Qeios

Can the electromagnetic fields form tensors if $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu$?

Jerrold Franklin*
Department of Physics
Temple University, Philadelphia, PA 19122

Abstract

One of the bases of relativistic electromagnetism is that the electromagnetic fields **E** and **B** are components of an electromagnetic field tensor $F^{\mu\nu}$. There is a simple proof of this in the absence of a polarizable medium. It has sometimes been assumed that the fields, **E** and **B**, and their associated fields, $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu$, are also components of tensors in a linearly polarizable medium. In this paper, we question that assumption, and show that a proof that they are components of a tensor fails. Consequently, the fields **E**, **B**, **D** and **H** are not components of a relativistic tensor in a linearly polarizable medium.

1 Introduction

It is a standard basis of relativistic electromagnetism that the electromagnetic fields **E** and **B** combine as elements in the electromagnetic field tensor[1]

$$[\mathrm{F}] = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{1}$$

*Internet address: Jerry.F@TEMPLE.EDU
[1]We are using Gaussian units with $c = 1$.

https://doi.org/10.32388/OQNE8N

Then, a Lorentz transformation to a system moving with velocity $\mathbf{v}$ with respect to the original system results in a transformed tensor

$$[\mathrm{F}'] = \begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & -B'_z & B'_y \\ E'_y & B'_z & 0 & -B'_x \\ E'_z & -B'_y & B'_x & 0 \end{pmatrix}, \tag{2}$$

where the primed elements are components of the electromagnetic fields in the moving system.

It has sometimes been assumed (without proof) that the associated fields, $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu$, (for a linearly polarizable medium) are also components of a relativistic tensor [1–4],

$$[\mathrm{G}] = \begin{pmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & -H_z & H_y \\ D_y & H_z & 0 & -H_x \\ D_z & -H_y & H_x & 0 \end{pmatrix}. \tag{3}$$

However, while there is a simple proof in textbooks that the $\mathbf{E}$ and $\mathbf{B}$ matrix in Eq. (1) represents a relativistic tensor if the constants ϵ and μ =1, I have seen no such proof for the corresponding $\mathbf{D}$ and $\mathbf{H}$ matrix, or even for the $\mathbf{E}$ and $\mathbf{B}$ matrix in a polarizable medium. The purpose of this paper is to see if such a proof can be made.

2 Electromagnetic Fields in a Linearly Polarizable Medium

First, we give a brief summary of the proof that $F^{\mu\nu}$ is a second-rank tensor if ϵ and μ =1. Starting from Maxwell's two homogenous equations, the electric and magnetic fields can be given in terms of scalar and vector potentials as

$$\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}, \tag{4}$$
$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{5}$$

Putting these fields into Maxwell's inhomogeneous equations, in the Lorenz gauge,

$$\nabla \cdot \mathbf{A} + \partial_t\phi = 0, \tag{6}$$

leads to wave equations for ϕ and $\mathbf{A}$:

$$(\partial_t^2 - \nabla^2)\phi = 4\pi\rho \tag{7}$$
$$(\partial_t^2 - \nabla^2)\mathbf{A} = 4\pi\mathbf{j}. \tag{8}$$

The wave equations can be put into four-component form as

$$\partial_\nu\partial^\nu A^\mu = 4\pi j^\mu, \tag{9}$$
$$\text{with} \quad \partial_\mu = (\partial_t, \nabla), \tag{10}$$
$$A^\mu = (\phi, \mathbf{A}), \tag{11}$$
$$\text{and} \quad j^\mu = (\rho, \mathbf{j}). \tag{12}$$

At this point only ∂_μ is known to be a covariant four-vector because the calculus rule, $dx'^\nu = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right) dx^\mu$, for the transformation of the coordinate differentials dx^μ, shows that $\partial_\mu dx^\mu$ is a scalar.

The charge and current densities must satisfy the continuity equation,

$$\nabla \cdot \mathbf{j} + \partial_t\rho = \partial_\mu j^\mu = 0, \tag{13}$$

to conserve charge in any Lorentz frame. This means that $\partial_\mu j^\mu$ must be a Lorentz scalar. Then, j^μ must be a four-vector because its contraction with the four-vector ∂_μ is a scalar. Then, we see from Eq. (9) that the four-potential A^μ must be a four-vector since $\partial_\nu\partial^\nu$ is a scalar and $\mathbf{j}^\mu$ is a four-vector.

This means that forming

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{14}$$

as the difference of the direct products of two four-vectors shows that $F^{\mu\nu}$ is a second rank tensor. Putting Eq. (14) into matrix form shows that the second-rank tensor $F^{\mu\nu}$ in Eq. (14) is the one given by Eq. (1).

Can we complete a similar proof for a linearly polarizable medium where $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu$? Then, Maxwell's equations are:

$$\boldsymbol{\nabla}\cdot(\epsilon\mathbf{E}) = 4\pi\rho \tag{15}$$
$$\boldsymbol{\nabla}\times\mathbf{E} = -\partial_t\mathbf{B} \tag{16}$$
$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \tag{17}$$
$$\boldsymbol{\nabla}\times(\mathbf{B}/\mu) = 4\pi\mathbf{j} + \partial_t(\epsilon\mathbf{E}). \tag{18}$$

We can still use the fact that $\mathbf{B}$ has zero divergence to introduce a vector potential $\mathbf{A}$, in terms of which $\mathbf{B}$ is given by

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}, \tag{19}$$

and the curl $\mathbf{E}$ equation still leads to

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \partial_t\mathbf{A}. \tag{20}$$

We note that, if the assumption that $F^{\mu\nu}$ in Eq. (1) and $G^{\mu\nu}$ in Eq. (3) were relativistic tensors was correct, then Eqs. (15)-(20) would be the same in every Lorentz system, even if some are not manifestly covariant.

Putting Eqs. (19) and (20) into the two Maxwell's equations with sources, now leads to new equations for the potentials,

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\epsilon\mathbf{E}) = 4\pi\rho &= -\epsilon\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi + \partial_t\mathbf{A}), \\
4\pi\rho &= -\epsilon[\nabla^2\phi + \boldsymbol{\nabla}\cdot(\partial_t\mathbf{A})]. \qquad (21)\\
\boldsymbol{\nabla}\times(\mathbf{B}/\mu) &= 4\pi\mathbf{j} + \partial_t(\epsilon\mathbf{E}), \\
4\pi\mathbf{j} &= \frac{1}{\mu}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) + \epsilon\partial_t[\boldsymbol{\nabla}\phi + \partial_t\mathbf{A}], \\
4\pi\mathbf{j} &= \frac{1}{\mu}[\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^{\mathbf{2}}\mathbf{A}] + \epsilon[\partial_t\boldsymbol{\nabla}\phi + \partial_t^2\mathbf{A}]. \qquad (22)
\end{aligned}$$

The Lorenz gauge condition of Eq. (6) can still be applied (It would be invariant if $F^{\mu\nu}$, as given by Eq. (14), were a tensor.), and reduces Eqs. (21) and (22) to

$$\begin{aligned}
\epsilon[\nabla^2\phi - \partial_t^2\phi] &= -4\pi\rho \qquad (23)\\
\frac{1}{\mu}[\nabla^2\mathbf{A} - \epsilon\mu\partial_t^2\mathbf{A} - (1-\epsilon\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot A)] &= -4\pi\mathbf{j}. \qquad (24)
\end{aligned}$$

These two equations are wave equations for the scalar and vector potentials, modified to include the permittivity and permeability constants. However, while the combination $(\rho, \mathbf{j})$ on the right-hand side of Eqs. (23) and (24) forms a four-vector, the operators acting on ϕ and $\mathbf{A}$ on the left-hand sides do not constitute a four-scalar. Consequently, Eqs. (23) and (24) show that the combination $(\phi, \mathbf{A})$ cannot be a four-vector.

If A^ν is not a relativistic four-vector, the product, $\partial^\mu A^\nu$, of A^ν with the relativistic four-vector, ∂^μ, cannot be part of a second rank tensor. Consequently, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is not a tensor for a linearly polarizable material.

This means that even **E** and **B** cannot now be Lorentz transformed to a moving frame. The matrix formed by **D** and **H** also cannot be a tensor, because they, too, are based on the product, $\partial^\mu A^\nu$, just with each term modified by either ϵ or $1/\mu$.

3 Conclusion

Our conclusion is that the electromagnetic fields **E,B,D,H** cannot be combined in relativistic tensors if **D** and **H** are related to **E** and **B** by the linear constitutive relations $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu$. This follows because the potential $(\phi, \mathbf{A})$ is then not a relativistic four-vector. Although the electromagnetic field matrix [F] can appear to be Lorentz transformed to a matrix [F′], the vectors $\mathbf{E}'$ and $\mathbf{B}'$ in Eq. (2) would not be the electric and magnetic fields in the moving frame.

This means there is no way in special relativity to calculate the electromagnetic fields in any case where there is a moving linearly polarizable medium. Even calculating the **E** and **B** fields in the rest system of the polarizable medium, and then trying to Lorentz transforming the fields to a moving system would not work because the **E** and **B** fields themselves cannot be combined in a relativistic tensor in the presence of a polarizable medium. Some method other than the Lorentz transformation would have to be used to treat the electromagnetism of a moving polarizable medium. For instance, non-relativistic treatment to first order in the velocity could be used

References

[1] Minkowski, H: Die Grundgleichungen far die elektromagnetischen Vorgänge in bewegten Körpern. Nachrichten von der Kgl. Gesellschaft der Wissenschaften zu Göttingen. 53-111 (1908)

[2] Abraham, M.: /https:On the Electrodynamics of Minkowski.en.wikisource.org/wiki/Translation. (1910)

[3] Panofsky, W.K.H, Phillips, M.: Classical Electricity and Magnetism, 2nd Edition, Section 18-5 (Addison-Wesley), (1962)

[4] Jackson, J.D.: Classical Electrodynamics, 3rd Edition, page 557 Wiley (1998)