Qeios

Some New Aspects of Quantum Gravity

Harish Parthasarathy
Electronics and Communication division
Netaji Subhas University of Technology
New Delhi, India

March 27, 2024

We have proposed the quantization of the gravitational field in a synchronous reference frame taking as independent position fields, the six spatial components of the metric tensor. The Einstein-Hilbert Lagrangian is quadratic in the space time derivatives of these metric tensor components and hence in particular, the momentum fields become linear functions of the space-time derivatives of the position fields. It is this fact that gives a simple form to the Hamiltonian density of the gravitational field in a synchronous frame, this simple form of the Hamiltonian being a quadratic function of the momentum fields with a shift that is linear in the spatial derivatives of the metric, very much like the Hamiltonian of a non-relativistic particle moving in a vector potential. Differential equations for the gravitational field propagator are derived and we explain how approximations to this propagator can be derived and used to deduce the graviton propagator corrections caused by nonlinear interactions of the graviton field with itself. We explain how this corrected graviton propagator can be used to deduce how much mass the graviton acquires due to these self-interactions of cubic and higher order. We then consider the important problem involving the coupling of a nonlinear field theory described by its Lagrangian density to a quantum noisy bath and explain how the resulting Hamiltonian of the field plus bath can be used to derive the Hudson-Parthasarathy noisy Schrodinger equation (HPS) which is a quantum stochastic differential equation for the joint unitary evolution of the field interacting with the noisy bath. We explain this in the context of gravity coupled to a noisy bath like a noisy electromagnetic field. The HPS equation contains linear as well as quadratic terms in the white bath noise with the linear terms representing quantum annihilation and creation/quantum Brownian motion process differentials and the quadratic terms representing quantum conservation/Poisson processes differentials. Finally we explain how using Feynman path integrals for fields for evaluating the quantum effective action produced by higher order cumulants of the current field, we can calculate corrections to the quantum effective action produced by higher order cumulants of the current field and hence demonstrate how gauge symmetries of the classical action get broken when we pass over to the quantum effective action

https://doi.org/10.32388/PB17OH

with additional symmetry breaking terms produced by the presence of higher order cumulants of the current. This kind of approximate symmetry breaking is known to give masses to massless particles or more generally, corrections to the masses of already massive particles and we illustrate this idea in the context of interactions of the gravitational field with a random electromagnetic field being regarded as the current. This interaction is the standard Maxwell action used in general relativity. The drawback of our approach to quantum gravity is that is its not diffeomorphic invariant since we have chosen our frame to be always synchronous. Further work on how one can incorporate interactions of the gravitational field with a random non-Abelian gauge field is in progress which becomes important because it generates non only quadratic but also cubic and fourth degree terms in the gauge field when it interacts with gravity.

1 Quantum gravity in the synchronous frame, some perturbative calculations for the equal time commutators

The independent components of the metric are just six in number: $\phi = (\phi_r)_{r=1}^6 = \{g_{rs} : 1 \leq r \leq s \leq 3\}$ since our coordinates are synchronous, ie, chosen so that the four conditions $g_{00} = 1, g_{0r} = 0, r = 1, 2, 3$ are satisfied. The Lagrangian density for ϕ, namely the Einstein-Hilbert Lagrangian has the form

$$L(\phi, \phi_{,0}, \phi_{,r}) = (1/2)F_{1,rs}(\phi)\phi_{r,0}\phi_{s,0}+(1/2)F_{2rksm}(\phi)\phi_{r,k}\phi_{s,m}+F_{3rsm}(\phi)\phi_{r,0}\phi_{s,m}$$

The position fields are $\phi = (\phi_r)_{r=1}^6$. The momentum field conjugate to the position field ϕ_r is

$$P_r = \partial L/\partial \phi_{r,0} = F_{1rs}(\phi)\phi_{s,0} + F_{3rsm}(\phi)\phi_{s,m}$$

let

$$((G_{rs}(\phi))) = ((F_{1rs}(\phi)))^{-1}$$

Then, we find that the velocity fields are given by

$$\phi_{r,0} = G_{rs}(\phi)P_s - G_{rk}(\phi)F_{3ksm}(\phi)\phi_{s,m}$$

or in matrix notation,

$$\phi_{,0} = G(\phi)P - G(\phi)F_3(\phi)(\nabla \otimes \phi)$$

where $\nabla = (\partial_r)_{r=1}^3$ is the spatial gradient operator. In order to make $\phi_{,0}$ self-adjoint after quantization, we replace the above by

$$\phi_{r,0} = (G_{rs}(\phi)P_s + P_s G_{rs}(\phi))/2 - G_{rk}(\phi)F_{3ksm}(\phi)\phi_{s,m}$$

The equal time CCR's in matrix notation are

$$[\phi(t, r), P(t, r')^T] = i\delta^3(r - r')$$

so it follows that

$$[\phi(t,r),\phi_{,0}(t,r')^T] = iG(\phi(t,r))\delta^3(r-r')$$

The field equations

$$\partial_0 \partial L/\partial\phi_{k,0} + \partial_r \partial L/\partial\phi_{k,r} - \partial L/\partial\phi_k = 0$$

can be expressed as

$$(\partial_0^2 - c(r,s)\partial_r\partial_s - b_1(r)\partial_0\partial_r - b_2\partial_0 - b_3(r)\partial_r)\phi = N(\phi)$$

where $N(\phi)$ is of the form

$$N(\phi) = A_{1rs}(\phi)\phi_{,rs} + A_{2r}(\phi)(\phi_{,0}\otimes\phi_{,r}) + A_3(\phi)(\phi_{,0}\otimes\phi_{,0}) + A_{4r}(\phi)(\phi_{,0}\otimes\phi_{,r})$$

$$+A_{5rs}(\phi)(\phi_{,r}\otimes\phi_{,s})$$

We can expand the nonlinear functional $\phi \to N(\phi)$ as Volterra series:

$$N(\phi(x)) = \sum_{n\geq 1}\delta^n K_n(\phi)(x)$$

where

$$K_n(\phi)(x) = \int K_n(x,x_1,...,x_n)\otimes_{m=1}^n \phi(x_m)dx_1...dx_n$$

δ is a small perturbation parameter. We also expand the field solution as

$$\phi(x) = \phi_0(x) + \sum_{n\geq 1}\delta^n\phi_n(x)$$

where ϕ_0, the zeroth order field satisfies

$$\Box\phi_0(x) = 0, \Box = \partial_0^2 - c(r,s)\partial_r\partial_s - b_1(r)\partial_0\partial_r - b_2\partial_0 - b_3(r)\partial_r$$

and for $n \geq 1$

$$\Box\phi_n(x) =$$

coefficient of δ^n in $N(\phi)(x)$. Writing

$$\phi_0(x) = u(r)exp(-i\omega t)$$

gives us

$$(-\omega^2 - c(k,s)\partial_k\partial_s + i\omega b_1(k)\partial_k + i\omega b_2 - b_3(k)\partial_k)u(r) = 0$$

After applying the boundary conditions on $u(r)$ the possible "eigenfrequencies" ω assume only discrete values and we can express the solution as

$$\phi_0(x) = \phi_0(t,r) = \sum_n c(n)exp(-i\omega(n)t)u_n(r)$$

Note that if $u_n(r)$ is an "eigen-solution" corresponding to the frequency eigenvalue $\omega(n)$, then $\bar{u}_n(r)$ is an eigensolution corresponding to the frequency eigenvalue $-\omega(n)$. Thus, taking into account the fact that $\phi_0(x)$ is self-adjoint (which corresponds in classical field theory to a real field), we can better express the above expansion as

$$\phi_0(x) = \sum_{n\geq 1}[c(n)u_n(r)exp(-i\omega(n)t) + c(n)^*\bar{u}_n(r)exp(i\omega(n)t)]$$

The zeroth order term in the commutation relation

$$[\phi(t,r), \phi_{,0}(t,r')^T] = iG(\phi(t,r))\delta^3(r-r')$$

is given by

$$[\phi_0(t,r), \phi_{0,0}(t,r')^T] = iG(\phi_0(t,r))\delta^3(r-r')$$

Remark: We are assuming that Planck's constant h is very small and actually appears on the rhs of the above commutation relation. Hence, $G(\phi_0)$ must actually be replaced by $G(<\phi_0(t,r)>)$ where $<\phi_0>$ is the Vacuum expectation of ϕ_0. Writing

$$G_0(t,r) = G_0(x) = G(<\phi_0(t,r)>)$$

it then follows that the zeroth order commutation relation is

$$[\phi_0(t,r), \phi_{0,0}(t,r')^T] = iG_0(t,r)\delta^3(r-r')$$

where now $G_0(t,r)$ is a c-number field. Let us consider the particular case when $<\phi_0(t,r)>$ is independent of time. Then we have

$$[\phi_0(t,r), \phi_{0,0}(t,r')^T] = iG_0(r)\delta^3(r-r')$$

It follows then that writing

$$[c(n), c(m)] = 0, [c(n), c(m)^*] = \lambda(n)\delta[n-m]$$

that the above commutation relations are satisfied iff

$$\sum_{n\geq 1}\lambda(n)\omega(n)u_n(r)u_n(r')^* = G_0(r)\delta^3(r-r')$$

Now observe that the $u_n's$ satisfy

$$(\omega(n)^2 + c(r,s)\partial_r\partial_s - i\omega(n)b_1(k)\partial_k - i\omega(n)b_2 + b_3(k)\partial_k)u_n(r) = 0, n \geq 1$$

We are assuming that the $\omega(n)'s$ are real. Taking the conjugate of this equation gives

$$(\omega(m)^2 + c(k,s)\partial_k\partial_s + i\omega(m)b_1(k)\partial_k + i\omega(m)b_2 + b_3(k)\partial_k)\bar{u}_m(r) = 0$$

Multiply the first equation by $\bar{u}_m(r)$, integrate over space, then multiply the second equation by $u_n(r)$, integrate over space, subtract the second from the first, integrate by parts and assume that the $u_n's$ vanish on the boundary. We get

$$(\omega(n)^2-\omega(m)^2) < u_m, u_n > +i(\omega(n)-\omega(m))b_1(k) < \partial_k u_m, u_n >$$

$$-ib_2(\omega(n)+\omega(m)) < u_m, u_n > -2b_3(k) < \partial_k u_m, u_n >= 0$$

Taking $m \neq n$ gives us

$$< u_m, u_n >= 0$$

and if either $b_1(k) \neq 0$ or $b_3(k) \neq 0$, then also

$$< \partial_k u_m, u_n >= 0$$

Taking $m = n$ gives us

$$b_2 = 0,$$

For simplicity, we assume that $b_1(k) = b_3(k) = b_2 = 0$ and hence

$$\square = \partial_0^2 - c(r,s)\partial_r\partial_s$$

so that the field equation reads

$$\square\phi = N(\phi)$$

Therefore since now $c(r,s)\partial_r\partial_s$ is a self adjoint operator and $-\omega(n)^2$ are its eigenvalues with corresponding normalized eigenfunctions $u_n(r)$, we have the result from the spectral theorem that $u_n's$ form a complete orthonormal basis for the spatial domain within which the field is enclosed. Then,

$$\sum_n \lambda(n)\omega(n)u_n(r)u_n(r')^* = G_0(r)\delta^3(r-r')$$

and

$$\sum_n u_n(r)u_n(r')^* = I_6.\delta^3(r-r')$$

These equations imply

$$\lambda(n)\omega(n)\delta[n-m] =< u_n, G_0 u_m >= \int u_n(r)^* G_0(r) u_m(r) dr$$

which is possible iff

$$G_0(r)u_m(r) = \lambda(n)\omega(n)u_m(r)\forall m$$

which is impossible to satisfy unless $G_0(r) = \lambda(n)\omega(n) = cI$ where c is a constant ie $G_0(r) = cI$ and $\lambda(n) = 1/\omega(n)$. We therefore modify the CCR to

$$[c(n), c(m)^*] = \lambda(n,m)$$

and derive

$$[\phi_0(t,r), \phi_{0,0}(t,r')^T] = \sum_{n,m} \lambda(n,m)\omega(m)u_n(r)u_m(r')^* = G_0(r)\delta^3(r-r')$$

In view of the orthonormality of the $u_n's$, this is equivalent to requiring that

$$< u_n, G_0 u_m >= \lambda(n,m)\omega(m) \forall n,m$$

This is equivalent to requiring that

$$G_0(r)u_m(r) = \omega(m)\sum_n \lambda(n,m)u_n(r) \forall n,m$$

2 Quantum gravity in N dimensional space-time, Hamiltonian formulation

The metric tensor is $g_{\mu\nu}(x)$ where $0 \leq \mu, \nu \leq N-1$. x^0 is time and $x^r, r = 1, 2..., N-1$ are the spatial coordinates. There are N coordinate conditions and these coordinate system can be chosen so that $g_{00} = 1$ and $g_{0r} = 0, 1 \leq r \leq N-1$. Having done so, the metric tensor now has just $\binom{N}{2}$ independent component which we denote by $\phi_r(x), r = 1, 2, ..., N(N-1)/2$. This is called a synchronous system of coordinates and the proper time element in these coordinates is given by

$$d\tau^2 = dt^2 - g_{rs}(x)dx^r dx^s$$

We write

$$\phi(x) = ((\phi_r(x))_{r=1}^K, K = N(N-1)/2$$

The Einstein-Hilbert Lagrangian density then has the form

$$L(\phi, \phi_{,0}, \phi_{,r}) =$$

$$(1/2)\phi_{,0}^T F_1(\phi)\phi_{,0} - (1/2)(\nabla \otimes \phi)^T F_2(\phi)(\nabla \otimes \phi)$$

$$+\phi_{,0}^T F_3(\phi)(\nabla \otimes \phi)$$

where $F_1(\phi)$ is an $K \times K$ symmetric (real) matrix which is a function of $\phi(x)$ and not its space-time partial derivatives. $F_2(\phi)$ is an $(N-1)K \times (N-1)K$ symmetric real matrix which is once again a function of $\phi(x)$ alone. Finally $F_3(\phi)$ is a $K \times K(N-1)$ real matrix that is again a function of $\phi(x)$ alone. Note that $\nabla = (\partial_r)_{r=1}^{N-1}$ is the spatial gradient operator. The canonical momentum vector (ϕ is the canonical position field) is given by

$$P = \partial L/\partial \phi_{,0} = F_1(\phi)\phi_{,0} + F_3(\phi)(\nabla \otimes \phi)$$

Thus

$$\phi_{,0} = F_1(\phi)^{-1}(P - F_3(\phi)(\nabla\otimes\phi))$$

and hence the Hamiltonian density is

$$H = P^T\phi_{,0} - L = (1/2)\phi_{,0}^T F_1(\phi)\phi_{,0} + (1/2)(\nabla\otimes\phi)^T F_2(\phi)(\nabla\otimes\phi)$$

$$= (1/2)(P - F_3(\phi)(\nabla\otimes\phi))^T F_1(\phi)^{-1}(P - F_3(\phi)(\nabla\otimes\phi))$$
$$+(1/2)(\nabla\otimes\phi)^T F_2(\phi)(\nabla\otimes\phi)$$

The Euler-Lagrange field equations are

$$\partial_0\partial L/\partial\phi_{,0} + \partial_r\partial L/\partial\phi_{,r} = \partial L/\partial\phi$$

These give

$$(F_1(\phi)\phi_{,0})_{,0} - (\nabla^T\otimes I)(F_2(\phi)(\nabla\otimes\phi))$$
$$+(F_3(\phi)(\nabla\otimes\phi))_{,0} + (\nabla^T\otimes I)(F_3(\phi)^T\phi_{,0}) = 0$$

This equation expands to give (ie we, separate out the components in this field equation that are linear in the space-time partial derivatives and those that are nonlinear in the same)

$$\phi_{,00} - F_1^{-1}(\phi)F_2(\phi)(\nabla\otimes\nabla\phi) + F_1(\phi)^{-1}F_3(\phi)(\nabla\otimes\phi_{,0})$$
$$= N_1(\phi)$$

where

$$N_1(\phi) = -F_1(\phi)^{-1}F_1'(\phi)(\phi_{,0}\otimes\phi_{,0})$$
$$+F_1(\phi)^{-1}F_2'(\phi)((\nabla\otimes\phi)\otimes(\nabla\otimes\phi))$$
$$-2F_1(\phi)^{-1}F_3'(\phi)(\phi_{,0}\otimes(\nabla\otimes\phi))$$

We shall be assuming that the term $N_1(\phi)$ that is quadratic in the field space-time partial derivatives are small. We shall in addition, be assuming that

$$F_1(\phi)^{-1}F_2(\phi) = C + N_2(\phi)$$

where $C = ((C(r,s)))_{1\le r,s\le(N-1)}$ is a block structured $(N-1)K\times(N-1)K$ matrix with $C(r,s)$ being a $K\times K$ matrix. C is assumed to be large while $N_2(\phi)$ is assumed to be small. Without loss of generality, $C(r,s)^T = C(s,r)$. We shall also assume that

$$F_1(\phi)^{-1}F_3(\phi) = B + N_3(\phi)$$

where B is a large constant matrix decomposed as $((B(r)))_{r=1}^{N-1}$ and $N_3(\phi)$ is small. Then, the above field equations can be expressed as

$$(\partial_0^2 - C(r,s)\partial_s\partial_s + B(r)\partial_0\partial_r)\phi = N(\phi)$$

where

$$N(\phi) = N_1(\phi) + N_2(\phi)(\nabla\otimes\nabla\phi) - N_3(\phi)(\nabla\otimes\phi_{,0})$$

Now we come to the discussion of the CCR. Since for $x^0 = y^0$, we have

$$[\phi(x), P(y)^T] = i\delta^3(x-y)$$

it follows that

$$[\phi(x), \phi_{,0}(y)^T] = iF_1(\phi(x))^{-1}\delta^3(x-y)$$

It is clear that by assuming C and B to be respectively the constant parts of $F_1(\phi)^{-1}F_2(\phi)$ and of $F_1(\phi)^{-1}F_3(\phi)$, it follows that $N(\phi)$ has an expansion in ϕ that begins with quadratic terms in ϕ, ie, we can write

$$N(\phi)(x) = \sum_{n\geq 2} \delta^{n-1} K_n(\phi)(x)$$

where K_n has an n^{th} order Volterra expansion:

$$K_n(\phi)(x) = \int K_n(x, x_1, ..., x_n)(\otimes_{k=1}^n \phi(x_k))dx_1...dx_n$$

3 How do you take into account quantum noise while formulating a quantum theory of gravity ?

hint: Consider the metric field $\phi(x) \in \mathbb{R}^6$ in a synchronous frame with Lagrangian density

$$L(\phi(x), \phi_{,0}(x), \nabla \otimes \phi(x))$$

that is a quadratic form in $\phi_{,0}, \nabla\phi$ with coefficients that can be complicated nonlinear functions of ϕ. Express it as

$$L = (1/2)\phi_{,0}^T F_1(\phi)\phi_{,0} - (1/2)(\nabla \otimes \phi)^T F_2(\phi)(\nabla \otimes \phi)$$

$$+\phi_{,0}^T F_3(\phi)(\nabla \otimes \phi)$$

To add quantum noise to this Lagrangian, replace $\phi(x)$ by

$$\phi(x) + W(x) + W(x)^*$$

where $W(x) = W(t, r)$ is quantum Brownian noise satisfying the CCR

$$[W(t, r), W(t', r')^*] = min(t, t')F(r, r')$$

This noise CCR is equivalent to requiring that

$$[\partial_t W(t, r), \partial_{t'} W(t', r')^*] = \delta(t-t')F(r, r')$$

Taking the adjoint of this equation, it immediately follows that

$$\bar{F}(r,r') = F(r',r)$$

ie F is a Hermitian kernel and hence admits the spectral expansion

$$F(r,r') = \sum_k \lambda(k)\phi_k(r)\bar{\phi}_k(r')$$

where $\lambda(k) \in \mathbb{R}$ and $\phi_k's$ form an orthonormal basis for $L^2(\mathbb{R}^3)$. Now let $A_k(t), k \geq 1$ be an infinite sequence of annihilation processes so that they satisfy the CCR

$$[A_k(t), A_m(t')^*] = \delta(k,m).min(t,t')$$

and hence also the quantum Ito formula

$$dA_k(t)dA_m(t)^* = \delta(k,m)dt$$

Then, we can write

$$W_k(t,r) = \sum_k \sqrt{\lambda(k)}A_k(t)\phi_k(r)$$

provided that we assume $\lambda(k) \geq 0 \forall k$. Now we compute the noise modified Lagrangian of the gravitational field as

$$L(\phi + W, \phi_{,0} + W_{,0}, \nabla \otimes \phi + \nabla \otimes W)$$

where the unperturbed Lagrangian is

$$L(\phi, \phi_{,0}, \nabla \otimes \phi) = (1/2)\phi_{,0}^T A_0(\phi)\phi_{,0}$$

$$+\phi_{,0}A_1(\phi)(\nabla \otimes \phi) - (1/2)(\nabla \otimes \phi)^T A_2(\phi)(\nabla \otimes \phi)$$

Assuming that the amplitude of quantum noise is small, it follows that upto quadratic orders in the noise amplitude, assuming that this Lagrangian density as above is of quadratic orders in the spatial and temporal derivatives of the field ϕ,the Hamiltonian density can be computed using the Legendre transform as follows:

$$P = \partial L/\partial \phi_{,0} = A_0(\phi + W)(\phi_{,0} + W_{,0}) + A_1(\phi + W)(\nabla \otimes \phi + \nabla \otimes W)$$

so that upto linear terms in the noise,

$$P = A_0(\phi)\phi_{,0} + A_0(\phi)W_{,0} + A_0'(\phi)(W \otimes \phi_{,0}) +$$

$$+A_1(\phi)(\nabla \otimes \phi) + A_1(\phi)(\nabla \otimes W)$$

$$+A_1'(\phi)(W \otimes \nabla \otimes \phi)$$

$$\mathcal{H} = (P, \phi_{,0}) - L =$$

$$(1/2)\phi_{,0}^T A_0(\phi+W)\phi_{,0} + (1/2)W_{,0}^T A_0(\phi+W)W_{,0}$$

$$-W_{,0}^T A_1(\phi+W)(\nabla\otimes\phi) - (1/2)(\nabla\otimes\phi+\nabla\otimes W)^T A_2(\phi+W)(\nabla\otimes\phi+\nabla\otimes W)$$

Retaining upto linear terms in the noise, this Hamiltonian density approximates to the form

$$\mathcal{H} =$$

$$(1/2)(P - A_1(\phi))(\nabla\otimes\phi))^T(P - A_1(\phi)(\nabla\otimes\phi))$$

$$+(1/2)(\nabla\otimes\phi)^T A_2(\phi)(\nabla\otimes\phi)$$

$$+\mathcal{H}_1(\phi, \nabla\phi, P, W_{,0}, \nabla\otimes W)$$

where the last term $\mathcal{H}_1$ is linear in $W, W_{,0}, \nabla W$ and linear quadratic in $\phi, \nabla\otimes\phi, P$. It follows that the Schrodinger equation taking into account the linear components in the noise will have the form after spatial discretization,

$$dU(t) = (-i(H_0(\phi,P)+S(\phi,P))dt+f_1(\phi,P)\otimes dB(t)-f_1(\phi,P)^*\otimes dB(t)^*+f_2(\phi,P)\otimes B(t)dt-f_2(\phi,P)^*\otimes B(t)^*dt)$$

where $B(t)$ is a quantum annihilation process in the language of Hudson and Parthasarathy. It should be noted that $W_{,0}$ is white noise and hence $W_{,0}dt = dB(t)$ after spatial discretization. This means that the Hudson-Parthasarathy qsde now contains apart from quantum Brownian differentials in addition terms proportional to the Brownian motion processes themselves which reflect the presence of the terms W and $\nabla\otimes W$. Here, ϕ is the position vector and P rhe corresponding momentum vector arising from spatial discretization of the fields. $S(\phi, P)$ is the quantum Ito correction term and is given by

$$S(\phi,P) = (1/2)f_1(\phi,P)f_1(\phi,P)^*$$

and is required to ensure unitarity of the evolution. Here,

$$H_0 = \int[(1/2)(P - A_1(\phi))(\nabla\otimes\phi))^T(P - A_1(\phi)(\nabla\otimes\phi))$$

$$+(1/2)(\nabla\otimes\phi)^T A_2(\phi)(\nabla\otimes\phi)]d^3x$$

is the gravitational Hamiltonian in the absence of noise and by calculating ϕ, P from the linearized solution to the Einstein field equations, these can be expressed as polynomials in the graviton creation and annihilation operators.

Remark: We can also take into account quadratic terms in the noise to give a more accurate description of the unitary evolution. Quadratic terms in the noise can be expressed in terms of the conservation process of the Hudson-Parthasarathy quantum stochastic calculus. If we take Fermionic quantum noise also into account arising from contributions from the Dirac action, then we obtain a supersymmetric signal and noise theory for the joint evolution of system and bath.

4 Stochastic problems in quantum general relativity based on the quantum effective action, symmetry breaking caused by higher order cumulants of the random current

If $J(x)$ is a random current field with mean $M_1(x) = \mathbb{E}J(x)$ and higher moments

$$M_r(x_1,...,x_r) = \mathbb{E}(J(x_1) \otimes ... \otimes J(x_r)), r \geq 1$$

The quantum effective action should then be based on the functional

$$Z(M_r, r \geq 1) = \int \mathbb{E}[exp(iS[\phi] + iJ.\phi)]D\phi$$

where

$$J.\phi = \int J(x)\phi(x)d^4x$$

Note that we can write

$$Z(M_r, g \geq 1) = Z(M_1, C_r, r \geq 2) =$$

$$\int exp(iS[\phi] + iM_1.\phi)\mathbb{E}(exp(i(J - M_1).\phi)D\phi$$

with

$$\mathbb{E}(exp(i(J - M_1).\phi) = 1 + \sum_{r \geq 2}(i^r/r!)\mathbb{E}[((J - M_1).\phi)^r]$$

$$= 1 + \sum_{r \geq 2}(i^r/r!)\int C_r(x_1,...,x_r)(\phi(x_1) \otimes ... \otimes \phi(x_r))d^4x_1...d^4x_r$$

where

$$C_r(x_1,...,x_r) = \mathbb{E}[(J(x_1) - M_1(x_1)) \otimes ... \otimes (J(x_r) - M_1(x_r))], t \geq 2$$

are the central moments of the random field J. The computation of the quantum effective action for a field ϕ having action $S[\phi]$ should be based on $Z(M_1, C_r, r \geq 2)$ by fixing $C_r, r \geq 2$ and taking the Legendre transform of Z w.r.t M_1. Equivalently, in terms of the cumulants of J,

$$\mathbb{E}[exp(iJ.\phi)] = exp(\sum_{r \geq 1}(i^r/r!)\int D_r(x_1,...,x_r)^T(\phi(x_1) \otimes ... \otimes \phi(x_r))d^4x_1...d^4x_r)$$

and we could write

$$Z(D_r, r \geq 1) = \mathbb{E}[\int exp(iS[\phi] + iJ.\phi)D\phi]$$

$$= \int exp(iS[\phi] + F[\phi, \mathbf{D}])D\phi$$

where

$$\mathbf{D} = ((D_r))_{r\geq 1}$$

are the cumulants of the random field J and

$$F[\phi, \mathbf{D}] = log\mathbb{E}[exp(iJ.\phi)] = \sum_{r\geq 1}(i^r/r!)\int D_r(x_1,...,x_r)^T(\phi(x_1)\otimes...\otimes\phi(x_r))d^4x_1...d^4x_r$$

is the cumulant generating functional of J. Note that $D_1(x) = M_1(x)$ is the mean of $J(x)$. We could now calculate the quantum effective action for the field ϕ for fixed values of $D_r, r \geq 2$ by applying the Legendre transform w.r.t D_1 alone. The quantum effective action is defined by

$$\Gamma(\phi_0, D_r, r \geq 2) = ext_{D_1}(-i.W(D_r, r \geq 1) - D_1.\phi_0)$$

where

$$D_1.\phi_0 = \int D_1(x)^T\phi_0(x)d^4x, W(D_r, r \geq 1) = ln(Z(D_r, r \geq 1))$$

The extremum above is attained when

$$i\delta W(\mathbf{D})/\delta D_1(x) + \phi_0(x) = 0 - - - (1)$$

It is clear that the optimal value of D_1, namely D_{10} is expressible as a function of $\phi_0, D_r, r \geq 2$. We now derive as usual the quantum equations of motion for the quantum effective action and prove a result that defines the amount of gauge symmetry that is broken when the classical action without the current has a gauge symmetry, in terms of the cumulants $D_r, r \geq 2$. This model then gives us a method to introduce approximate symmetry breaking due to the presence of randomness in the current field and hence to calculate the masses acquired by particles that represent the field ϕ in terms of the cumulants $D_r, r \geq 2$. We first derive the quantum equations of motion:

$$\delta\Gamma(\phi_0, D_r, r \geq 2)/\delta\phi_0(x) =$$

$$-i\int(\delta W/\Delta D_1(y)))(\delta D_1(y)/\delta\phi_0(x))d^4y$$

$$-D_1(x) - \int(\delta D_1(y)/\delta\phi_0(x)).\phi_0(y)d^4y$$

$$= -D_1(x)$$

in view of the defining relation for D_1 that extremises $iW + D_1.\phi_0$ as in (1). Suppose now that the classical action $S[\phi]$ as well as the path measure $D\phi$ are invariant under the infinitesimal gauge transformation

$$\phi \to \phi + \epsilon.\Delta(\phi) = \phi'$$

Then we get by replacing the path integration variable ϕ by ϕ' that

$$\mathbb{E}\int exp(iS[\phi]+i.J.(\phi+\epsilon\Delta(\phi)))D\phi = \mathbb{E}\int exp(iS[\phi]+i.J.\phi)D\phi$$

$$= Z$$

Since $\epsilon \to 0$, this gives us

$$\mathbb{E}\int exp(iS[\phi]+iJ.\phi)(J.\Delta(\phi))D\phi = 0$$

If J were a non-random field, then this equation could be expressed as

$$J. < \Delta(\phi) >_J = 0$$

ie,

$$\int J(x). < \Delta(\phi)(x) >_J \, d^4x = 0$$

which in view of the quantum equations of motion derived above for the special case of nonrandom J, would give

$$\int (\delta\Gamma(\phi_0))/\delta\phi_0(x)). < \Delta(\phi)(x) >_{J_0} \, d^4x = 0$$

where J_0 is that current for which $< \phi >_{J_)} = \phi_0$. This means that the quantum effective action is invariant under the infinitesimal gauge transformation

$$\phi_0 \to \phi_0 + \epsilon. < \Delta(\phi) >_{J_0}$$

Only when $\Delta(\phi)$ is a linear functional of ϕ does it follow from this equation that the quantum effective action is invariant under the same gauge transformation $\Delta(\phi_0)$ for which the classical action is invariant since then

$$< \Delta(\phi) >_{J_0} = \Delta(< \phi >_{J_0}) = \Delta(\phi_0)$$

Note that

$$\delta W(J)/\delta J(x) = i < \phi(x) >_J$$

In the random case, even the nonlinear gauge symmetry of the quantum effective action is broken. To estimate by how much this is broken, we write

$$F[\phi,\mathbf{D}] = \sum_{r\geq 1}(i^r/r!)\int D_r(x_1,...,x_r)^T(\phi(x_1)\otimes ... \otimes \phi(x_r))d^4x_1...d^4x_r$$

$$= iD_1.\phi + F_2[\phi,\mathbf{D}']$$

where

$$F_2[\phi,\mathbf{D}'] = \sum_{r\geq 2}(i^r/r!)\int D_r(x_1,...,x_r)^T(\phi(x_1)\otimes ... \otimes \phi(x_r))d^4x_1...d^4x_r$$

$$= \sum_{r \geq 2} (i^r/r!)(D_r.\phi^{\otimes r})$$

where we have defined

$$\mathbf{D'} = (D_r : r \geq 2)$$

Then, by a change of path integration variable and using invariance of the classical action and the path measure under the gauge transformation $\epsilon.\Delta(\phi)(x)$, we get

$$\int exp(iS[\phi] + F[\phi + \epsilon.\Delta(\phi), \mathbf{D}])D\phi =$$

$$\int exp(iS[\phi] + F[\phi, \mathbf{D}])D\phi$$

or equivalently,

$$\int exp(iS[\phi] + F[\phi, \mathbf{D}])(F[\phi + \epsilon.\Delta(\phi)] - F[\phi, \mathbf{D}])D\phi$$

$$+O(\epsilon^2) = 0$$

Dividing by ϵ and taking the limit $\epsilon \to 0$, we get

$$\int < (\delta F[\phi, \mathbf{D}]/\delta\phi(x))\Delta(\phi)(x) >_{\mathbf{D}} d^4x = 0$$

where for any functional $f[\phi]$ of the field ϕ, we have defined

$$< f[\phi] >_{\mathbf{D}} = Z(\mathbf{D})^{-1} \int exp(iS[\phi] + iF[\phi, \mathbf{D}])f[\phi]D\phi$$

On the other hand, we observe that by the quantum equations of motion derived above in the random current field case,

$$\delta F[\phi, \mathbf{D}]/\delta\phi(x) = iD_1(x) + \delta F_2[\phi, \mathbf{D'}]/\delta\phi(x)$$

$$= -i\delta\Gamma(\phi_0, \mathbf{D'})/\delta\phi_0(x) + \delta F_2[\phi, \mathbf{D'}]/\delta\phi(x)$$

so we get

$$-i \int (\delta\Gamma(\phi_0, \mathbf{D'})/\delta\phi_0(x)) < \Delta(\phi)(x) >_{D_1} d$$

$$+ \int < (\delta F_2[\phi, \mathbf{D'}]/\delta\phi(x))\Delta(\phi)(x) >_{D_1} d^4x = 0$$

where D_1 has been computed in terms of $\phi_0, \mathbf{D'}$ as above, ie, in such a way that the classical-quantum average of ϕ equals ϕ_0. Equivalently, we can write

$$\int (\delta\Gamma(\phi_0, \mathbf{D'})/\delta\phi_0(x)) < \Delta(\phi)(x) >_{D_1} d^4x$$

$$= -i \int < (\delta F_2[\phi, \mathbf{D'}]/\delta\phi(x))\Delta(\phi)(x) >_{D_1} d^4x$$

The lhs of this equation gives us the change in the quantum effective action under the quantum gauge transformation $\phi_0 \to \phi_0 + \epsilon. < \Delta(\phi)(x) >_{D_1}$ and therefore the rhs gives us the amount by which randomness in the current field causes the gauge invariance of the quantum effective action to be broken. The rhs can thus be used as a correction to the quantum effective action that leads to massless particles acquiring masses or massive particles to have their masses changed.

[2] Consider the Einstein-Hilbert action for the gravitational field. As seen earlier, it has the form

$$S_0[\phi] = \int L_0 d^4x,$$

where

$$L_0 = (1/2)\phi_{,0}^T A_{rs}(\phi)\phi_{,0} - (1/2)\phi_{,s}^T B_{rs}(\phi)\phi_{,r}$$
$$+\phi_{,0}^T C_r(\phi)\phi_{,r}$$

summation being over the spatial indices $r, s = 1, 2, 3$. Here, we are assuming a synchronous frame so that $g_{00} = 1, g_{0r} = 0$ which implies that the metric has just six independent components which we denote by ϕ. The interaction of the metric field with a random electromagnetic field can be represented by the interaction Lagrangian

$$L_1(\phi) = (-1/4)\sqrt{-g}F^{\mu\nu}F_{\mu\nu}$$
$$= (-1/4)\sqrt{-g}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}$$
$$= F^T D(\phi)F$$

where $D(\phi(x))$ is a function of only the metric field $\phi(x)$ and not its partial derivatives while $F = ((F_{\mu\nu}))$ is the random electromagnetic field. The total Lagrangian of the gravitational field ϕ interacting with a fixed external random electromagnetic field F is thus

$$L(\phi, \phi_{,0}, \phi_{,r}|F) = L_0 + L_1$$

Our aim is to calculate the quantum effective action for the gravitational field by assuming that the electromagnetic field F as mean zero and hence

$$\mathbb{E}(F(x) \otimes F(x)) = Vec(Cov(F(x))$$

In view of this problem, it is instructive to deal with the problem of defining the quantum effective action of a field when it interacts with a classical random current field with all the cumulants of the current field being specified.

Spontaneous symmetry breaking and approximate symmetry breaking in quantum gravity.

5 Propagator computation of a nonlinear field theory using differential equations for time ordered vacuum expectations

The equations of motion for the gravitational field in a synchronous frame are expressible in the form

$$\phi_{k,00}(x) - \sum_{m,r,s} C_1(k,m,r,s,\phi(x))\phi_{m,rs}(x) - \sum C_2(k,m,r,\phi(x))\phi_{m,r0}(x)$$

$$-F_k(\phi_m(x), \phi_{m,0}(x), \phi_{m,r}(x)) = 0$$

with the CCR

$$[\phi_k(t,r), \phi_{m,0}(t,r')] = iG_{km}(\phi(t,r))\delta^3(r-r'),$$

$$[\phi_k(t,r), \phi_m(t,r')] = 0, [\phi_k(t,r), \phi_{m,s}(t,r')] = 0$$

Define the gravitational propagator

$$\Delta_{km}(x,y) =< T(\phi_k(x)\phi_m(y)) >= \theta(x^0-y^0) < \phi_k(x)\phi_m(y) > +\theta(y^0-x^0) < \phi_m(y)\phi_k(x) >$$

Then, we get

$$\partial_0\Delta_{km}(x,y) = \delta(x^0-y^0) < [\phi_k(x), \phi_m(y)] <$$

$$+ < T(\phi_{k,0}(x)\phi_m(y)) >=< T(\phi_{k,0}(x)\phi_m(y)) >$$

Thus,

$$\partial_0^2\Delta_{km}(x,y) = \delta(x^0-y^0) < [\phi_{k,0}(x), \phi_m(y)] >$$

$$+ < T(\phi_{k,00}(x))\phi_m(y)) >$$

$$= -i\delta^4(x-y) < G_{km}(\phi(x)) > + < T(\phi_{k,00}(x)\phi_m(y)) >$$

Now we can express $\phi_{k,00}(x)$ appearing within the time ordered vacuum expected value on the rhs in terms of $\phi_{m,r0}(x)$ and $\phi_m(x), \phi_{m,r}(x), \phi_{m,rs}(x), \phi_{m,0}(x)$. The troublesome factor here is $\phi_{m,r0}(x)$ because it involves a time derivative which causes it not to commute with the other factors. However, this factor occurs linearly in the field equations and hence its contribution can be evaluated easily as follows:

$$\partial_0\partial_r\Delta_{km}(x,y) = \partial_0 < T(\phi_{k,r}(x)\phi_m(y)) >= \delta(x^0-y^0) < [\phi_{k,r}(x), \phi_m(y)] >$$

$$+ < T(\phi_{k,r0}(x)\phi_m(y)) >=< T(\phi_{k,r0}(x)\phi_m(y)) >$$

The other troublesome factor here is the one involving $\phi_{m,0}(x)$ which appears nonlinearly in the term $F_k(\phi_m(x), \phi_{m,0}(x), \phi_{m,r}(x))$. Taking all this into consideration, we obtain the following pde for the graviton propagator as

$$\partial_0^2 \Delta_{km}(x,y) =$$

$$= -i\delta^4(x-y) < G_{km}(\phi(x)) > +$$

$$\sum_{n,r,s} < T(C_1(k,n,r,s,\phi(x))\phi_{n,rs}(x)\phi_m(y)) > + \sum_{n,r} < T(C_2(k,n,r,\phi(x))\phi_{m,r0}(x)\phi_m(y)) >$$

$$+ < T(F_k(\phi_n(x), \phi_{n,0}(x), \phi_{n,r}(x))\phi_m(y)) >$$

We can now make the following approximations: [a] Replace $\phi(x)$ by $< \phi(x) >$ in C_1, C_2, (b) Consider only terms linear in $\phi_n, \phi_{n,0}, \phi_{n,r}$ in F_k. This approximation corresponds to neglecting higher that quadratic products in the propagator. However, to determine corrections to graviton mass, we must compute cubic and higher order correction terms also in the propagator. These contributions are in practice evaluated using perturbation theory to express the solution for $\phi(x)$ to the field equations in terms of the linearized solution which is expressed in terms of Bosonic creation and annihilation operators satisfying the CCR. However in the quadratic approximation to the propagagtor, we have

$$< T(C_1(k,n,r,s,\phi(x))\phi_{m,rs}(x)\phi_m(y)) > \approx$$

$$C_1(k,n,r,s,< \phi(x) >)\partial_r\partial_s\Delta_{nm}(x,y)$$

$$< T(C_2(k,n,r,\phi(x))\phi_{n,r0}(x)\phi_m(y)) >$$

$$\approx C_2(k,n,r,< \phi(x) >)\partial_r\partial_0\Delta_{nm}(x,y)$$

and finally, linearizing F_k to express it as

$$F_k(\phi_n(x), \phi_{n,0}(x), \phi_{n,r}(x))$$

$$\approx F_{ks}(< \phi_n(x) >, \partial_r < \phi_n(x) >)\phi_{s,0}(x)$$

we get

$$< T(F_k(\phi_n(x), \phi_{n,0}(x), \phi_{n,r}(x))\phi_m(y)) >$$

$$\approx F_{ks}(< \phi_n(x) >, \partial_r < \phi_n(x) >)\partial_0\Delta_{sm}(x,y)$$

Thus, we obtain finally the linearized approximate propagator equation as

$$\partial_0^2\Delta_{km}(x,y) - \sum_{nrs} C_1(k,n,r,s,< \phi(x) >)\partial_r\partial_s\Delta_{nm}(x,y)$$

$$- \sum_{nr} C_2(k,n,r,< \phi(x) >)\partial_r\partial_0\Delta_{nm}(x,y)$$

$$- \sum_{s} F_{ks}(< \phi_n(x) >, \partial_r < \phi_n(x) >)\partial_0\Delta_{sm}(x,y)$$

$$= -i\delta^4(x-y) < G_{km}(\phi(x)) >$$

This approximate graviton propagator equation can be used to evaluate approximately the contribution of nonlinear self interaction of the graviton field to the generation of graviton mass.

Acknowledgements: This paper was presented by me in a workshop on gravitation and cosmology organized at the NSUT by my friend and colleague Professor J.K.Singh. I am grateful to him for giving me the opportunity to present this paper.

6 References

[1] K.R.Parthasarathy, "An introduction to quantum stochastic calculus", Birkhauser, 1992.

[2] Steven Weinberg, "The quantum theory of fields, vols. I,II,III.

[3] Thomas Thiemann, "Modern canonical quantum general relativity", Cambridge University Press.

[4] Harish Parthasarathy, Developments in Mathematical and Conceptual Physics:Concepts and Applications for Engineers, Springer Nature, Singapore, 2020.

[5] Harish Parthasarathy, Supersymmetry and Superstring Theory with Engineering Applications, Nov.14,2022, CRC press, Taylor and Francis.

[6] Timothy Eyre, "Quantum stochastic calculus and representations of Lie super-algebras", Springer lecture notes in mathematics.