (1969). "The Degree of Knottedness of Tangled Vortex Lines." J Fluid Mech. **35**:117.*

12. ^ *Kraichnan RH (1973). "Helical Turbulence and Absolute Equilibrium." J Fluid Mech. **59**:745.*

13. a, b, c *Zhu JZ. "Structures and Dynamics of Lone Schur Flows with Vorticity but no Swirls." Preprint.*

14. ^ *Keylock CJ (2022). "Studying Turbulence Structure Near the Wall in Hydrodynamic Flows: An Approach Based on the Schur Decomposition of the Velocity Gradient Tensor." J Hydrodyn. **34**:806.*

15. ^ *Li Z (2024). "Schur Forms and Normal-Nilpotent Decompositions." Appl Math Mech [Applied Mathematics and Mechanics]. **45**(9):1200–1211.*

16. a, b, c, d *Zhu JZ (2026). "Component-Wise Dimensionally Reduced Flows and Helicity Conservation." Preprint.*

17. ^ *Arnold VI (1992). Ordinary Differential Equations. 3rd ed. Springer.*

18. ^ *Shchurov VA (2011). Vector Acoustics of the Ocean. Translated by Jia ZF. Beijing: National Defense Industry Press.*

19. ^ *Khesin B, Chekanov Yu V (1989). "Invariants of the Euler Equations for Ideal or Barotropic Hydrodynamics and Superconductivity in D Dimensions." Phys D. **40**:119–131.*

20. ^ *Frisch U (1995). Turbulence. Cambridge: Cambridge University Press.*

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