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Research Article

Unified Inversion Method for Solving Polynomial Equations: A Reverse Detour to the Common Procedure

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There are many solutions to polynomial equations that have been developed by mathematicians over the centuries. These methods adopt different approaches such as substitution, complex number algebra, trigonometry, reduction to depressed form, elimination, and decomposition of the original polynomial into solvable products of polynomials of lesser degree. In this paper, a historical preview of the methods used to solve polynomial equations is provided, together with a review of recent methods demonstrated for solving polynomial equations. This paper also proposes a new unified method of solving polynomial equations based on the inversion of the n^{th} roots of variables that will explicitly determine the root. The method is applicable to all polynomials within the limits of solvability of polynomials by radicals. The method follows a reverse route to the common methods and logically finds roots that are algebraically expressed as radicals of real numbers, although the formulation of the solution starts with inversion by finding the n^{th} root of either real or complex numbers. By contrast, methods such as Cardan's solution to cubic equations give solutions that have cube roots of complex numbers, whereas the roots are real numbers. The proposed method is simple and intuitive to understand and use. Examples have been provided to demonstrate the application of the proposed method.

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1. Introduction

The techniques for solving polynomial equations, including quadratic and cubic equations, have been recorded since ancient times, with the Babylonians around 2000 BC. The algebraic solutions to cubic and quartic equations were successfully established during the Renaissance period (1450-1630). Scipione Del Ferro (1465-1526) found the solution for the cubic equation formulated in reduced form, but his solution was kept secret (Conner, 1956). Tartaglia also developed the solution to the cubic equation, which was also not published but only told to Cardano. Girolamo Cardano (1501-1576) published the first public method of solving cubic equations, crediting Del Ferro for the method. Francois Viete (1540-1603) similarly established a method for solving cubic equations using a two-step transformation involving one variable only, rather than the two variables involved in Cardano's method. The original solutions for cubic equations by both Cardano and Viete are not exactly intuitive and look somehow like magical discoveries. Later attempts at more explicit and intuitive approaches have been forwarded (Mukundan, 2010). Simplifications of the solution using derivatives have also been used (Abesheck Das, 2014; Tiruneh, 2020).

Joseph Luis Lagrange (1736-1813) used a combination of symmetric functions that are enough to specify the polynomial equations in reduced form and thus solve them. Lagrange's solution, as such, implicitly used the Fourier transform, though the Fourier transform was not yet established during that time (Jansen, 2009).

Lagrange's method is also said to be a precursor to the Group theory credited to Evariste Gallois (1811-1832).

The solution to quartic polynomial equations was first established by Ferrari (1522-1565). However, since the method involves solving a resolvent cubic equation, Ferrari's method became public only when the method for solving cubic equations was established (Dickson, 1920). Rene Descartes (1596-1650) and a number of other mathematicians also suggested methods of solving quartic polynomial equations (Dickson, 1914). The occurrence of repeating roots in quartic equations could be apparent when the resolvent cubic also has a repeating root (Neumark, 1965). Leonard Euler (1707-1783) made use of the fact that the sum of the four roots is equal to zero for the reduced quartics and hence was able to offer a solution by solving a resolvent cubic arising out of the three variables (Nickalls, 2009). Fathi and Sharifan (2013) provided a new method of solving quartic equations by expressing the original root x as a sum of three transformed variables u, v, and w in a manner similar to the solution provided by Cardano. Kulkarni (2006) suggested a unified method for solving polynomial equations that has a more explicit and intuitive form compared to earlier methods.

Over the centuries since the attempts at solving cubic and quartic equations became successful, numerous methods have been proposed for solving polynomial equations of degree less than five. These methods demonstrate the dynamics of the different ways in which polynomial equations can be solved and the intelligence of the authors who came up with solutions to the polynomial equations. While the complexity of the solutions proposed over the years varies, there are certain aspects of the solutions that are apparent in each method. For example, for solving a polynomial equation of degree N, the methods involve solving a resolvent polynomial equation of degree N-1. This means cubic equations involve solving a quadratic of a transformed variable, and quartic equations involve solving a cubic resolvent equation.

The attempt at solving quintic and higher degree polynomial equations using the same techniques as those of lesser degree polynomials was not successful, and mathematicians successfully established the condition for the solvability of polynomials, which proved that all polynomials of degree less than five are solvable in terms of radicals, as the historical development of the solutions also suggests. Early attempts at solving quintic equations using methods similar to those of quartic and cubic equations resulted in a resolvent polynomial equation that is a six-degree polynomial, which is greater than the original five-degree polynomial. This provided an early hint to mathematicians like Lagrange that quintic and higher degree polynomial equations may be impossible to solve in terms of radicals like those of lesser degree polynomials. This led to attempts at proving the general non-solvability of polynomials of degree five and greater. On the other hand, there were apparently quintic and higher degree polynomial equations such as $p(x) = x^5-1=0$, that can be solved as they occur in solvable form. The condition for the solvability of such polynomials has been provided through Galois Group Theory (Nguyen and Ruan, 2024). The proof and demonstration of the non-solvability of certain polynomials of degree five and higher have earlier been provided by Abel and Ruffini (Tignol, 2016).

Examples of recent demonstrations of methods for solving polynomials are discussed below, including unified approaches that apply to all polynomial equations, such as ones that are given by Vieira (2011) and Kulkarni (2006). In the case of the method shown by Vieira (2011), which is related to the use of complex numbers by Lagrange, the direct application of complex numbers for solving quadratic, cubic, and quartic equations is demonstrated.

i. Unified approach using complex number substitution (Vieira (2011)

The solution to the quadratic, cubic, and quartic equations in reduced form given below is solved by substituting $z= x + w*y$, where x and y are real numbers and w is the n^{th} root of negative one (for quadratic and cubic equations) and the n^{th} root of one for quartic equations

$$z^2 + az + b = 0\ ;\ z = x + wy\ ;\ w = \sqrt{-1} = i$$

$$z^3 + az + b = 0\ ; z = x + wy\ ;\ w = \sqrt[3]{-1} = \frac{1 \pm \sqrt{-3}}{2}$$

$$z^4 + az^2 + bz + c = 0\ ;\ z = x + wy;\ w = \sqrt[4]{1} = \pm i$$

For each of the equations, substituting $z = x + w*y$ and separating the real and imaginary parts of the equations gives a system of two equations in two unknowns, i.e., x and y. Eliminating y from the equations gives a single equation in x that can be solved in explicit form.

ii. Unified method for solving polynomials by Kulkarni (2010),

A given polynomial of degree N is decomposed into the form shown below:

$$\frac{[V_M(x)]^K - p^K [V_M(x)]^K}{1 - p^K}$$

Where $V_M(x)$ and $W_M(x)$ are polynomials of degree $M<N$, and p is an unknown constant to be determined. The decomposed polynomial has degree $KM = N$. There are a total of 2M+1 unknowns, m coefficients of the polynomial from each of $V_M(x)$ and $W_M(x)$, and the additional unknown, which is the p value. In order for the system to

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be solvable, 2M+1 = N for odd values of N, and 2M+1 = N+1 for even N.

The following decomposed forms of the polynomials have been suggested for the quadratic, cubic, and quartic equations:

Quadratic:

$$\frac{(x_0+b_0))^2-\ p^2(x_0+c_0)^2}{1-p^2}\ ;\ c_0=0$$

Cubic:

$$\frac{(x_0+b_0))^3-\ p^3(x_0+c_0)^3}{1-p^3}$$

Quartic:

$$\frac{\left(x^2+b_1x+b_0\right)\right)^2-\ p^2\left(x^2+c_1x+c_0\right)^2}{1-p^2}\ ;\ c_1\ =0$$

The choices such as c_0=0 and c_1=0 are taken so that the number of unknowns in the decomposed polynomial is equal to the number of coefficient terms in the given polynomial. Once the undetermined coefficients of the decomposed polynomial, together with the p value, are determined explicitly by solving a polynomial equation of degree N-1 (for an equation involving a polynomial of degree N), the solution proceeds by decomposing the polynomial and equating each term to zero. The decomposable polynomial can be decomposed into the following:

$$\frac{\left[V_M(x)\right]^K-\ p^K\left[V_M(x)\right]^K}{1-p^K}$$
$$=\ \left(\frac{V_M(x)-\ pV_M(x)}{1-p}\right)\left(\frac{F_{N-M}(x)}{\sum_{i=0}^{K-1}p^i}\right)$$

Where $F_{N-M}(x)$ is a decomposed polynomial of degree N-M. Each of the polynomials in brackets in the above expression is of degree less than N (the degree of the given polynomial) and is solved to determine the roots of the polynomial. Kulkarni (2006) further suggested a similar procedure for solving quintics by transforming them into sextic equations, which probably leads to coefficients of the original polynomials being dependent on each other, making them solvable quintics. This, however, cannot be taken as a general solution as it belongs to only a certain class of quintic polynomials in which the coefficients of the polynomial are not independent from each other.

iii. Solution to cubic equations by Mochimaro (2015)

A given equation $x^3+Ax^2+Bx+C=0$ is transformed through a new variable y such that $x=y+\beta$ to the form:

$$y^3+ay^2+by+c=0$$

$$a=A+3\beta$$

$$b=3\beta^2+2A\beta+B$$

$$c=\ \beta^3+A\beta^2+B\beta+C$$

Further condition is attached to the transformation such that $b^2=3ac$. This condition imposed on the b and c coefficients of y will give the following quadratic equation that determines the variable transformation constant β:

$$\left(A^2-3B\right)\beta^2+\ (AB-9C)\beta+B^2-3AC=0$$

The coefficients a, b, and c are now worked out once β is determined, and it is easy to see that the new cubic equation in y with the condition $b^2=3ac$ imposed can be reformulated as follows:

$$\left(\frac{b}{ay}+1\right)^3=1-\ \frac{3b}{a^2}$$

Finally, the three roots of y are determined from the following equation that also makes use of De Movire's theorem;

$$\frac{b}{ay}\ =\ -1+\omega\left(1-\frac{3b}{a^2}\right)^{\frac{1}{3}}\ ;\ \omega=\ e^{2n\pi i/3}$$

Special conditions where either of a, b, or c=0 are handled differently in the above method by Mochimaro (2015).

iv. Quartic equation solution by Tehrani (2020)

Given a quartic equation in depressed form:

$$f(z)=\ z^4+az^2+bz+c=0$$

An equivalent polynomial is constructed in decomposed form, which is given by:

$$f(z)=\ \ (z-R_1)(z-R_2)\left(z^2+pz+q\right)$$
$$=\ \left[z^2-(R_1+R_2)z+\ R_1R_2\right]\left[z^2+pz+q\right]$$

Comparison of the coefficients with the original equation gives:

$$p=\ R_1+R_2=\ \gamma$$

$$a=q+\lambda-\gamma^2$$

$$b=\ \gamma(\lambda-q)$$

$$c=q\lambda$$

The three by three non-linear equation in q, γ, and λ above, when solved for γ^2, will give a cubic equation that always has a real number solution. After this, the other unknowns q and λ are determined from the three by three equations above. This will, in turn, provide the roots R_1 and R_2 from the quadratic equations of $R_1+R_2=\gamma$ and $R_1R_2=\lambda$. The other roots are likewise determined by solving $z^2+pz+q=0$.

v. Sousa's Solution (2021)

Jose Risomar Sousa (2021) provided a solution to cubic equations based on completing the cube in a manner similar to completing the squares that lead to the quadratic equation solution. According to Sousa (2021), the solution to the cubic equation can be directly worked on without conversion to a depressed form. For a given cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

The equation is transformed by defining a new variable π such that

$$x \ = \ \pi + m$$

This results in the equation in π:

$$a\pi^3 + \pi^2(3am + b) + \pi\left(3am^2 + 2bm + c\right) am^3 + bm^2 + cm + d = 0$$

$$\Delta^3 \ = am^3 + bm^2 + cm + d$$

$$3\Delta^2\delta = 3am^2 + 2bm + c$$

$$3\Delta\delta^2 = 3am + b$$

The following expression is true from Sousa's Solution:

$$\frac{\Delta^3}{3\Delta^2\delta} \ = \ \frac{1}{3}\left(\frac{3\Delta^2\delta}{3\Delta\delta^2}\right)$$

Inserting the expression above:

$$\frac{am^3 + bm^2 + cm + d}{3am^2 + 2bm + c} \ = \ \frac{1}{3}\left(\frac{3am^2 + 2bm + c}{3am + b}\right)$$

Simplifying the above expression further:

$$\left(b^2 - 3ac\right)m^2 + \ (bc - 9ad)m + c^2 - 3bd = 0$$

The solution for m is obtained using the quadratic formula:

$$m = \ \frac{-(bc - 9ad) \pm \sqrt{(bc - 9ad)^2 - 4\left(b^2 - 3ac\right)\left(c^2 - 3bd\right)}}{2(bc - 9ad)}$$

The transformed cubic equation in π is now transformed as:

$$a\pi^3 + 3\Delta(\delta\pi)^2 + 3\Delta^2\delta\pi + \Delta^3 = 0$$

Completing the cube is done by adding and subtracting $(\delta\pi)^3$ as follows:

$$a\pi^3 - (\delta\pi)^3 + (\delta\pi)^3 + 3\Delta(\delta\pi)^2 + 3\Delta^2\delta\pi + \Delta^3 = 0$$

$$(\delta\pi + \Delta)^3 = \ -\pi^3\left(a - \delta^3\right)$$

The above equation is solved for π, which will eventually give the solution in x as follows:

$$\pi = \ \frac{-\Delta}{\delta + \omega\left(a - \delta^3\right)^{1/3}}$$

$$x = \ \pi + m = m - \frac{-\Delta}{\delta + \omega\left(a - \delta^3\right)^{1/3}}$$

Where:

$$\Delta = \sqrt[3]{am^3 + bm^2 + cm + d}$$

And

$$\delta = \frac{3am^2 + 2bm + c}{3\Delta^2}$$

However, the author did not specify how to handle the situation where (bc-9ad)=0, which will not allow the quadratic solution for the m value.

2. Method Development

The unified procedure for finding the roots of polynomial equations is based on inversion through finding the roots of the following systems of equations:

$$(A + B)^N \ = C + D; \ (1)$$

$$(A - B)^N \ = \ C - D \ (2)$$

The variables A, B, C, and D in Equations (1) and (2) each can be a real (without a complex part) or a complex number (without a real part). The root of a polynomial equation is related through the inversion procedure that solves the system of equations above for the variable A, i.e.,

$$A = \ \frac{1}{2}\left(\sqrt[N]{C + D}\right) \ \pm \sqrt[N]{C - D}) \ (3)$$

The above solution is solved separately for two cases, which are shown below:

2.1. Case I: Both C and D are either purely real numbers or purely complex numbers

When both C and D are either real numbers or both complex numbers, the solution is solved without involving complex numbers. This is seen from the following relation:

$$both\ real\ numbers: \ (a + b)^N \ = c + d;$$

If C and D are both real numbers, a and b should also be real numbers since by definition they cannot have a mixed part (they are both either purely real numbers or purely complex numbers).

Similarly, if both C and D are purely complex numbers, the following relation holds true:

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$$both\ complex\ numbers:\ (ai+bi)^N = \pm i((a+b)^N = ci+di = i(c+d)$$

$$(a+b)^N = \pm(c+d)$$

It is clear that both a and b should also be both purely complex numbers since a mixed number such as a+bi when raised to the power of n will also have a mixed number c+di, which contradicts the assumption that both c and d are complex numbers of the form ci and di. Therefore, the system reverts to a real number form since the complex term i is cancelled from both sides of the equation. The solution belonging to case I is solved as the roots of real numbers, whose solution can be a real number or a complex number depending on the discriminant, as will be shown further.

2.2. Case II: Either C or D is a complex number and the other a real number

When either C or D is a complex number and the other a real number, for convenience, we switch the real part designation as C=c and the complex part as D=di. Therefore,

$$(a+bi)^N = c+di$$

It is clear that a and b should also be mixed, i.e., when one is real, the other should be a complex number. For convenience, the real part is designated as A=a and the complex part as B=bi. The system is solved by finding the roots of complex numbers, which, as will be shown further, turn out to be real numbers. The application of De Movier's Theorem is relevant for this case where the solution involves finding the cube roots of a complex number.

The detail of the development of the method for quadratic, cubic, and quartic equations is now provided below.

2.3. Method for Quadratic Equations

For a quadratic equation, n=2, the solution for A is given from the equation:

$$(A+B)^2 = C+D$$

$$(A-B)^2 = C-D$$

Taking the square roots of each term on either side of both equations,

$$(A+B) = \pm\sqrt{C+D}$$

$$(A-B) = \pm\sqrt{C-D}$$

The solution for A is found by adding the equations together:

$$A = \frac{1}{2}\left(\sqrt{C+D} \pm \sqrt{C-D}\right) \ (4)$$

As will be shown further, the above solution gives two independent roots, although it initially appears that there are four solutions from the combination of the plus and minus signs. This is due to the fact that the variable A is related to the variable in the quadratic equation X through X = A^2, and the values of A that are equal in magnitude but opposite in sign will both give the same value of X when squared through X= A^2.

To relate the above solution A to the quadratic equation, consider the system of equations again:

$$(A+B)^2 = C+D$$

$$(A-B)^2 = C-D$$

Adding and subtracting the above equations in turns gives:

$$A^2 + B^2 = C\ and\ 2AB = D$$

Eliminating B from the equation containing C gives:

$$A^2 + \left(\frac{D}{2A}\right)^2 - C = 0$$

Rearranging gives:

$$A^4 - CA^2 + \frac{D^2}{4} = 0$$

Let X = A^2 so that:

$$X^2 - CX + \frac{D^2}{4} = 0 \ (5)$$

Given the quadratic equation: X^2 + RX +S and equating the constants gives:

$$C = -R\ and\ D = \sqrt{4S}$$

From the solution obtained above for A, i.e.,

$$A = \frac{1}{2}\left(\sqrt{C+D} \pm \sqrt{C-D}\right)$$

The value of X is then obtained:

$$X = A^2 = \frac{1}{4}\left(\sqrt{C+D} \pm \sqrt{C-D}\right)^2 \ (6)$$

Now we consider the solution for the two cases.

<u>Case I: Solution for quadratic in the real number domain of both C and D</u>

In this case, A=a, b=b, c= C, and D=d are all taken as real numbers.

Given the quadratic equation: X^2 + RX +S and

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$$X^2 - CX + \frac{D^2}{4} = 0$$

Equating the constants gives:

$$C = -R \; and \; D = \sqrt{4S}$$

$$X = A^2 = \frac{1}{4}\left(\sqrt{C+D} \pm \sqrt{C-D}\right)^2$$

Substituting the values of $C = -R \; and \; D = \sqrt{4S}$ in the above equation gives the quadratic formula as shown below:

$$X = A^2 = \frac{1}{4}\left(\sqrt{-R+\sqrt{4S}} \pm \sqrt{-R-\sqrt{4S}}\right)^2$$

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$$X = A^2 = \frac{1}{4}\left(-R + \sqrt{4S} - R - \sqrt{4S} \pm \sqrt{((-R)^2 - 4S)}\right)$$

$$X = A^2 = \frac{1}{2}\left(-R \pm \sqrt{R^2 - 4S}\right)$$

This gives the familiar quadratic formula. The discriminant:

$$R^2 - 4S = C^2 - D^2$$

Will determine if the system has a real or complex root. If C is less than D, the roots are both complex, and if C is greater than D, the roots are both real.

Case II: Solution for quadratic in the Complex number domain of c+di

This case occurs when C and D are mixed, i.e., one is real and the other complex. For convenience, C=c is taken to be the real part and D= di the complex part. The application of De Moivre's theorem is used as follows:

$$X = A^2 = \frac{1}{4}\left(\sqrt{c+di} \pm \sqrt{c-di}\right)^2$$

Let $r = \sqrt{c^2 + d^2}$ and $\theta = Cos^{-1}\left(\frac{c}{r}\right)$

$$\sqrt{c+di} = r^{1/2}\left(Cos\left(\frac{\theta}{2}\right) + i\ Sin\left(\frac{\theta}{2}\right)\right)$$

$$\sqrt{c-di} = r^{1/2}\left(Cos\left(\frac{-\theta}{2}\right) + i\ Sin\left(\frac{-\theta}{2}\right)\right)$$

$$X = A^2$$
$$= \frac{1}{4}\left(\left(r^{1/2}\left(Cos\left(\frac{\theta}{2}\right) + i\ Sin\left(\frac{\theta}{2}\right)\right)\right)\right)$$
$$\pm \left(r^{1/2}\left(Cos\left(\frac{-\theta}{2}\right) + i\ Sin\left(\frac{-\theta}{2}\right)\right)\right)\Bigg)^2$$

$$X = A^2 = \left\{r\left(Cos\left(\frac{\theta}{2}\right)\right)^2,\ -r\left(Sin\left(\frac{\theta}{2}\right)\right)^2\right\} \quad (7)$$

It turns out that both roots are always real numbers. This is seen from the relation

$$C = -R \; and \; D = \sqrt{4S}$$

For D to be a complex number, S should be less than zero, or a negative number. The discriminant $R^2 - 4S = c^2 + d^2$ will always be positive, as is shown below, guaranteeing the roots are both real numbers.

$$R^2 - 4S = C^2 - D^2 = c^2 - (di)^2 = c^2 + d^2$$

2.4. Method for Cubic Equations

A similar procedure is followed for cubic equations whereby the variable A is solved by taking the cubic roots of C+D, i.e.,

$$(A+B)^3 = C + D$$

$$(A-B)^3 = C - D$$

Taking the cube roots of the expressions on either side of the above equations:

$$A + B = \sqrt[3]{C+D}$$

$$A - B = \sqrt[3]{C-D}$$

Eliminating B by adding the two equations yields the solution for A, i.e.,

$$A = \frac{1}{2}\sqrt[3]{C+D} + \frac{1}{2}\sqrt[3]{C-D} \quad (8)$$

Now we consider the two cases for the solution to cubic equations.

Case I: Solution for cubic equation in the real number domain of C and D

As has been mentioned above, when both C and D are either purely real or purely complex, the solution occurs in the real number domain. Expanding the cubic power of both equations gives:

$$(A+B)^3 = a^3 + 3a^2B + 3aB^2 + B^3 = C + D$$

$$(A-B)^3 = a^3 - 3a^2B + 3aB^2 - B^3 = C - D$$

Adding and subtracting the above two equations gives:

$$a^3 + 3aB^2 = C \; ; \; 3a^2B + B^3 = D$$

Eliminating the variable B from the above two equations will give the equation in a as follows:

$$a^9 - \frac{3}{4}ca^6 - \frac{\left[15c^2 - 27d^2\right]}{64}a^3 - \frac{c^3}{64} = 0$$

Using the substitution m = a^3 to convert the above equation into a general cubic equation gives;

$$m^3 - \frac{3}{4}cm^2 - \frac{\left[15c^2 - 27d^2\right]}{64}m - \frac{c^3}{64} = 0$$

The above equation is converted into depressed form and equated to the given polynomial equation in x. To do this, the usual variable transformation equation to depressed cubic form is used, i.e.,

$$m = x - \frac{1}{3}\left(\frac{-3}{4}c\right) = x + \frac{1}{4}c$$

Using this transformation, the cubic equation in m is transformed into the x variable as follows:

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$$x^3 - \left[\frac{27}{64}\left(c^2 - d^2\right)\right] x - \left[\frac{27}{256}\left(c^3 - d^2 c\right)\right] = 0$$

Equating the terms of the above equation to that of the given equation: $x^3 + Rx + S = 0$,

$$R = -\left[\frac{27}{64}\left(c^2 - d^2\right)\right] ; S = -\left[\frac{27}{256}\left(c^3 - d^2 c\right)\right]$$

Solving for c and d in terms of R and S will eventually give:

$$c = \frac{4S}{R} ; d = \pm \frac{4}{3\sqrt{3}\,R} \sqrt{\left(4R^3 + 27S^2\right)} \ (9)$$

In order for Case I to be true, both c and d should be real numbers. For the given cubic equation:

$$x^3 + Rx + S = 0$$

Since c = 4S/R, the condition for c to be a real number is automatically satisfied since R and S are both assumed to be real numbers in the given cubic equation. From the expression for d given in terms of the coefficients of the cubic equation, for d to be a real number, the following condition shall be satisfied:

$$\left(4R^3 + 27S^2\right) < 0$$

In other words, the discriminant has to be positive. It is interesting to note that this condition is similar to that of Cardan's solution where the discriminant is positive in case the solution does not involve manipulating complex numbers or where there is no need to apply De Moivre's Theorem.

The solution in the real number domain of c = real and D = real proceeds first by computing the values of c and d from the coefficients of the given equation: $x^3 + Rx + S = 0$

$$c = \frac{4S}{R} ; d = \pm \frac{4}{3\sqrt{3}\,R} \sqrt{\left(4R^3 + 27S^2\right)}$$

The roots of the cubic equation are then given by:

$$x = a^3 - \frac{S}{R} \ (10)$$

<u>Case II: Solution for cubic equation in the complex number domain of C +D = c+di</u>

The solution to the equation when C+D takes the form c+di is obtained through the cubic root of a complex number c + di such that:

$$(a + bi)^3 = c + di$$

Expanding the $(a+bi)^3$ term and equating it to c+di gives:

$$\left[a^2 - 3ab^2\right] + \left[3a^2 b - b^3\right] i = c + di$$

From which it is apparent that:

$$a^2 - 3ab^2 = c ; 3a^2 b - b^3 = d$$

Eliminating the complex coefficient b and expressing the above equation in terms of the real part of a+bi, i.e., a only, gives:

$$a^9 - \frac{3}{4} c a^6 - \frac{\left[15c^2 + 27d^2\right]}{64} a^3 - \frac{c^3}{64} = 0$$

Using the substitution $m = a^3$ to convert the above equation into a general cubic equation gives;

$$m^3 - \frac{3}{4} c m^2 - \frac{\left[15c^2 + 27d^2\right]}{64} m - \frac{c^3}{64} = 0$$

The above equation is converted into depressed form and equated to the given polynomial equation in x. To do this, the usual variable transformation equation to depressed cubic form is used, i.e.,

$$m = x - \frac{1}{3}\left(\frac{-3}{4} c\right) = x + \frac{1}{4} c$$

Using this transformation, the cubic equation in m is transformed into the x variable as follows:

$$x^3 - \left[\frac{27}{64}\left(c^2 + d^2\right)\right] x - \left[\frac{27}{256}\left(c^3 + d^2 c\right)\right] = 0$$

Equating the terms of the above equation to that of the given equation: $x^3 + Rx + S = 0$,

$$R = -\left[\frac{27}{64}\left(c^2 + d^2\right)\right] ; S = -\left[\frac{27}{256}\left(c^3 + d^2 c\right)\right]$$

Solving for c and d in terms of R and S will eventually give:

$$c = \frac{4S}{R} ; d = \pm \frac{4}{3\sqrt{3}\,R} \sqrt{-\left(4R^3 + 27S^2\right)}$$

Now, working backwards from c and d to the equation in the x variable, since a +bi is the cube root of c +di, the value of a is computed using De Moivre's Theorem as the real part of the cube root of the complex number c +di as follows:

$$a = r^{1/3} \left[Cos\left(\frac{\theta + 2n\pi}{3}\right)\right] \quad n = 0,\ 1,\ 2 \ (11)$$

Where the values of r and θ are given by:

$$r = \sqrt{c^2 + d^2} ; \theta = Cos^{-1}\left(\frac{c}{r}\right) \ (12)$$

The roots of the cubic equation are given by:

$$x = m - \frac{1}{4} c$$

Using the relation:

$$m = a^3 ; c = \frac{4S}{R}$$

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gives:

$$x = a^3 - \frac{S}{R}$$

It is apparent from the two cases that the same formula is used with different signs of the discriminant in the square root. Which of the cases applies depends on the discriminant

$$\left(4R^3 + 27S^2\right)$$

The expression in the square root is negative for it to be formulated through the complex number of the c+di, which requires applying De Moivre's Theorem. This is seen from the formula for c and d:

$$c = \frac{4S}{R}\ ;\ d = \pm \frac{4}{3\sqrt{3}\,R}\sqrt{-\left(4R^3 + 27S^2\right)}$$

While c is automatically a real number in both cases, to get a real number from the square root, the discriminant should be negative, i.e.,

$$\left(4R^3 + 27S^2\right) < 0$$

This means that the expression for d is always a real number, whereas in Cardan's method, d can have a complex number part although the solution is a real number whereby the complex parts cancel each other. The table below provides a comparison of the conditions between Cardan's Method and the proposed method.

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Discriminant	Cardan's method		Proposed method	
Discriminant	Positive	Negative	Positive	Negative
$4R^3+27S^2$	Solution formulated as average of cube roots of real numbers C+D and C-D	Solution involves complex numbers and formulated using De Movire's Theorem	Solution formulated as average of cube roots of the real numbers C+D and C-D same as Cardan's Method	Solution starts with complex numbers but eventually formulated in real number form

Table 1. Formulation of solution to cubic equation based on the sign of discriminant: Comparison of the proposed method with Cardan's solution

2.5. Method for Quartic equations

The quartic equation is solved as a combination of the roots of two quadratic equations as follows. For a given four-degree polynomial equation that is expressed in depressed form:

$$x^4 + px^2 + qx + r = 0 \ (13)$$

The solution to the equation is obtained through the square roots of a pair of complex numbers A, B, C, D, E, F, and H such that:

$$(A+B)^2 = C+D;\ (A-B)^2 = C-D \ (14)$$

$$(E+F)^2 = -C+H;\ (E-F)^2 = -C-H \ (15)$$

To solve this quartic equation as a product of two quadratics, the following relationships are established whereas E, F, C, and H are defined similarly like AB, C, and D to have real or complex number form:

$$(A+B)^2 = C+D$$

$$(E+F)^2 = -C+H$$

With the definition X=A^2 and E= X^2, two sets of quadratic equations are formed:

$$X^2 - CX + \frac{D^2}{4} = 0$$

$$X^2 + CX + \frac{H^2}{4} = 0$$

Multiplying the two quadratics and using lower case symbols gives:

$$\left(x^2 - xc + \frac{d^2}{4}\right)\left(x^2 + xc + \frac{h^2}{4}\right) = 0 \ (16)$$

Equation (16) is expanded further as follows:

$$x^4 - \left(\frac{d^2}{4} + \frac{h^2}{4} + c^2\right)x^2 + \left(\frac{h^2}{4} - \frac{d^2}{4}\right)(c)x + \frac{h^2d^2}{16} = 0 \ (17)$$

Equating the coefficients p, q, and r of the given quartic polynomial equation given in Equation (13) with those of equation (17) will give the following:

$$\frac{d^2}{4} + \frac{h^2}{4} + c^2 = -p \ (18)$$

$$\left(\frac{h^2}{4} - \frac{d^2}{4}\right)(c) = q \ (19)$$

$$\frac{h^2d^2}{16} = r \ (20)$$

Solving Equations 18, 19, and 20 simultaneously for the variable c gives the following six-degree equation:

$$c^6 + 2\,p\,c^4 + \left(p^2 - 4r\right)\ c^2 - q^2 = 0 \ (21)$$

and using the substitution:

$$y = c^2 \ (22)$$

Gives a cubic equation in y, i.e.,

$$y^3 + 2py^2 + \left(p^2 - 4r\right)y - q^2 \ (23)$$

Once the value of y is found by solving the cubic equation given in Equation (23), the values of c, d, and h are found as given by the equations stated in Section 2.1:

$$c = \sqrt{y} \ (24)$$

The values of d and h are found from:

$$d = \sqrt{-2\left(\frac{q}{c}\right) - 2p - 2c^2} \ (25)$$

$$h = \sqrt{+2\left(\frac{q}{c}\right) - 2p - 2c^2} \ (26)$$

The values of d and h serve as the discriminant of the quartic equations with the following relationship to the roots:

1. If both D and H are real numbers, all the roots of the quartic equation are real numbers, and De Movier's formula can be applied.
2. If both D and H are complex numbers, the roots of the quartic equations are either real or complex numbers, and interestingly, De Movie's formula cannot be used as the solution is expressed in real number coefficients, whereas the roots may turn out to be real numbers or complex numbers depending on the occurrence of the square root of a positive number or a negative number, respectively.
3. If either of D or H is a complex number, the quartic equation has two real number and two complex conjugate solutions. De Movier's Formula can be applied to the part (d or h) that is a real number.

Case 1: Solution when D and H are complex numbers

This corresponds to Case I above in which both D and H are real numbers, all the roots are real numbers, and De Movier's formula can be applied.

The values of a and e to the left sides of equations (14) and (15) are determined using De Moivre's Formula.

Let $r_A = \sqrt{c^2 + d^2}$ and $\theta_A = Cos^{-1}\left(\frac{c}{r_A}\right)$

$$X_A = A^2 = \left\{ r_A \left(Cos\left(\frac{\theta_A}{2}\right)\right)^2, \ -r_A \left(Sin\left(\frac{\theta_A}{2}\right)\right)^2 \right\}$$

Let $r_E = \sqrt{c^2 + h^2}$ and $\theta_E = Cos^{-1}\left(\frac{c}{r_E}\right)$

$$X_E = E^2 = \left\{ r_E \left(Cos\left(\frac{\theta_E}{2}\right)\right)^2, \ -r_E \left(Sin\left(\frac{\theta_E}{2}\right)\right)^2 \right\}$$

Case 2: Solution when both D and H are real numbers

This is the case in which the roots can be real or complex. In this case, De Movier's formula cannot be applied, and the roots are found from the following solutions:

$$X_A = A^2 = \frac{1}{2}\left(C \pm \sqrt{C^2 - D^2}\right)$$

$$X_E = E^2 = \frac{1}{2}\left(C \pm \sqrt{C^2 - H^2}\right)$$

Case 3: Solution when either D or H (but not both) is either a real number or a complex number

This is a mixed case which results in roots, at least two of which are real numbers corresponding to either D or H being a complex number as for Case I above. The remaining two roots can be real or complex conjugates like it was for Case II above. If, for example, D is a complex number and H is a real number, the solution is obtained as follows:

$$X_A = A^2 = \left\{ r_A \left(Cos\left(\frac{\theta_A}{2}\right)\right)^2, \ -r_A \left(Sin\left(\frac{\theta_A}{2}\right)\right)^2 \right\}$$

$$X_E = E^2 = \frac{1}{2}\left(C \pm \sqrt{C^2 - H^2}\right)$$

In general, the expressions for d and h given by:

$$d = \sqrt{-2\left(\frac{q}{c}\right) - 2p - 2c^2}$$

$$h = \sqrt{+2\left(\frac{q}{c}\right) - 2p - 2c^2}$$

will determine the nature of the roots being real, complex, or a combination of real and complex numbers.

3. Application Example

3.1. Quadratic Equation Examples

Example 3.1.1. x^2-2x-3 = 0; R = -2 and S = -3

$$C = -R = 2 \ and \ d = \sqrt{4S} = \sqrt{4 * -3} = \sqrt{12}i$$

This is Case II, where C+D is in the complex form: C+D = c+di

$$X = A^2 = \frac{1}{4}\left(\sqrt{C+D} \pm \sqrt{C-D}\right)^2$$

$$X = \frac{1}{4}\left(\sqrt{2+\sqrt{12}i} \pm \sqrt{2-\sqrt{12}i}\right)^2$$

$$= \frac{1}{4}\left(2+\sqrt{12}i + 2 - \sqrt{12}i \ \pm 2 * \left(\sqrt{2^2 - (\sqrt{12}i)^2}\right)\right)$$

$$= \frac{1}{4}(4 \pm 8) = \{3, \ -1\}$$

The application of De Moivre's Theorem is shown below for this example:

Let $r = \sqrt{c^2 + d^2}$ and $\theta = Cos^{-1}\left(\frac{c}{r}\right)$

$$r = \sqrt{2^2 + \sqrt{12}^2} = 4$$

$$\theta = Cos^{-1}\left(\frac{2}{4}\right) = 60^0$$

$$X = A^2 = \left\{ r \left(Cos\left(\frac{\theta}{2}\right)\right)^2, \ -r \left(Sin\left(\frac{\theta}{2}\right)\right)^2 \right\}$$

$$X = A^2 = \left\{ 4\left(Cos\left(30^0\right)\right)^2, \ -4\left(Sin\left(30^0\right)\right)^2 \right\}$$

$$X = A^2 = \left\{ 4\left(\frac{\sqrt{3}}{2}\right)^2, \ -4\left(\frac{1}{2}\right)^2 \right\}$$

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$$X = A^2 = \{3, -1\}$$

Example 3.1.2. $x^2+2x+10 = 0$; R = 2 and S = 10

$$C = -R = -2 \; and \; d = \sqrt{4S} = \sqrt{4*10} = \sqrt{40}$$

This is Case I, where C+D is in the real number form: C+D = c+d

$$X = A^2 = \frac{1}{4}\left(\sqrt{C+D} \pm \sqrt{C-D}\right)^2$$

$$X = \frac{1}{4}\left(\sqrt{-2+\sqrt{40}} \pm \sqrt{-2-\sqrt{40}}\right)^2$$

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$$X = \frac{1}{4}\left(-2+\sqrt{40}+-2-\sqrt{40} \pm 2*\left(\sqrt{(-2)^2-(\sqrt{40})^2}\right)\right)$$

$$X = \frac{1}{4}(-4 \pm 12i) = \{-1+3i, \ -1-3i\}$$

3.2. Cubic equations Examples

The method developed is tested through three cubic equation examples having discriminants negative, zero, and positive respectively. The solutions are worked out for each case as provided below:

Example 1: $x^3 - 6x + 4 = 0$

In this equation, R = -6 and S = 4. The Discriminant

$$Discr = 4R^3 + 27S^2 = 4(-6)^3 + 27(4)^2 = -432 \leq 0$$

This corresponds to Case II of C+D being in the form C+D = c+di, i.e., complex number domain. All the solutions of the cubic equations must be real numbers.

The values of c and d are given by:

$$c = \frac{4S}{R} = \frac{4*4}{-6} = -\frac{8}{3};$$

$$d = \pm\frac{4}{3\sqrt{3}\,R}\sqrt{-\left(4R^3+27S^2\right)} = \pm\frac{4}{3\sqrt{3}\,(-6)}\sqrt{-(-432)} = \pm\frac{8}{3}$$

The value of a is computed using De Moivre's Theorem as the real part of the cube root of the complex number c + di as follows:

$$a = r^{1/3}\left[Cos\left(\frac{\theta+2n\pi}{3}\right)\right] \quad n = 0, \ 1, \ 2$$

Where the values of r and θ are given by:

$$r = \sqrt{c^2 + d^2} = \frac{8\sqrt{2}}{3};$$

$$\theta = Cos^{-1}\left(\frac{c}{r}\right) = Cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$$

The values of a are worked out as follows:

$$a_1 = r^{1/3}\left[Cos\left(\frac{\theta}{3}\right)\right] = \left(\frac{8\sqrt{2}}{3}\right)^{1/3}\left[Cos\left(\frac{\pi}{4}\right)\right] = \left(\frac{8\sqrt{2}}{3}\right)^{1/3}\left(\frac{1}{\sqrt{2}}\right)$$

$$m_1 = {a_1}^3 = \frac{8\sqrt{2}}{3}\left(\frac{1}{(\sqrt{2})^3}\right) = \frac{4}{3}$$

$$a_2 = r^{\frac{1}{3}}\left[Cos\left(\frac{\theta+2\pi}{3}\right)\right] = \left(\frac{8\sqrt{2}}{3}\right)^{\frac{1}{3}}\left[Cos\left(\frac{11\pi}{12}\right)\right] = \left(\frac{8\sqrt{2}}{3}\right)^{1/3}(-0.965925826) = -1.503505501$$

$$m_2 = {a_2}^3 = -3.398717474$$

$$a_3 = r^{\frac{1}{3}}\left[Cos\left(\frac{\theta+2\pi}{3}\right)\right] = \left(\frac{8\sqrt{2}}{3}\right)^{\frac{1}{3}}\left[Cos\left(\frac{19\pi}{12}\right)\right] = \left(\frac{8\sqrt{2}}{3}\right)^{1/3}(0.258819045) = 0.402863084$$

$$m_3 = {a_3}^3 = 0.06538414$$

The roots of the cubic equation are then given by:

$$x_1 = {a_1}^3 - \frac{S}{R} = \frac{4}{3} - \left(\frac{4}{-6}\right) = 2$$

$$x_2 = {a_2}^3 - \frac{S}{R} = -3.398717474 - \left(\frac{4}{-6}\right) = -2.732050808$$

$$x_3 = {a_3}^3 - \frac{S}{R} = 0.06538414 - \left(\frac{4}{-6}\right) = 0.732050807$$

Example 2: $x^3 - 3x - 2 = 0$

In this equation, R = -3 and S = -2. The Discriminant

$$Discr = 4R^3 + 27S^2 = 4(-3)^3 + 27(-2)^2 = 0$$

This cubic equation has repeating roots since Discr = 0.

The values of c and d are given by:

$$c = \frac{4S}{R} = \frac{4*-2}{-3} = \frac{8}{3};$$

$$d = \pm\frac{4}{3\sqrt{3}\,R}\sqrt{-\left(4R^3+27S^2\right)} = \pm\frac{4}{3\sqrt{3}\,(-3)}\sqrt{-(0)} = 0$$

The value of a is computed using De Moivre's Theorem as the real part of the cube root of the complex number c + di as follows:

$$a = r^{1/3}\left[Cos\left(\frac{\theta+2n\pi}{3}\right)\right] \quad n = 0, \ 1, \ 2$$

Where the values of r and θ are given by:

$$r = \sqrt{c^2 + d^2} = \frac{8}{3};$$

$$\theta = Cos^{-1}\left(\frac{c}{r}\right) = Cos^{-1}(1) = 0$$

The values of a are worked out as follows:

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$$a_1 = r^{1/3}\left[Cos\left(\frac{\theta}{3}\right)\right] = \left(\frac{8}{3}\right)^{1/3}[Cos(0)] = \left(\frac{8}{3}\right)^{1/3}(1)$$

$$m_1 = {a_1}^3 = \frac{8}{3}(1^3) = \frac{8}{3}$$

$$a_2 = r^{\frac{1}{3}}\left[Cos\left(\frac{\theta+2\pi}{3}\right)\right] = \left(\frac{8}{3}\right)^{\frac{1}{3}}\left[Cos\left(\frac{2\pi}{3}\right)\right]$$

$$= \left(\frac{8}{3}\right)^{1/3}\left(\frac{-1}{2}\right)$$

$$m_2 = {a_2}^3 = \left(\frac{8}{3}\right)\left(\frac{-1}{2}\right)^3 = -\frac{1}{3}$$

$$a_3 = r^{\frac{1}{3}}\left[Cos\left(\frac{\theta+4\pi}{3}\right)\right] = \left(\frac{8}{3}\right)^{\frac{1}{3}}\left[Cos\left(\frac{4\pi}{3}\right)\right]$$

$$= \left(\frac{8}{3}\right)^{1/3}\left(\frac{-1}{2}\right)$$

$$m_3 = {a_3}^3 = \left(\frac{8}{3}\right)\left(\frac{-1}{2}\right)^3 = -\frac{1}{3}$$

The roots of the cubic equation are then given by:

$$x_1 = {a_1}^3 - \frac{S}{R} = \frac{8}{3} - \left(\frac{-2}{-3}\right) = 2$$

$$x_2 = {a_2}^3 - \frac{S}{R} = -\frac{1}{3} - \left(\frac{-2}{-3}\right) = -1$$

$$x_3 = {a_3}^3 - \frac{S}{R} = -\frac{1}{3} - \left(\frac{-2}{-3}\right) = -1$$

The repeating root is x=1 as the solution indicates.

Example 3: *x^3 - 2x + 4 = 0*

In this equation, R = -2 and S = 4. The Discriminant

$$Discr = 4R^3 + 27S^2 = 4(-2)^3 + 27(4)^2 = 400 > 0$$

This corresponds to Case I where C+D is in the real number domain C+D = c+d, both c and d are real numbers. This equation has one real root and two complex roots. To get the real root, the formula given in Case 2 of the Methods section is applied.

The values of c and D are given by:

$$C = \frac{4S}{R} = \frac{4*4}{-2} = -8\,;$$

$$D = \pm\frac{4}{3\sqrt{3}\,R}\sqrt{\left(4R^3+27S^2\right)}$$

$$= \pm\frac{4}{3\sqrt{3}\,(-2)}\sqrt{400} = \mp\frac{40}{3\sqrt{3}} = \mp 7.698003589$$

Both +D and –D give the same result; hence, choose D = 7.698003589

$$\sqrt[3]{C+D} = \sqrt[3]{-8+7.698003589} = -0.670914627$$

$$\sqrt[3]{C-D} = \sqrt[3]{-8-7.698003589} = -2.503887477$$

The value of a is given by:

$$a = \frac{1}{2}\sqrt[3]{C+D} + \frac{1}{2}\sqrt[3]{C-D}$$

$$a = \frac{1}{2}(-0.670914627 - 2.503887477) = -1.58740105$$

$$m_3 = {a_3}^3 = (-1.587401052)^3 = -4$$

The real root of the cubic equation is then given by:

$$x = a^3 - \frac{S}{R} = -4 - \left[\frac{4}{-2}\right] = -4+2 = -2$$

To obtain the other (complex) roots, synthetic division of the cubic equation by x + 2 gives:

$$\frac{x^3-2x+4}{x+2} = x^2 - 2x + 2$$

The roots of the quadratic equation using function evaluation:

$$z = -\frac{-2}{2} = 1$$

$$f(z) = 1^2 - 2(1) + 1 = 1$$

$$x = z \pm \sqrt{-f(z)} = 1 \pm i$$

Therefore, the other complex roots of the cubic equation are 1 + it and 1 - i.

3.3. Quartic equations

Let us now try to solve the following quartic polynomial equation that is given in depressed form as follows (to avoid the hustle of having to convert other forms to the depressed form):

$$x^4 - 19.375x^2 - 20.625\,x + 26.05078 = 0$$

In Equation (37), p = -19.375; q = -20.625; and r = 26.05078. Now we try to solve the resolving cubic equation given by Equation (16), i.e.,

$$y^3 + 2py^2 + \left(p^2 - 4r\right)y - q^2 = 0$$

$$y^3 + 2*(-19.375)*y^2 + \left((-19.375)^2 - 4*26.05078\right)y - (-20.625)^2 = 0$$

$$y^3 - 38.75*y^2 + 271.1875*y - 425.390625 = 0$$

Taking one of the roots of the above cubic equation in y,

$$y = 2.25\,,\; c = \sqrt{y} = \sqrt{2.25} = \pm 1.5$$

The values of d and h are found from Equations (18) and (19)

$$d = \sqrt{-2\left(\frac{q}{c}\right) - 2p - 2c^2} \; (18)$$

$$d = \sqrt{-2\left(\frac{-20.625}{\pm 1.5}\right) - 2(-19.375) - 2*(\pm 1.5)^2} = \{7.85812,\ 2.5980\}$$

$$h = \sqrt{+2\left(\frac{q}{c}\right) - 2p - 2c^2} \; (19)$$

$$h = \sqrt{+2\left(\frac{-20.625}{\pm 1.5}\right) - 2(-19.375) - 2*(\pm 1.5)^2} = \{2.5980,\ 7.85812\}$$

The above pair of values show that instead of four, only two solutions are unique. The solutions for a and e are therefore interchangeable and can proceed with either of these as follows. Choosing the real part of a +bi as the alternative:

$$r_a = \sqrt{c^2 + d^2}\,;\ \theta_a = Cos^{-1}\left(\frac{c}{r_a}\right) \; (21)$$

Using {c, d} = {1.5, 7.85812},

$$r_a = \sqrt{1.5^2 + 7.85812^2} = 8\,;\ \theta_a = Cos^{-1}\left(\frac{1.5}{8}\right) = 1.38218$$

$$X_A = A^2 = \left\{r_A\left(Cos\left(\frac{\theta_A}{2}\right)\right)^2,\ -r_A\left(Sin\left(\frac{\theta_A}{2}\right)\right)^2\right\}$$

$$x_{a1} = 8 * Cos^2\left(\frac{1.38218}{2}\right) = 4.75$$

$$x_{a2} = -8 * Sin^2\left(\frac{1.38218}{2}\right) = -3.25$$

Using {-c. h} = {-1.5, 2.5980}

$$r_e = \sqrt{1.5^2 + 2.5980^2} = 3\,;\ \theta_e = Cos^{-1}\left(\frac{-1.5}{3}\right) = 2.094395$$

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$$\{X_{e1},\ X_{e2}\} = \ E^2$$
$$= \left\{ 3 * \left(Cos\left(\frac{2.094395}{2}\right)\right)^2,\ -3 * \left(Sin\left(\frac{2.094395}{2}\right)\right)^2 \right\}$$

The solution set, therefore, is:

$$X = \ \{4.75,\ 3.25,\ 0.75,\ -2.25\}$$

4. Conclusion

Methods for solving polynomial equations have been developed over the years that adopt variable approaches and involve varying degrees of complexity. The methods are broad in approach, involving substitution, complex number algebra, and trigonometry, reduction to depressed form, elimination, and decomposition of the original polynomial into solvable products of polynomials of lesser degree. Some methods are unified in that they apply to the broader range of the degree of polynomial equations, while others are specific, such as applying to cubic or quartic equations only.

This paper presented and discussed a unified approach for solving polynomial equations of degrees 2, 3, and 4. The method uses an inversion of the roots of variables that allows explicit determination of the roots within the limits of solvability of polynomials by radicals. The approach is simple to develop, understand, and even formulate the solution, as the discussion on method development shows. The method, in addition, follows a reverse route to the common methods of solving polynomials, starting with the dependent variable of the polynomials and inverting through the nth root to find an explicit solution of the roots of the equations.

In following up through finding the roots of the equations in this method, it is noticed that the final solution to the roots of the equation eventually appears in the form they have to appear. In other words, real roots appear as real numbers, and complex roots appear as complex numbers. As a comparison, Cardan's solution starts with real numbers and arrives at a solution that involves complex number manipulation, whereas the roots are eventually real numbers. In this proposed method, the solution starts with a general form irrespective of whether the solutions are real or complex but arrives at the solutions that are always expressed as radicals of real numbers. Moreover, this method proceeds from complex to real numbers and hence takes a reverse detour to Cardan's Method. The use of complex number arithmetic for solving equations that may eventually be expressed in real number forms is also further demonstrated. This approach is one further example of the many ways in which polynomial equations can be solved.

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