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CLOSED SUBSPACES OF MULTI-INDEXED SEQUENCES OVER A FIELD

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Abstract. This paper investigates the closed subsets of the vector space of multi-indexed sequences over a field with respect to the pointwise convergence topology. These closed subsets are shown to be the orthogonal complements of the subspaces of sequences with finite support in the vector space. The notion of orthogonality is established using a natural scalar product on these vector spaces.

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Introduction

In this paper, we investigate the closed subspaces of the vector space $\mathbb{F}^{\mathbb{N}^r}$, where $\mathbb{F}$ is a commutative field and $r \geqslant 1$ is an integer, by exploring their closed subsets. These closed subspaces serve as valuable tools for characterizing discrete linear dynamical systems [5, 7, 8]. The fundamental concept we employ is duality [1, 4, 5], which we apply to the vector spaces $\mathbb{F}^{\mathbb{N}^r}$ and $\mathbb{F}^{(\mathbb{N}^r)}$, utilizing a scalar product [1, 3, 4, 5].

The structure of this paper is organized as follows:

In Section 1, we introduce the vector spaces $\mathbb{F}^{\omega}$ and $\mathbb{F}^{(\omega)}$, along with the scalar product that induces duality. We then define the orthogonals and conclude the section by presenting several properties of these orthogonals.

Moving on to Section 2, we describe the topology on $\mathbb{F}^{\mathbb{N}^r}$, which is the product of the discrete topology on $\mathbb{F}$. The first result, Theorem 2.1, allows for the construction of 0-bases, derived from the definition of 0-bases in a product topology.

In Section 3, we state and prove our main theorem, Theorem 3.1. This theorem establishes that the closed subspaces of $\mathbb{F}^{\mathbb{N}^r}$ are the orthogonals of subsets of $\mathbb{F}^{(\mathbb{N}^r)}$. More precisely, if $V \subset \mathbb{F}^{\mathbb{N}^r}$ is closed, then

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$V^{\perp\perp} = V$. A key component of the proof is Lemma 3.2, which provides an approximation of an element q in $V^{\perp\perp}$ through a sequence $(q_n)_{n\in\mathbb{N}}$ of elements from V, defined on finite subsets G_n of $\mathbb{F}^{(\mathbb{N}^r)}$.

1. Duality of Vector Spaces

Let $\mathbb{F}$ be a field. All vector spaces considered in this paper will be over $\mathbb{F}$. For two vector spaces E and F, we denote $\mathrm{Hom}_{\mathbb{F}}(E, F)$ as the set of all linear mappings from E to F; this set also forms an $\mathbb{F}$-vector space. For an integer $r \geqslant 1$, $\mathbb{F}^{\mathbb{N}^r}$ denotes the vector space of all mappings

$$\begin{aligned} y : \mathbb{N}^r &\longrightarrow \mathbb{F} \\ \alpha &\longmapsto y(\alpha) = y_\alpha. \end{aligned}$$

Let $\mathbb{F}^{(\mathbb{N}^r)}$ be the vector subspace of $x \in \mathbb{F}^{\mathbb{N}^r}$ with finite support:

$$\mathbb{F}^{(\mathbb{N}^r)} = \{x \in \mathbb{F}^{\mathbb{N}^r} \mid \{\alpha \in \mathbb{N}^r \mid x_\alpha \neq 0\} \text{ is finite}\}.$$

For $\alpha \in \mathbb{N}^r$, let δ_α be the element of $\mathbb{F}^{(\mathbb{N}^r)}$ defined by

$$\delta_\alpha(\beta) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

for $\beta \in \mathbb{N}^r$. In other words, $\delta_\alpha(\beta) = \delta_{\alpha\beta}$, where $\delta_{\alpha\beta}$ is the Kronecker delta symbol. The set $(\delta_\alpha)_{\alpha\in\mathbb{N}^r}$ forms an $\mathbb{F}$-basis of $\mathbb{F}^{(\mathbb{N}^r)}$, and for $x \in \mathbb{F}^{(\mathbb{N}^r)}$,

$$x = \sum_{\alpha\in\mathbb{N}^r} x_\alpha \cdot \delta_\alpha, \tag{1}$$

with the sum being finite.

Now, an element $y \in \mathbb{F}^{\mathbb{N}^r}$ is represented by $(y_\alpha)_{\alpha\in\mathbb{N}^r}$. We express y as a *formal sum*

$$y = \sum_{\alpha\in\mathbb{N}^r} y_\alpha \delta_\alpha,$$

indicating that for all $\beta \in \mathbb{N}^r$, the value of $y(\beta)$ is given by

$$y(\beta) = \sum_{\alpha\in\mathbb{N}^r} y_\alpha \delta_\alpha(\beta),$$

where the sum is finite.

For $f \in \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^{(\mathbb{N}^r)}, \mathbb{F})$ and $x \in \mathbb{F}^{(\mathbb{N}^r)}$, using equation (1), we have

$$f(x) = \sum_{\alpha\in\mathbb{N}^r} x_\alpha \cdot f(\delta_\alpha).$$

Thus, f is defined by the vector $(f(\delta_\alpha))_{\alpha\in\mathbb{N}^r} \in \mathbb{F}^{\mathbb{N}^r}$. Conversely, an element $f = (f_\alpha)_{\alpha\in\mathbb{N}^r} \in \mathbb{F}^{\mathbb{N}^r}$ defines an element of $\mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^{(\mathbb{N}^r)}, \mathbb{F})$ through the mapping $\delta_\alpha \mapsto f_\alpha$ for all $\alpha \in \mathbb{N}^r$. This can be expressed as follows:

$$\forall x \in \mathbb{F}^{(\mathbb{N}^r)}, \quad f(x) = \sum_{\alpha \in \mathbb{N}^r} x_\alpha \cdot f_\alpha.$$

These observations lead to the following proposition, which we proved in [1] using different notations:

1.1. **Proposition.** *The mapping*

$$\begin{aligned} \langle -, - \rangle : \mathbb{F}^{(\mathbb{N}^r)} \times \mathbb{F}^{\mathbb{N}^r} &\longrightarrow \mathbb{F} \\ (x, y) &\longmapsto \langle x, y \rangle = \sum_{\alpha \in \mathbb{N}^r} x_\alpha \cdot y_\alpha, \end{aligned} \tag{2}$$

(where the sum is finite) satisfies the following properties:
(1) The mappings

$$\begin{aligned} \mathbb{F}^{(\mathbb{N}^r)} &\longrightarrow \operatorname{Hom}_{\mathbb{F}}(\mathbb{F}^{\mathbb{N}^r}, \mathbb{F}) \\ x &\longmapsto \left\{ \begin{array}{rl} \langle x, - \rangle : & \mathbb{F}^{\mathbb{N}^r} \longrightarrow \mathbb{F} \\ & y \longmapsto \langle x, y \rangle \end{array} \right. \end{aligned} \tag{3}$$

and

$$\begin{aligned} \mathbb{F}^{\mathbb{N}^r} &\longrightarrow \operatorname{Hom}_{\mathbb{F}}(\mathbb{F}^{(\mathbb{N}^r)}, \mathbb{F}) \\ y &\longmapsto \left\{ \begin{array}{rl} \langle -, y \rangle : & \mathbb{F}^{(\mathbb{N}^r)} \longrightarrow \mathbb{F} \\ & x \longmapsto \langle x, y \rangle \end{array} \right. \end{aligned} \tag{4}$$

are injective.
(2) The mapping (4) *is a monomorphism and also an isomorphism of vector spaces.*

We refer to $\langle -, - \rangle$ as a *scalar product*, and the vector spaces $\mathbb{F}^{\mathbb{N}^r}$ and $\mathbb{F}^{(\mathbb{N}^r)}$ as *dual spaces.*

1.2. **Definition.** For $P \subset \mathbb{F}^{\mathbb{N}^r}$ and $Q \subset \mathbb{F}^{(\mathbb{N}^r)}$, we define the orthogonals $P^\perp$ and $Q^\perp$ as follows:

$$P^\perp = \{x \in \mathbb{F}^{(\mathbb{N}^r)} \mid \langle x, y \rangle = 0 \ \forall y \in P\} \subset \mathbb{F}^{(\mathbb{N}^r)}, \tag{5}$$

$$Q^\perp = \{y \in \mathbb{F}^{\mathbb{N}^r} \mid \langle x, y \rangle = 0 \ \forall x \in Q\} \subset \mathbb{F}^{\mathbb{N}^r}. \tag{6}$$

The following proposition is straightforward:

1.3. **Proposition.** *The set $P^\perp$ (resp. $Q^\perp$) is a vector subspace of $\mathbb{F}^{(\mathbb{N}^r)}$ (resp. $\mathbb{F}^{\mathbb{N}^r}$).*

1.4. **Proposition** ([3, 5]). *Let P, P' be subsets of $\mathbb{F}^{\mathbb{N}^r}$, and Q, Q' be subsets of $\mathbb{F}^{(\mathbb{N}^r)}$. Then:*

$$P \subset P' \Longrightarrow P^\perp \supset P'^\perp \quad \textit{and} \quad Q \subset Q' \Rightarrow Q^\perp \supset Q'^\perp, \tag{7}$$

$$P \subset P^{\perp\perp} \quad \textit{and} \quad Q \subset Q^{\perp\perp}, \tag{8}$$

$$P^{\perp\perp\perp} = P^\perp \quad \textit{and} \quad Q^{\perp\perp\perp} = Q^\perp. \tag{9}$$

Proof. (7): Let $x \in P'^\perp$. If $y \in P$, then $y \in P'$ by assumption. Therefore, $\langle x, y\rangle = 0$. Since this holds for all $y \in P$, it follows that $x \in P^\perp$. The proof for the second equation is similar.
(8): Let $y \in P$. If $x \in P^\perp$, then $\langle x, y\rangle = 0$. Since this is true for all $x \in P^\perp$, it follows that $y \in P^{\perp\perp}$. The proof for the second equation is analogous.
(9): Using (8) and applying (7) with P' replaced by $P^{\perp\perp}$, we obtain $P^{\perp\perp\perp} \subset P^\perp$. Similarly, applying (8) and (7) with P replaced by $P^\perp$, we have $P^\perp \subset P^{\perp\perp\perp}$. Thus, $P^\perp = P^{\perp\perp\perp}$. The proof for the second equation is analogous. ☐

The orthogonals defined in Proposition 1.4 represent *Galois correspondences* ([3, 5]). This implies that they have reverse inclusions and satisfy $R \subset R^{\perp\perp}$ for the appropriate set R.

1.5. **Corollary** ([3, 5]). *Consider the sets*

$$\mathcal{L} = \{P^\perp \mid P \subset \mathbb{F}^{\mathbb{N}^r}\} \quad \textit{and} \quad \mathcal{CL} = \{Q^\perp \mid Q \subset \mathbb{F}^{(\mathbb{N}^r)}\}.$$

Then the map

$$\begin{aligned} \mathcal{L} &\longrightarrow \mathcal{CL} \\ P^\perp &\longmapsto P^{\perp\perp} \end{aligned} \tag{10}$$

is a bijection, with its inverse given by

$$\begin{aligned} \mathcal{CL} &\longrightarrow \mathcal{L} \\ Q^\perp &\longmapsto Q^{\perp\perp}. \end{aligned} \tag{11}$$

2. Topology on $\mathbb{F}^{\mathbb{N}^r}$

In this section and the following ones, $\leqslant$ denotes the usual total ordering on $\mathbb{N}$, and $r \geqslant 1$ is an integer. The relation $\leqslant_+$ is the partial ordering on $\mathbb{N}^r$ defined by

$$\alpha = (\alpha_1, \ldots, \alpha_r) \leqslant_+ \beta = (\beta_1, \ldots, \beta_r) \Longleftrightarrow \alpha_i \leqslant \beta_i \quad \text{for } i = 1, \ldots, r.$$

Let ψ_r be the mapping

$$\begin{aligned} \psi_r : \mathbb{N} &\longrightarrow \mathbb{N}^r \\ n &\longmapsto \psi_r(n) = \underbrace{(n, n, \ldots, n)}_{r \text{ times}}. \end{aligned} \tag{12}$$

Considering $\mathbb{F}$ as a topological vector space equipped with the *discrete topology* [2], a fundamental system of neighborhoods of 0 (the 0-basis) is the set $\{\{0\}\}$. A sequence $(a_n)_{n\in\mathbb{N}^r}$ of elements of $\mathbb{F}$ converges to an element $a \in \mathbb{F}$ if there exists an integer $N \in \mathbb{N}$ such that

$$n \geqslant N \Longrightarrow a_n = a.$$

The vector space $\mathbb{F}^{\mathbb{N}^r}$ is endowed with the *product topology* derived from that of $\mathbb{F}$, making it a topological vector space as well ([2, 5]). According to the definition of the product topology, we have the following result, which modifies the statement found in [5]:

2.1. **Theorem.** *An* 0*-basis for* $\mathbb{F}^{\mathbb{N}^r}$ *is given by the family of sets* $(V_n)_{n\in\mathbb{N}^*}$, *where*

$$V_n = \{y \in \mathbb{F}^{\mathbb{N}^r} \mid y_\alpha = 0 \text{ for } \alpha \leqslant_+ \psi_r(n)\} \quad \text{for } n \in \mathbb{N}^*. \tag{13}$$

This family has the property

$$V_1 \supset V_2 \supset V_3 \supset \cdots \supset V_n \supset V_{n+1} \supset \cdots.$$

Proof. We will use some notations. If I is a set of indices and $(E_i)_{i\in I}$ is a family of sets indexed by I, then $\prod_{i\in I} A$ denotes the Cartesian product $\prod_{i\in I} E_i$ when $E_i = A$ for $i \in I$. It is also the set A^I of mappings from I to A:

$$A^I = \prod_{i\in I} A.$$

In particular, for $A = \mathbb{F}$ and $I = \mathbb{N}^r$, we have

$$\mathbb{F}^{\mathbb{N}^r} = \prod_{\alpha\in\mathbb{N}^r} \mathbb{F}.$$

The sets $\mathcal{O}$ defined by

$$\mathcal{O} = \prod_{\alpha\in I}\{0\} \times \prod_{\alpha\in\mathbb{N}^r\setminus I} \mathbb{F}, \tag{14}$$

where I is a finite subset of $\mathbb{N}^r$, are open sets in $\mathbb{F}^{\mathbb{N}^r}$ containing 0. Other open subsets of $\mathbb{F}^{\mathbb{N}^r}$ are unions of such subsets ([5, 6]).

If $n \in \mathbb{N}^*$ and $y \in V_n$, we have $y_\alpha = 0$ for $\alpha \leqslant_+ \psi_r(n)$. Let $\mathcal{O}$ be the set defined by

$$\mathcal{O} = \prod_{\alpha\leqslant_+\psi_r(n)} \{0\} \times \prod_{\alpha\nleqslant_+\psi_r(n)} \mathbb{F}.$$

If $z \in \mathcal{O}$, it follows that $z_\alpha = 0$ for $\alpha \leqslant_+ \psi_r(n)$, hence $z \in V_n$. It follows that

$$V_n \supset \mathcal{O}. \tag{15}$$

Let Δ be the set defined by

$$\Delta = \{\alpha \in \mathbb{N}^r \mid \alpha \leqslant_+ \psi_r(n)\} \subset \mathbb{N}^r. \tag{16}$$

We have

$$\mathcal{O} = \prod_{\alpha\in\Delta}\{0\} \times \prod_{\alpha\in\mathbb{N}^r\setminus\Delta} \mathbb{F},$$

so that $\mathcal{O}$ takes the form described in (14). By (15), the set V_n is a neighborhood of 0.

If $\mathcal{U}$ is another neighborhood of 0, there exists a finite subset I of $\mathbb{N}^r$ such that

$$\mathcal{O} = \prod_{\alpha\in I}\{0\} \times \prod_{\alpha\in\mathbb{N}^r\setminus I} \mathbb{F} \subset \mathcal{U}.$$

Let n be an integer strictly greater than the maximum of the coordinates of the α's in I:

$$\Big(\forall\alpha \in I\Big) \quad \Big[\alpha <_+ \psi_r(n)\Big].$$

Using the set Δ defined in (16), we have

$$I \subset \Delta \quad \text{and} \quad \mathbb{N}^r \setminus \Delta \subset \mathbb{N}^r \setminus I,$$

which implies that

$$\begin{aligned} y \in V_n &\Longrightarrow y_\alpha = 0 \quad \text{for } \alpha \leqslant_+ \psi_r(n), \\ &\Longrightarrow y \in \prod_{\alpha\leqslant_+\Delta}\{0\} \times \prod_{\alpha\in\mathbb{N}^r\setminus\Delta} \mathbb{F} \subset \prod_{\alpha\in I}\{0\} \times \prod_{\alpha\in\mathbb{N}^r\setminus I} \mathbb{F} = \mathcal{O}. \end{aligned}$$

It follows that $V_n \subset \mathcal{O}$ and therefore $V_n \subset \mathcal{U}$. We have shown that the family $(V_n)_{n\in\mathbb{N}^*}$ consists of neighborhoods of 0 such that any other neighborhood of 0 contains at least one element from this family. It is clear that the sequence $(V_n)_{n\in\mathbb{N}^*}$ is decreasing. □

2.2. **Corollary.** *A sequence $(f_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{F}^{\mathbb{N}^r}$ converges to an element $f \in \mathbb{F}^{\mathbb{N}^r}$ if and only if, for every $\alpha \in \mathbb{N}^r$, the sequence $(f_{n\alpha})_{n\in\mathbb{N}}$ converges to f_α in $\mathbb{F}$.*

Proof. Suppose that the sequence $(f_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{F}^{\mathbb{N}^r}$ converges to $f \in \mathbb{F}^{\mathbb{N}^r}$, and let α be an element of $\mathbb{N}^r$. There exists $m \in \mathbb{N}$ such that $\alpha \leqslant_+ \psi_r(m)$. Let $n \in \mathbb{N}^*$ be such that $n \geqslant m$, and let V_n be the corresponding neighborhood of 0 in $\mathbb{F}^{\mathbb{N}^r}$. There exists $N \in \mathbb{N}$ such that

$$\begin{aligned} k \geqslant N &\Longrightarrow (f_k - f) \in V_n \\ &\Longrightarrow f_{k\beta} = f_\beta \quad \text{for } \beta \leqslant_+ \psi_r(n). \end{aligned} \tag{17}$$

Applying the second equation in (17) for the case $\beta = \alpha$, we find

$$k \geqslant N \Longrightarrow f_{k\alpha} = f_\alpha,$$

i.e., the sequence $(f_{n\alpha})_{n\in\mathbb{N}}$ converges to f_α in $\mathbb{F}$.

Conversely, suppose that for every $\alpha \in \mathbb{N}^r$, the sequence $(f_{n\alpha})_{n\in\mathbb{N}}$ converges to f_α in $\mathbb{F}$. Given $\alpha \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that

$$k \geqslant N \Longrightarrow f_{k\alpha} - f_\alpha = 0. \tag{18}$$

Let V_n be a neighborhood of 0 in $\mathbb{F}^{\mathbb{N}^r}$ and let Δ be the set defined by

$$\Delta = \{\alpha \in \mathbb{N}^r \mid \alpha \leqslant_+ \psi_r(n)\} \subset \mathbb{N}^r. \tag{19}$$

Let m be the cardinality of Δ, and let $\alpha_1, \dots, \alpha_m$ be the elements of Δ. By applying (18) to each of the elements of Δ, we find that there exist $N_1, \dots, N_m \in \mathbb{N}$ such that

$$k \geqslant N_i \Longrightarrow f_{k\alpha_i} - f_{\alpha_i} = 0,$$

for $i = 1, \dots, m$. Taking $N = \max\{N_1, \dots, N_m\}$, we then have

$$k \geqslant N \Longrightarrow f_{k\alpha_i} - f_{\alpha_i} = 0,$$

for $i = 1, \dots, m$. In other words,

$$k \geqslant N \Longrightarrow f_{k\alpha} = f_\alpha \quad \text{for } \alpha \leqslant_+ \psi_r(n).$$

This shows that

$$k \geqslant N \Longrightarrow (f_k - f) \in V_n.$$

Thus, $(f_n)_{n\in\mathbb{N}}$ converges to f in $\mathbb{F}^{\mathbb{N}^r}$. □

The topology of $\mathbb{F}^{\mathbb{N}^r}$ is therefore the topology of *pointwise convergence.*

3. Closed Subspaces of $\mathbb{F}^{\mathbb{N}^r}$

We will now investigate the closed subspaces of $\mathbb{F}^{\mathbb{N}^r}$.

3.1. **Theorem.** *An $\mathbb{F}$-vector subspace V of $\mathbb{F}^{\mathbb{N}^r}$ is closed if and only if there exists $G \in \mathbb{F}^{(\mathbb{N}^r)}$ such that*

$$V = G^\perp. \tag{20}$$

Proof. Let G be a non-empty subset of $\mathbb{F}^{(\mathbb{N}^r)}$. We need to show that $G^\perp$ is closed in $\mathbb{F}^{\mathbb{N}^r}$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $G^\perp$ that converges to $f \in \mathbb{F}^{\mathbb{N}^r}$ with respect to the topology of $\mathbb{F}^{\mathbb{N}^r}$.

Given $x \in G$, the following property holds: for $k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for $n \geqslant N$, we have $(f_n - f) \in V_k$, i.e., $(f_n - f)_\alpha = 0$ whenever $\alpha \leqslant_+ \psi_r(k)$. Since

$$\begin{aligned} f(x) &= \sum_{\alpha \leqslant_+ \psi_r(k)} x_\alpha f_\alpha + \sum_{\alpha \nleqslant_+ \psi_r(k)} x_\alpha f_\alpha \\ &= \sum_{\alpha \leqslant_+ \psi_r(k)} x_\alpha (f_n)_\alpha + \sum_{\alpha \nleqslant_+ \psi_r(k)} x_\alpha f_\alpha, \end{aligned} \tag{21}$$

and since f_n has finite support, we can choose k sufficiently large such that

$$0 = f_n(x) = \sum_{\alpha \leqslant_+ \psi_r(k)} x_\alpha f_{n\alpha}.$$

It follows that

$$f(x) = \sum_{\alpha \nleqslant_+ \psi_r(k)} x_\alpha f_\alpha.$$

However, since x has finite support, we can increase k, if necessary, so that $x_\alpha = 0$ for α satisfying $\alpha \nleqslant_+ \psi_r(k)$, which implies that $f(x) = 0$. Since this holds for any arbitrary $x \in G$, we conclude that $f \in G^\perp$.

Conversely, suppose that V is closed in $\mathbb{F}^{\mathbb{N}^r}$. We will show that

$$V^{\perp\perp} = V = \overline{V}. \tag{22}$$

It suffices to demonstrate the non-trivial inclusion $V^{\perp\perp} \subset V$ (see (8)). Let $\{f_\lambda \mid \lambda \in \Lambda\}$ be a generating set of V:

$$\begin{aligned} f_\lambda : \mathbb{N}^r &\longrightarrow \mathbb{F} \\ \alpha &\longmapsto f_{\lambda\alpha}. \end{aligned}$$

Let $q = (q_\alpha)_{\alpha \in \mathbb{N}^r} \in V^{\perp\perp}$, and let $n \in \mathbb{N}$. Define G_n as the finite-dimensional vector subspace of $\mathbb{F}^{(\mathbb{N}^r)}$ given by

$$G_n = \bigoplus_{\alpha \in \Delta} \mathbb{F}\delta_\alpha, \tag{23}$$

where $\Delta = \{\alpha \in \mathbb{N}^r \mid \alpha \leqslant_+ \psi_r(n)\}$ (i.e., G_n is the subspace of $\mathbb{F}^{(\mathbb{N}^r)}$ generated by $\{\delta_\alpha \mid \alpha \leqslant_+ \psi_r(n)\}$). From classical linear algebra, we have the isomorphism

$$\begin{aligned} \mathbb{F}^\Delta &\longleftrightarrow \operatorname{Hom}_{\mathbb{F}}(G_n, \mathbb{F}) \\ x = (x_\alpha)_{\alpha \in \Delta} &\longleftrightarrow \left\{ \begin{array}{ll} \varphi : & G_n \to \mathbb{F} \\ & \delta_\alpha \mapsto x_\alpha. \end{array} \right. \end{aligned}$$

For $\beta \in \Delta$, we define the element $\gamma_\beta \in \mathrm{Hom}_{\mathbb{F}}(G_n, \mathbb{F})$ by

$$\begin{aligned} \gamma_\beta : G_n &\longrightarrow \mathbb{F} \\ \delta_\alpha &\longmapsto \gamma_\beta(\delta_\alpha) = \delta_{\beta\alpha}, \end{aligned}$$

where $\delta_{\beta\alpha}$ is the Kronecker symbol. The family $\{\gamma_\beta \mid \beta \in \Delta\}$ forms an $\mathbb{F}$-basis for $\mathrm{Hom}_{\mathbb{F}}(G_n, \mathbb{F})$. Therefore, if $\varphi \in \mathrm{Hom}_{\mathbb{F}}(G_n, \mathbb{F})$, we can express φ as

$$\varphi = \sum_{\beta \in \Delta} \varphi_\beta \gamma_\beta \tag{24}$$

with $\varphi_\beta \in \mathbb{F}$.

Additionally, we have the isomorphism:

$$\begin{aligned} \Phi : \mathbb{F}^\Delta &\longleftrightarrow \mathrm{Hom}_{\mathbb{F}}(\mathrm{Hom}_{\mathbb{F}}(G_n, \mathbb{F}), \mathbb{F}) \\ y = (y_\alpha)_{\alpha \in \Delta} &\longmapsto \begin{cases} \Phi(y) : \mathrm{Hom}_{\mathbb{F}}(G_n, & \mathbb{F}) \to \mathbb{F} \\ & \gamma_\alpha \mapsto \Phi(y)(\gamma_\alpha) = y_\alpha. \end{cases} \end{aligned} \tag{25}$$

Consider the restrictions:

$$\begin{aligned} f_\lambda|_{G_n} &= f_\lambda^{(n)} \quad \text{with } f_\lambda^{(n)}(\delta_\alpha) = f_{\lambda\alpha}, \\ q|_{G_n} &= q_n \quad \text{with } q(\delta_\alpha) = q_\alpha \end{aligned} \tag{26}$$

for $\alpha \in \Delta$. We will need the following lemma:

3.2. **Lemma.** *The property $q_n \in \langle (f_\lambda^{(n)})_{\lambda \in \Lambda} \rangle$ holds.*

Proof: We have $q_n \in \mathrm{Hom}_{\mathbb{F}}(G_n, \mathbb{F})$. Suppose that $q_n \notin \langle (f_\lambda^{(n)})_{\lambda \in \Lambda} \rangle$. In this case, there exists $\Theta \in \mathrm{Hom}_{\mathbb{F}}(\mathrm{Hom}_{\mathbb{F}}(G_n, \mathbb{F}), \mathbb{F})$ such that $\Theta(f_\lambda^{(n)}) = 0$ for $\lambda \in \Lambda$ and $\Theta(q_n) = 1$. By the isomorphism in (25), there exists $y = (y_\alpha)_{\alpha \in \Delta} \in \mathbb{F}^\Delta$ such that $\Phi(y) = \Theta$, i.e., $\Theta(\gamma_\alpha) = y_\alpha$ for $\alpha \in \Delta$.
Define

$$g = \sum_{\alpha \in \Delta} y_\alpha \delta_\alpha \in G_n.$$

Then,

$$
\begin{aligned}
f_\lambda^n(g) = f_{\lambda|G_n}(g) &= f_\lambda\left(\sum_{\alpha\in\Delta} y_\alpha\delta_\alpha\right)\\
&= \sum_{\alpha\in\Delta} f_\lambda(y_\alpha\delta_\alpha)\\
&= \sum_{\alpha\in\Delta} y_\alpha f_\lambda(\delta_\alpha)\\
&= \sum_{\alpha\in\Delta} y_\alpha f_{\lambda\alpha}\\
&= \sum_{\alpha\in\Delta} \Theta(\gamma_\alpha) f_{\lambda\alpha}\\
&= \sum_{\alpha\in\Delta} \Theta(f_{\lambda\alpha}\gamma_\alpha)\\
&= \Theta\left(\sum_{\alpha\in\Delta} f_{\lambda\alpha}\gamma_\alpha\right)\\
&= \Theta(f_\lambda^{(n)}) = 0.
\end{aligned} \tag{27}
$$

This implies that

$$(\forall \lambda\in\Lambda) \quad \Big[f_\lambda(g) = 0\Big],$$

which shows that $g\in V^\perp$. However, we also have:

$$
\begin{aligned}
q(g) = q|_{G_n}(g) = q_n(g) &= q_n\left(\sum_{\alpha\in\Delta} y_\alpha\delta_\alpha\right)\\
&= \sum_{\alpha\in\Delta} y_\alpha q_\alpha = \sum_{\alpha\in\Delta}\Theta(\gamma_\alpha)q_\alpha = \Theta\left(\sum_{\alpha\in\Delta}\gamma_\alpha q_\alpha\right).
\end{aligned}
$$

By (24),

$$q_n = \sum_{\alpha\in\Delta}\gamma_\alpha q_{n_\alpha} = \sum_{\alpha\in\Delta}\gamma_\alpha q_\alpha,$$

which leads to

$$q(g) = \Theta(q_n) = 1.$$

This implies that $g\notin V^{\perp\perp\perp} = V^\perp$, which is a contradiction. We conclude that necessarily $q_n \in \langle (f_\lambda^{(n)})_{\lambda\in\Lambda}\rangle$. $\square$

Proof of Theorem 3.1 (continued): For $n \in \mathbb{N}^*$, there exists a family $(\mu_\lambda^{(n)})_{\lambda\in\Lambda}$ with finite support such that

$$q|_{G_n} = q_n = \sum_{\lambda\in\Lambda} \mu_\lambda^{(n)} f_\lambda^{(n)}.$$

Consider the element $q'_n = \sum_{\lambda\in\Lambda} \mu_\lambda^{(n)} f_\lambda \in V$, which is an extension of q_n to $\mathbb{F}^{\mathbb{N}^r}$. For $\alpha \in \Delta$, we have, by (26),

$$q(\alpha) = q'_n(\alpha),$$

so that $q - q'_n \in V_n$. Therefore, the sequence $(q'_n)_{n\in\mathbb{N}^*}$ converges to q in $\mathbb{F}^{\mathbb{N}^r}$. Finally, we have $q \in \overline{V}$. Thus, $V^{\perp\perp} \subset V$, and the equality holds. Taking $G = V^\perp$, we obtain $G^\perp = V^{\perp\perp} = V$, and the theorem is proved. $\square$

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