Research Article

Stability and Synchronization of Kuramoto Oscillators

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Imagine a group of oscillators, each endowed with their own rhythm or frequency, be it the ticking of a biological clock, the swing of a pendulum, or the glowing of fireflies. While these individual oscillators may seem independent of one another at first glance, the true magic lies in their ability to influence and synchronize with one another, like a group of fireflies glowing in unison.

The Kuramoto model was motivated by this phenomenon of collective synchronization, when a group of a large number of oscillators spontaneously lock to a common frequency, despite vast differences in their individual frequencies (A.T. Winfree 1967,[1]). Inspired by Kuramoto's groundbreaking work in the 1970s, this model captures the essence of how interconnected systems, ranging from biological networks to power grids, can achieve a state of synchronization.

This work aims to study the stability and synchronization of Kuramoto oscillators, starting off with an introduction to Kuramoto Oscillators and it's broader applications. We then at a graph theoretic formulation for the same and establish various criterion for the stability, synchronization of Kuramoto Oscillators. Finally, we broadly analyze and experiment with various physical systems that tend to behave like Kuramoto oscillators followed by further simulations.

(Note: this work was done while at IIT Madras)

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1. Introduction

The Kuramoto model was proposed to study huge populations of coupled limit-cycle oscillators whose natural frequencies are known a priori. Tha background work for Kuramoto model was done by Winfree (1967), where he proposed that **'each oscillator was coupled to the collective rhythm generated by the whole population'**, analogous to a mean-field approximation in Physics (where we

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model certain random variables in terms of the mean of their variation). So, Winfree's proposed model for a system of N oscillators was as follows:

$$\dot{\theta}_i = \omega_i + K\left(\sum_{j=1}^{N} A_{ji} X\left(\theta_j\right)\right) Z\left(\theta_i\right)$$

where, K is the coupling strength and A_{ji} is a measure of the communication capacity between different channels and $X(\theta_j)$ is the influence of other channels on the i^{th} channel.

But this model was not widely accepted by the scientific community because it lacked certain symmetries like that of translational invariance when both phases are slightly perturbed, the model does not remain invariant. The classical Kuramoto model proposed in 1975 is as follows:

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{n} \Gamma_{ij}\left(\theta_j - \theta_i\right)$$

where θ_i are the phases and ω_i are the limit cycle frequencies of the oscillators. Kuramoto studied a further simplification of this model. He used a sine function to couple the oscillators, this simplified the analysis of the model as will be shown below:

$$\dot{\theta}_i = \omega_i + \frac{K}{n}\sum_{j=1}^{n} \sin(\theta_j - \theta_i)$$

When we have a high K, even if the system is initially incoherent it will first gain partial coherence and then become fully synchronized. But how high does of a coupling constant do we need? For a given coupling constant, how can we quantify the degree of synchronization in our system.

1.1. Order parameter

An interesting measure of synchronization in a Kuramoto model is the order parameter. The order parameter is the centroid of all oscillators represented as points on the unit circle in C^1. The magnitude r of the order parameter is a synchronization measure (or a measure of phase cohesiveness):

$$re^{i\psi} = \frac{1}{n}\sum_{j=1}^{n} e^{i\theta_j}$$

- if the oscillators are phase-synchronized, then r = 1;
- if the oscillators are spaced equally on the unit circle, then r = 0; and

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- for $r \in [0, 1]$ the associated configuration of oscillators have a level of phase cohesiveness; this extent of cohesiveness is higher for a higher value of r

Expressing our original equation in terms of the order parameter, we get:

$$\boxed{\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i)}$$

If we assume our critical coupling constant for synchronization to occur to be K_c, then the above plot gives us an idea of how order parameter evolves with time.

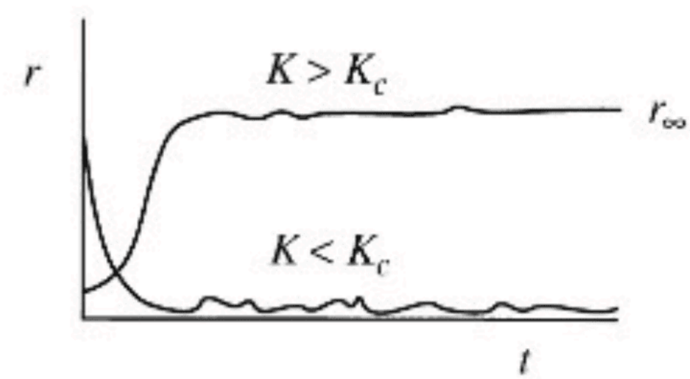

Figure 1. Evolution of $r(t)$

2. Stability — A primer on Stability of Kuramoto Oscillators

2.1. Mathematical Setup

Before getting into the stability analysis, it is essential for us to first set up the problem. I will first give a graph theoretic formulation of our model.

The incidence matrix B of an oriented graph $\mathcal{G}^\sigma$ with N vertices and e edges is the $N \times e$ matrix such that: $B_{ij} = 1$ if the edge j is incoming to vertex i, $B_{ij} = -1$ if edge j is outcoming from vertex i, and 0 otherwise. The symmetric $N \times N$ matrix defined as: $L = BB^T$ is called the Laplacian of $\mathcal{G}$ and is independent of the choice of orientation σ. It has several characteristics:

- L is always positive semidefinite with a zero eigenvalue;
- the algebraic multiplicity of its zero eigenvalue is equal to the number of connected components in the graph;

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- the N-dimensional eigenvector associated with the zero eigenvalue is the vector of ones, $\mathbf{1}_N$.
- the first non-zero eigenvalue $\lambda_2(L)$ gives a measure of algebraic connectivity of the graph.

- If we associate a positive number W_i to each edge and we form the diagonal matrix $W_{e\times e} := \operatorname{diag}(W_i)$, then the matrix $L_W(\mathcal{G}) = BWB^T$ is a weighted Laplacian which fulfills the above properties.

Now, our dynamics equation becomes:

$$\dot{\theta} = \omega - \frac{K}{N} B \sin\left(B^T\theta\right), \tag{1}$$

The order parameter is defined as follows:

$$r^2 = 1 - \frac{1}{N}\left[e^{j\theta}\right]^* L\left[e^{j\theta}\right] = 1 - \frac{1}{N}\left([\cos\theta]^T L[\cos\theta] + [\sin\theta]^T L[\sin\theta]\right)$$

We also define the generalized inverse, denoted by $\left(V^TB\right)^{\#}$, is equal to $B^TV\Lambda^{-1}$, where Λ is the $N-1$ diagonal matrix of the eigenvalues of the unweighted Laplacian. We therefore have the following expression

$$\left(\sin\left(B^T\theta\right)\right)_{R(B^T)} = B^TV\Lambda^{-1}V^T\frac{N\omega}{K}.$$

Noting that $L^{\#} = V\Lambda^{-1}V^T$, we have

$$\left(\sin\left(B^T\theta\right)\right)_{R(B^T)} = B^TL^{\#}B\sin\left(B^T\theta\right) = B^TL^{\#}\frac{N\omega}{K}$$

2.2. *Synchronization of Identical Coupled Oscillators*

Result 1: Consider the coupled oscillator model as defined earlier. If $\omega_i \neq \omega_j$ for some distinct $i, j \in \{1, \ldots, n\}$, then the oscillators cannot achieve phase synchronization.

Proof: We prove the lemma by contradiction. Assume that all oscillators are in phase synchrony $\theta_i(t) = \theta_j(t)$ for $t \geq 0$ and $i, j \in \{1, \ldots, n\}$. Then equating the dynamics, $\dot{\theta}_i(t) = \dot{\theta}_j(t)$, implies that $\omega_i = \omega_j$.

Without loss of generality, let's now take the above dynamics equation with all angular frequencies of the system set to zero. (rotation of axes would help us achieve this)

$$\dot{\theta} = -\frac{K}{N}B\sin\left(B^T\theta\right).$$

Result 2: If we take the unperturbed Kuramoto model defined over an arbitrary connected graph with incidence matrix B for any value of the coupling $K > 0$, all trajectories will converge to the set of equilibrium solutions. In particular the synchronized state is locally asymptotically stable. Moreover, the rate of approach to the synchronized state is no worse than $(2K/\pi N)\lambda_2(L)$, where $\lambda_2(L)$ is the first non-zero eigenvalue or the algebraic connectivity of the graph.

Proof 1: Consider the function $U_1(\theta) = 1 - r^2 = \frac{4\left\|\sin\left(\frac{B^T\theta}{2}\right)\right\|^2}{N^2}$, where r^2 is as defined above. Taking the derivative along trajectories wrt time and using the fact that $\nabla_\theta U = \left(2/N^2\right) B \sin\left(B^T\theta\right)$ leads to

$$\dot{U}(\theta) = \nabla_\theta U \dot{\theta} = -\frac{2}{KN}\dot{\theta}^T\dot{\theta} \leq 0.$$

Therefore, the positive function $U(\theta) \in [0,1]$ is a non-increasing function along the trajectories of the system. By using LaSalle's invariance principle we conclude that U is a Lyapunov function for the system, and that all trajectories converge to the set where $\dot{\theta}$ is zero, i.e., the fixed point solutions.

Proof 2: We could use a different Lyapunov function similar to the approach above, infact a small angle approximation of the above Lyapunov function and consider the quadratic Lyapunov function candidate $U_2 = \frac{1}{2}\sum_{i=1}^N \sum_{j=1}^N \left(\theta_i - \theta_j\right)^2 = \theta^T L_c \theta$, where $L_c = NI - \mathbf{1}\mathbf{1}^\mathbf{T}$ is the Laplacian matrix of a complete graph. Note that $B^T\mathbf{1} = 0$, therefore, taking derivatives wrt time we get:

$$\dot{U}_2 = -\frac{K}{N}\theta^T B \sin\left(B^T\theta\right) = -\frac{K}{N}\theta^T BW(\phi)B^T\theta \leq 0$$

We can also show that locally, the convergence is exponential with the rate determined by the smallest non-zero eigenvalue of the weighted Laplacian:

$$\dot{U}_2 \leq -\frac{K}{N}\lambda_2\left(BW(\phi)B^T\right)\|\theta_{\mathbf{1}^\perp}\|^2 \leq -\frac{2K}{\pi N}\lambda_2(L)\|\theta_{\mathbf{1}^\perp}\|^2$$

using $\lambda_2\left(BW(\phi)B^T\right) \leq (2/\pi)\lambda_2\left(BB^T\right)$ which proves the above result.

2.3. Existence and uniqueness of stable fixed points

The fixed point equation can be written as

$$\theta = \left(BW\left(B^T\theta\right)B^T\right)^{\#}\frac{N\omega}{K} = L_W^{\#}\left(B^T\theta\right)\frac{N\omega}{K}.$$

Using Brouwer's fixed point theorem (that states a continuous function that maps a non-empty compact, convex set X into itself has at least one fixed-point), we can develop conditions which guarantee the existence (but not uniqueness) of the fixed point. If a fixed-point exists in any compact subset of $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, it is stable, since this will ensure that $B^T\theta$ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. We therefore have to ensure that

$$K > \frac{4}{\pi} N \max_{|\theta_i| < \frac{\pi}{4}} \left\| L_W^{\#} \left(B^T \theta \right) \right\|_2 \|\omega\|_2 .$$

We are hence imposing bounds with respect to the 2-norm. So, we have:

$$\left\| \left(BW \left(B^T \theta \right) B^T \right)^{\#} \right\|_2 \frac{N\|\omega\|_2}{K} \leq \frac{\pi}{4} \sqrt{N}$$

Hence, a sufficient condition for synchronization of all oscillators can be determined in terms of a lower bound for K : where we used the fact that $\left\| \left(BW \left(B^T \theta \right) B \right)^{\#} \right\|_2 = \frac{1}{\lambda_2(L_W)}$, and λ_2 is the algebraic connectivity of the (weighted) graph. A lower bound on the minimum value of λ_2 occurs for the minimum value of the weight which is $\frac{2}{\pi}$. As a result,

$$K_L \geq 2 \frac{\sqrt{N}\|w\|_2}{\lambda_2(L)}$$

2.4. Bounds for existence of a unique fixed point:

Consider the Kuramoto model for non-identical coupled oscillators with different natural frequencies ω_i. For $K \geq K_L := 2\frac{\sqrt{N}\|w\|_2}{\lambda_2(L)}$, there exist at least one fixed-point for $|\theta_i| < \frac{\pi}{4}$ or $\left| \left(B^T \theta \right)_i \right| < \frac{\pi}{2}$. Moreover, for $K \geq \frac{\pi^2}{4} \frac{N \lambda_{\max}(L) \|w\|_2}{\lambda_2(L)^2}$ there is only one stable fixed-point (modulo a vector in the span of $\mathbf{1}_N$), and the order parameter is strictly increasing.

3. Synchronization — A primer on Synchronization of Kuramoto Oscillatoras

3.1. Notions of Synchronization

(Clarification: $1_n^{\perp}$ denotes perpendicular to 1_n) There are various notions of synchronization, described as follows:

1. **Frequency synchrony:** A solution $\theta : \mathbb{R}_{\geq 0} \to \mathbb{T}^n$ is frequency synchronized if $\dot{\theta}_i(t) = \dot{\theta}_j(t)$ $\forall$ t and $\forall$ i and j.
2. **Phase synchrony**: A solution $\theta : \mathbb{R}_{\geq 0} \to \mathbb{T}^n$ is phase synchronized if $\theta_i(t) = \theta_j(t)$ $\forall$ t and $\forall$ i and j. Before moving to the next notion of synchronization, let us look at a few mathematical preliminaries.

   A torus $\mathbb{T}^n$ is the set consisting of n-copies of the circle. Let $\mathbb{G}$ be an undirected, weighted graph, with $\gamma \in [0, \pi]$. Then:

- The arc subset $\bar{\Gamma}_{\text{arc}}(\gamma) \subset \mathbb{T}^n$ is the set of $(\theta_1, \ldots, \theta_n) \in \mathbb{T}^n$ such that there exists an arc of length γ in $\mathbb{S}^1$ containing all angles $\theta_1, \ldots, \theta_n$. The set $\Gamma(\gamma)$ is the interior of $\bar{\Gamma}_{\text{arc}}(\gamma)$.
- The cohesive subset $\Delta^G(\gamma) \subseteq \mathbb{T}^n$ is

$$\Delta^G(\gamma) = \{\theta \in \mathbb{T}^n || \theta_i - \theta_j | \leq \gamma, \quad \text{for all edges } (i,j)\}$$

3. **Phase cohesiveness**: A solution $\theta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{T}^n$ is phase cohesive with respect to $\gamma > 0$ if one of the following conditions holds $\forall\, t$:
   - $\theta(t) \in \Gamma_{\text{arc}}(\gamma)$; or
   - $\theta(t) \in \Delta^G(\gamma)$, for the graph G.
4. **Asymptotic Synchronization**: This happens in cases where one of the above criterion is asymptotically achieved. For example, a solution $\theta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{T}^n$ achieves phase synchronization if $\lim_{t \rightarrow \infty} |\theta_i(t) - \theta_j(t)| = 0$.

3.2. Results based on above notions

Res. 1 (Synchronization frequency). Consider the coupled oscillator model as defined originally with frequencies $\omega \in \mathbb{R}^n$ defined over a connected weighted undirected graph. If a solution achieves frequency synchronization, then it does so with a constant synchronization frequency equal to

$$\omega_{\text{sync}} \triangleq \frac{1}{n} \sum_{i=1}^{n} \omega_i = \text{average}(\omega).$$

Proof: This fact is obtained by summing all equations

$$\dot{\theta}_i = \omega_i + \frac{K}{n} \sum_{j=1}^{n} \sin(\theta_j - \theta_i)$$

for $i \in \{1, \ldots, n\}$.

Res. 2 Consider a coupled oscillator model with frequencies $\omega \in 1_n^{\perp}$ defined over a connected weighted undirected graph with incidence matrix B. The following statements hold:

1. (Jacobian:) the Jacobian of the coupled oscillator model at $\theta \in \mathbb{T}^n$ is

$$J(\theta) = -B \operatorname{diag}\Big(\{a_{ij} \cos(\theta_i - \theta_j)\}_{\{i,j\} \subset E}\Big) B^\top$$

2. ((local stability:) if there exists an equilibrium $\theta^* \in \Delta^G(\gamma), \gamma < \pi/2$, then
   - $-J(\theta^*)$ is a Laplacian matrix; and

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- the equilibrium set $[\theta^*]$ (the rotation set for an equilibrium point obtained during rotation by a certain angle) is locally exponentially stable;

3. ((frequency synchronization:) if a solution $\theta(t)$ is phase cohesive in the sense that $\theta(t) \in \Delta^G(\gamma), \gamma < \pi/2$, for all $t \geq 0$, then there exists a phase cohesive equilibrium $\theta^* \in \Delta^G(\gamma)$ and $\theta(t)$ achieves exponential frequency synchronization converging to $[\theta^*]$.

Proof: Given $\theta \in \mathbb{T}^n$, we define the undirected graph $G_{\text{cosine}}(\theta)$ with the same nodes and edges as G and with edge weights $a_{ij}\cos(\theta_i - \theta_j)$. Next, we compute

$$\frac{\partial}{\partial \theta_i}\left(\omega_i - \sum_{j=1}^{n} a_{ij}\sin(\theta_i - \theta_j)\right) = -\sum_{j=1}^{n} a_{ij}\cos(\theta_i - \theta_j),$$
$$\frac{\partial}{\partial \theta_j}\left(\omega_i - \sum_{k=1}^{n} a_{ik}\sin(\theta_i - \theta_k)\right) = a_{ij}\cos(\theta_i - \theta_j).$$

Therefore, the Jacobian is equal to minus the Laplacian matrix of the graph $G_{\text{cosine}}(\theta)$. And, if $\left|\theta_i^* - \theta_j^*\right| < \pi/2$ for all $\{i,j\} \in E$, then $\cos\left(\theta_i^* - \theta_j^*\right) > 0$ for all $\{i,j\} \in E$, so that $G_{\text{cosine}}(\theta)$ has strictly non-negative weights so first part of 2. can be proved.

For the next part, we use the property that $J(\theta^*)$ is negative semidefinite with the nullspace 1_n arising from the rotational symmetry. All other eigenvectors are orthogonal to 1_n and have negative eigenvalues. Let's take a coordinate transformation matrix $Q \in \mathbb{R}^{(n-1)\times n}$ with orthonormal rows orthogonal to 1_n,

$$Q 1_n = \mathbb{O}_{n-1} \text{ and } QQ^\top = I_{n-1}$$

and we note that $QJ(\theta^*)Q^\top$ has negative eigenvalues. So in our original coordinate system, the zero eigenspace 1_n is exponentially stable, and hence the set $[\theta^*]$ is locally exponentially stable.

For the last one, let's take a system $\dot{x}(t) = J(\theta(t))x(t)$. The associated undirected graph has time-varying yet strictly positive weights $a_{ij}\cos(\theta_i(t) - \theta_j(t)) \geq a_{ij}\cos(\gamma) > 0$ for each $\{i,j\} \in E$. Now let us look at the following theorem from Bullo's book on Networks and Systems[2].

"**Theorem 12.9 (Consensus for time-varying algorithms in continuous time)**. Let $t \mapsto A(t)$ be a time-varying adjacency matrix with associated time-varying digraph $t \mapsto G(t), t \in \mathbb{R}_{\geq 0}$. Assume each non-zero edge weight $a_{ij}(t)$ is larger than a constant $\varepsilon > 0$,
Then the solution to $\dot{x}(t) = -L(t)x(t)$ converges exponentially fast to $\left(w^\top x(0)\right)1_n$"

Now, we have a $J(\theta(t))$ here in place of our $-L(t)$ hence the result holds true.

3.3. Onset of Synchronization: (for non-identical oscillators)

Here, we will first calculate the necessary critical gain K_c for the onset of synchronization. As we are interested in studying the phase difference dynamics, we have

$$\dot{\theta}_i - \dot{\theta}_j = \omega_i - \omega_j + \frac{K}{N}\{-2\sin(\theta_i - \theta_j) + \sum_{k=1, k\neq i,j}^{N} (\sin(\theta_k - \theta_i) + \sin(\theta_j - \theta_k))\Bigg\}$$

If the oscillators are to at least asymptotically synchronize i.e. $\dot{\theta}_i - \dot{\theta}_j \to 0$ as $t \to \infty \forall i, j = 1, \ldots, N$, the R.H.S of the above equation must go to zero.

So, we derive a necessary (but not sufficient) condition for the onset of synchronization. Synchronization can only occur if our original Kuramoto equation has at least one fixed point, and hence we have:

$$\omega_j - \omega_i = \frac{K}{N}\Bigg\{2\sin(\theta_j - \theta_i) + \sum_{k=1}^{N} (\sin(\theta_k - \theta_i) + \sin(\theta_j - \theta_k))\}$$

Without loss of generality let's assume $w_j \geq w_i$, then we have to maximize the following expression:

$$E = 2\sin(\theta_j - \theta_i) + \sum_{k=1}^{N} \sin(\theta_k - \theta_i) + \sin(\theta_j - \theta_k)$$

Using elementary calculus, we can equate the first order derivatives wrt i, j, k to zero, which gives us:

$$\frac{\partial E}{\partial \theta_k} = \cos(\theta_k - \theta_i) - \cos(\theta_j - \theta_k) = 0$$

From the last equation, we get $\theta_k = \frac{\theta_i + \theta_j}{2}$ or $\theta_i = \theta_j$. Since the latter gives us E equals zero, we use the former condition and proceed.

$$\begin{aligned}
&2\cos(\theta_j - \theta_i) + \sum_{k=1, k\neq i,j}^{N} \cos\left(\frac{\theta_j - \theta_i}{2}\right) = 0 \\
&\Rightarrow 2\cos(\theta_j - \theta_i) + (N-2)\cos\left(\frac{\theta_j - \theta_i}{2}\right) = 0 \\
&\Rightarrow 4\cos^2\left(\frac{\theta_j - \theta_i}{2}\right) - 2 + (N-2)\cos\left(\frac{\theta_j - \theta_i}{2}\right) = 0
\end{aligned} \tag{2}$$

Solving the quadratic equation gives us:

$$\cos\left(\frac{\theta_j - \theta_i}{2}\right) = \frac{-(N-2) + \sqrt{(N-2)^2 + 32}}{8}$$

If this is our optimal $\theta_j - \theta_i$, say the maximum value of E is given by:

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$$E_{\max} = 2\sin\left(\theta_j - \theta_i\right)_{opt} + 2(N-2)\sin\left(\frac{\left(\theta_j - \theta_i\right)_{opt}}{2}\right)$$

Thus, the critical coupling gain desired for onset of synchronization is:

$$K_c = \frac{\left(\omega_j - \omega_i\right)N}{E_{\max}}$$

If the natural frequencies belong to a compact (closed, bounded) set, this becomes:

$$K_c = \frac{\left(\omega_{max} - \omega_{min}\right)N}{E_{\max}}$$

So, K_c is simply the critical gain below which synchronization cannot occur. Now, the value for critical coupling given in[3] is:

$$K_L = \frac{\left(\omega_{\max} - \omega_{\min}\right)N}{2(N-1)}$$

Comparing the denominators of the above bounds, we can say that E_{max} equals to $2(N-1)$ is not possible in our case, because in[3] the authors assumed that at $E_{\max}$, $|\theta_m - \theta_n| = \frac{\pi}{2} \quad \forall m, n = 1, \ldots, N$. This clearly is not possible as the phase differences $(\theta_m - \theta_n) \forall m, n = 1, \ldots, N$ are not independent. Thus the onset of synchronization is not possible for all coupling gains K satisfying $K_L \leq K < K_c$. **Hence, we have derived a stronger lower bound for the onset of synchronization**.

3.4. Sufficient condition for Synchronization

Now, we will derive a sufficient condition for synchronization. The assumption in the analysis is that the initial phase of all oscillators lie within the set:

$$\mathcal{D} = \left\{\theta_i, \theta_j \in R \mid |\theta_i - \theta_j| \leq \frac{\pi}{2} - 2\epsilon\right\}$$

We will find a lower bound on the coupling gain K denoted by K_{inv} which makes this set positively invariant for all oscillators, i.e. $\theta_i - \theta_j \in \mathcal{D}$ at $\mathrm{t} = 0 \Rightarrow \theta_i - \theta_j \in \mathcal{D} \forall t > 0$. Then having phase-locked the oscillators in $\mathcal{D}$, we will show that the oscillators synchronize. We now, have:

$$\dot{\theta}_i - \dot{\theta}_j = K\left\{\frac{\omega_i - \omega_j}{K} - \sin(\theta_i - \theta_j) + \frac{1}{N}\left(\sum_{k=1}^{N}\sin(\theta_i - \theta_j) + \sin(\theta_k - \theta_i) + \sin(\theta_j - \theta_k)\right)\right\} \quad (3)$$

Rewrite the term:

$$\frac{1}{N}\left(\sin(\theta_i - \theta_j) + \sin(\theta_k - \theta_i) + \sin(\theta_j - \theta_k)\right)$$

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as

$$\frac{1}{N}\sin(\theta_i-\theta_j)\left(1-\frac{\cos\left(\theta_k-\frac{(\theta_i+\theta_j)}{2}\right)}{\cos\left(\frac{(\theta_i-\theta_j)}{2}\right)}\right)=\frac{1}{N}\sin(\theta_i-\theta_j)C_k$$

where $C_k \in [0,1)$. Now, we have

$$\dot{\theta}_i-\dot{\theta}_j=K\left\{\frac{\omega_i-\omega_j}{K}-\sin(\theta_i-\theta_j)\left(1-\frac{1}{N}\sum_{k=1}^{N}C_k\right)\right\} \tag{4}$$

Now, let us state a result.

Result 1. Consider the system dynamics as described by (3). Let all initial phase differences at $\mathrm{t}=0$ be contained in the compact set $\mathcal{D}=\{\theta_i,\theta_j||\theta_i-\theta_j\mid\le\frac{\pi}{2}-2\epsilon\quad\forall i,j=1,\ldots,N\}$. Then there exists a coupling gain $K_{\text{inv}}>0$ such that $(\theta_i-\theta_j)\in\mathcal{D}\quad\forall t>0$.

Proof: Consider a positive definite Lyapunov function

$$V=\frac{1}{2K}(\theta_i-\theta_j)^2 \tag{5}$$

Taking the derivative of V along the trajectories of (3) wrt time, we get:

$$\begin{aligned}\dot{V}&=\frac{1}{K}(\theta_i-\theta_j)\left(\dot{\theta}_i-\dot{\theta}_j\right)\\&=(\theta_i-\theta_j)\left(\frac{\omega_i-\omega_j}{K}-\sin(\theta_i-\theta_j)\left(1-\frac{1}{N}\sum_{k=1}^{N}C_k\right)\right)\\&\le|\theta_i-\theta_j|\left|\frac{\omega_i-\omega_j}{K}\right|-(\theta_i-\theta_j)\sin(\theta_i-\theta_j)\left(1-\sum_{k=1}^{N}\frac{C_k}{N}\right)\\&\le|\theta_i-\theta_j|\left|\frac{\omega_i-\omega_j}{K}\right|-(\theta_i-\theta_j)\sin(\theta_i-\theta_j)\left(1-\frac{N-2}{N}\right)\end{aligned}$$

where we use $C_k<1$ and that $C_k=0$ for $k=i,j$. Thus the derivative can be written as

$$\dot{V}\le|\theta_i-\theta_j|\left|\frac{\omega_i-\omega_j}{K}\right|-(\theta_i-\theta_j)\sin(\theta_i-\theta_j)\frac{2}{N}$$

Hence, if $K>\frac{N|\omega_i-\omega_j|}{2\cos(2\epsilon)}$ the derivative of Lyapunov function is negative at $|\theta_i-\theta_j|=\frac{\pi}{2}-2\epsilon$ and thus the phase difference cannot leave the set $\mathcal{D}$. And to conclude, if $K=K_{\text{inv}}>\frac{N|\omega_{\max}-\omega_{\min}|}{2\cos(2\epsilon)}$, $\theta_i-\theta_j\quad\forall i=1,2,\ldots,N$ are positively invariant with respect to the compact set $\mathcal{D}$.

Now, having trapped the phase differences within the desired compact set $\mathcal{D}$ by choosing a desired coupling gain, we will mathematically prove the synchronization.

Result 2. Consider the system dynamics as described by (3). Let all initial phase differences at $t=0$ be contained in the compact set $\mathcal{D}$. If the coupling gain K is chosen such that $K=K_{inv}$, then all the oscillators asymptotically synchronize i.e. $\dot{\theta}_i - \dot{\theta}_j \to 0$ as $t \to \infty \quad \forall i,j=1,\ldots,N$

Proof: Consider the positive function,

$$S=\frac{1}{2}\dot{\theta}^T\dot{\theta}$$

where $\dot{\theta}=[\dot{\theta}_1\ldots\dot{\theta}_N]^T$ Taking the derivative of V along the trajectories of (3) wrt time, we get:

$$\begin{aligned}
\dot{S} &= \dot{\theta}_1\ddot{\theta}_1+\dot{\theta}_2\ddot{\theta}_2+\ldots+\dot{\theta}_n\ddot{\theta}_n \\
&= \frac{\dot{\theta}_1}{\beta}\Big(\cos(\theta_1-\theta_2)(\dot{\theta}_2-\dot{\theta}_1)+\ldots+\cos(\theta_n-\theta_1)(\dot{\theta}_n-\dot{\theta}_1)\Big) \\
&+ \frac{\dot{\theta}_2}{\beta}\Big(\cos(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2)+\ldots+\cos(\theta_n-\theta_2)(\dot{\theta}_n-\dot{\theta}_2)\Big) \\
&\vdots \\
&+ \frac{\dot{\theta}_n}{\beta}\Big(\cos(\theta_2-\theta_n)(\dot{\theta}_2-\dot{\theta}_n)+\ldots+\cos(\theta_1-\theta_n)(\dot{\theta}_1-\dot{\theta}_n)\Big)
\end{aligned}$$

where $\beta=\frac{N}{K}$. On rearranging terms and simplifying we have that,

$$\dot{S}=-\frac{K}{N}\sum_{j=1}^{N}\sum_{i=1}^{N}\cos(\theta_i-\theta_j)(\dot{\theta}_i-\dot{\theta}_j)^2$$

Due to Result 1, we have that $(\theta_i-\theta_j)\in\mathcal{D},\forall i,j$. This gives us that $\cos(\theta_i-\theta_j)>0\forall i,j$ and hence $\dot{S}\leq 0$. Hence all angular frequencies are bounded. Consider the set $E=\{\theta_i-\theta_j,\dot{\theta}_i\in R\forall i,j \mid \dot{S}=0\}$. The set E is characterized by all trajectories such that $\dot{\theta}_i=\dot{\theta}_j,\forall i,j$. Let M be the largest invariant set contained in E. Using **Lasalle's Invariance** Principle, all trajectories starting in $\mathcal{D}$ converge to M as $t\to\infty$. Hence, the oscillators synchronize asymptotically.

3.5. Exponential Synchronization

Here, we will make use of the graph theoretic view of Kuramoto Oscillators from earlier. Building up from the previous theorem,

$$S=\frac{1}{2}\dot{\theta}^T\dot{\theta}$$

The derivative of this function along trajectories of (1) can be written as

$$\begin{aligned}
\dot{S} &= -\frac{K}{N}\dot{\theta}^T B\mathrm{diag}(\cos(\phi))B^T\dot{\theta} \\
&= -\frac{K}{N}\dot{\theta}^T L_K(\mathcal{G})\dot{\theta}
\end{aligned}$$

The matrix $L_K(\mathcal{G}) = B\mathrm{diag}(\cos(\phi))B^T \in N \times N$ is the weighted Laplacian and is described as follows

$$L_W(\mathcal{G})_{ii} = \sum_{k=1, k\neq i}^{N} \cos(\theta_k - \theta_i) \quad \forall i = 1, \ldots, N$$
$$L_W(\mathcal{G})_{ij} = -\cos(\theta_i - \theta_j) \quad \forall i, j = 1, \ldots, N \quad i \neq j$$

Clearly, if all phase differences $\phi \in \mathcal{D}$, then the weighted Laplacian matrix $L_K(\mathcal{G})$ is positive-semidefinite, and hence the previous result follows. In the next result we extend this result by developing an exponential bound on the synchronization rate of the oscillators.

Result 3. Consider the dynamics of the system as described by (1). If the phase differences given by $\phi \in \mathcal{D}$ at $t = 0$ and the coupling gain is selected such that $K = K_{\text{inv}}$, then the oscillators synchronize exponentially at a rate no worse that $\sqrt{K\sin(2\epsilon)}$. Proof: It follows from the synchronization frequency result that:

$$\Omega = \frac{\sum_{i=1}^{N} \dot{\theta}_i}{N} = \frac{\sum_{i=1}^{N} \omega_i}{N}$$

We can write from the result in[3] that

$$\dot{\theta} = \Omega \mathbf{1} + \delta \tag{6}$$

where **1** is the N dimensional vector of ones associated with the zero eigenvalue of the weighted Laplacian $L_W(\mathcal{G}), \delta \in R^n$ satisfies $\sum_{i=1}^{N} \delta = 0 \left(as \sum_{i=1}^{N} \dot{\theta}_i = N\Omega \right)$. Substituting (6) in the positive definite function S as defined above, we have

$$\frac{d(\delta^T \delta)}{dt} = -\frac{K}{N} \delta^T L_W(\mathcal{G}) \delta \tag{7}$$

(the proof is obtained using the fact that Ω is invariant, and

$$\mathbf{1}^T L_W(\mathcal{G}) = 0$$

as 1 is an eigenvector associated with the zero eigenvalue of our weighted Laplacian matrix.) We can easily see from above that δ exponentially converges to origin, now as this δ will fall to zero, we can hence say from (6) that the oscillators start moving with mean frequency of the group.

As $\lambda_2(L_K(\mathcal{G}))$ is the smallest non-zero eigenvalue of the weighted Laplacian $\lambda_2(L_K(\mathcal{G}))$, we have from (7) that

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$$
\begin{aligned}
\frac{d(\delta^T\delta)}{dt} &\leq -\frac{K}{N}\delta^T\lambda_2(L_W(\mathcal{G}))\delta \\
&\leq -\frac{K}{N}\delta^T\lambda_2(\mathrm{Bdiag}(\cos(\phi))B^T)\delta \\
&\leq -\frac{K}{N}\delta^T\sin(2\epsilon)\lambda_2(BB^T)\delta \\
&\leq -K\sin(2\epsilon)\delta^T\delta
\end{aligned}
$$

as the $\min\{\cos(\phi)\} : \forall\phi\in\mathcal{D} = \cos\left(\frac{\pi}{2}-2\epsilon\right) = \sin(2\epsilon)$ and for an all-to-all connected topology $\lambda_2(BB^T) = N$. Thus the exponential convergence rate for synchronization is no worse that $\sqrt{K\sin(2\epsilon)}$.

4. Applications and Simulations

4.1. Kuramoto in a Power Network

I will be considering an AC power network here from Dorfler's paper.

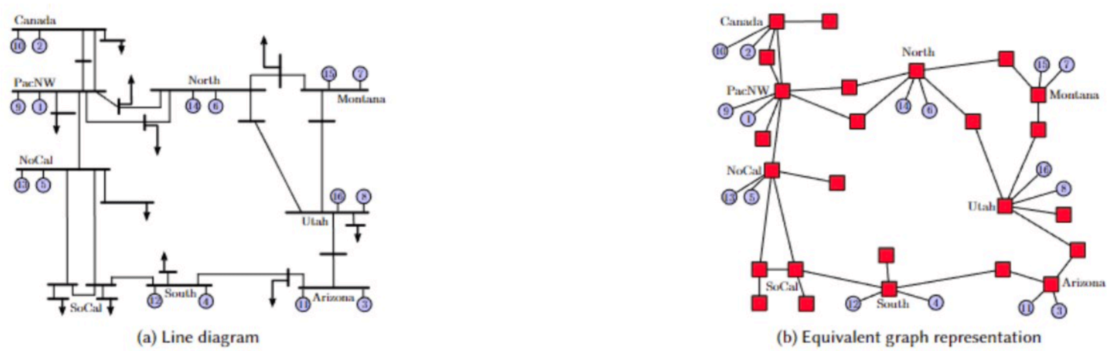

Figure 2. The power network

The transmission network is described by an admittance matrix $Y\in\mathbb{C}^{n\times n}$ that is symmetric and sparse with line impedances $Z_{ij} = Z_{ji}$ for each branch $\{i,j\}\in E$. The network admittance matrix is sparse matrix with nonzero off-diagonal entries $Y_{ij} = -1/Z_{ij}$ for each branch $\{i,j\}\in E$; the diagonal elements $Y_{ii} = -\sum_{j=1,j\neq i} Y_{ij}$ assure zero row-sums.

The static model is described by the following two concepts. Firstly, according to Kirchhoff's current law, the current injection at node i is balanced by the current flows from adjacent nodes:

$$
I_i = \sum_{j=1}^{n}\frac{1}{Z_{ij}}(V_i - V_j) = \sum_{j=1}^{n} Y_{ij}V_j.
$$

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Here, I_i and V_i are the phasor representations of the nodal current injections and nodal voltages, so that, for example, $V_i = |V_i|e^{i\theta_i}$ corresponds to the signal $|V_i|\ \cos\ (\omega_0 t + \theta_i)$. The complex power injection $S_i = V_i \cdot \bar{I}_i$ then satisfies the power balance equation

$$S_i = V_i \cdot \sum_{j=1}^{n} \bar{Y}_{ij}\bar{V}_j = \sum_{j=1}^{n} \bar{Y}_{ij}|V_i||V_j|e^{i(\theta_i - \theta_j)}$$

Next, for a lossless network:

$$\underbrace{P_i}_{\text{active power injection}} = \sum_{j=1}^{n} \underbrace{a_{ij} \cdot \sin(\theta_i - \theta_j)}_{\text{active power flow from } i \text{ to } j} \quad , \quad i \in \{1, \ldots, n\}$$

where $a_{ij} = |V_i||V_j||Y_{ij}|$ denotes the maximum power transfer over the transmission line $\{i, j\}$, and $P_i = \mathrm{Re}(S_i)$ is the active power injection into the network at node i which is positive for generators and negative for loads.

Now, let's describe a dynamical model for this network. Our assumption here is that every node is described by a first-order integrator with the following intuition: node i speeds up (i.e., θ_i increases) when the power balance at node i is positive, and slows down (i.e., θ_i decreases) when the power balance at node i is negative. This intuition leads to a Kuramoto-like equation as follows:

$$\boxed{\dot{\theta}_i = P_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)}$$

A small simulation (plot) can be found here. Code here.

4.2. Coupled Oscillator Network:

We start by studying a system of n dynamic particles constrained to rotate around a unit-radius circle and no collisions occur. (Figure 3)

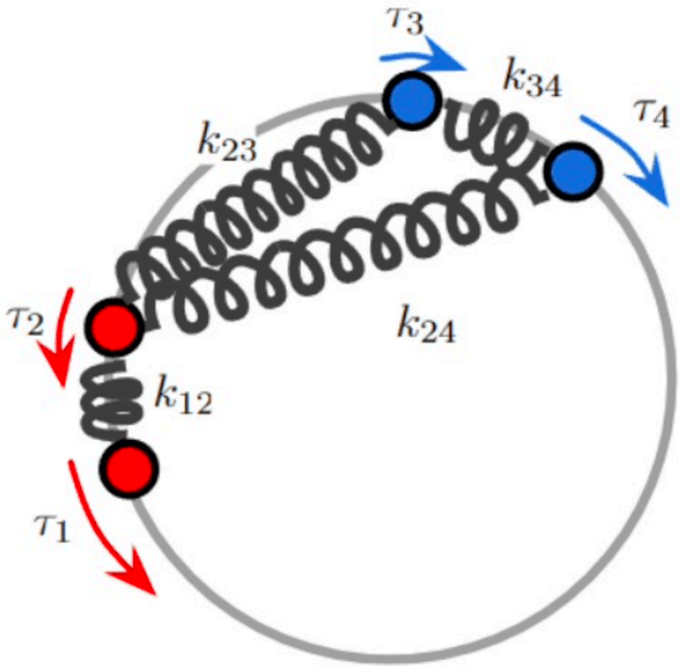

Figure 3. Springs on a ring

We assume that pairs of interacting particles i and j are coupled through elastic springs with stiffness $k_{ij} > 0$; we set $k_{ij} = 0$ if the particles are not interconnected. The elastic energy stored by the spring between particles at angles θ_i and θ_j is

$$\begin{aligned}\mathrm{U}_{ij}(\theta_i, \theta_j) &= \frac{k_{ij}}{2}(\text{distance})^2 = \frac{k_{ij}}{2}\left((\cos\theta_i - \cos\theta_j)^2 + (\sin\theta_i - \sin\theta_j)^2\right) \\ &= k_{ij}\left(1 - \cos(\theta_i)\cos(\theta_j) - \sin(\theta_i)\sin(\theta_j)\right) = k_{ij}\left(1 - \cos(\theta_i - \theta_j)\right)\end{aligned}$$

so that the elastic torque on particle i is

$$\mathrm{T}_i(\theta_i, \theta_j) = -\frac{\partial}{\partial\theta_i}\mathrm{U}_{ij}(\theta_i, \theta_j) = -k_{ij}\sin(\theta_i - \theta_j)$$

From Newton's second law, we have

$$m_i\ddot{\theta}_i + d_i\dot{\theta}_i = \tau_i - \sum_{j=1}^{n} k_{ij}\sin(\theta_i - \theta_j),$$

Assuming these springs are point masses, with high damping coefficients d, we get

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij}\sin(\theta_i - \theta_j), \quad i \in \{1, \ldots, n\}$$

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with natural rotation frequencies $\omega_i = \tau_i/d$ and with coupling strengths $a_{ij} = k_{ij}/d$.

4.3. Vehicle Coordination: Kuramoto-Vicsek Model

Another interesting example is the phenomenon of flocking and vehicle coordination. Let's assume that all particles have unit speed. The particle kinematics are then given by

$$\begin{aligned} \dot{r}_i &= e^{\mathrm{i}\theta_i}, \\ \dot{\theta}_i &= u_i(r, \theta), \end{aligned}$$

for $i \in \{1, \ldots, n\}$. If no control is applied, then particle i travels in a straight line with orientation $\theta_i(0)$, and if $u_i = \omega_i \in \mathbb{R}$ is a nonzero constant, then particle i traverses a circle with radius $1/|\omega_i|$

The interaction among the particles is modeled by a graph $G = (\{1, \ldots, n\}, E, A)$ determined by communication and sensing patterns. Say the controllers use only relative phase information between neighboring particles (as we are mimicking biological phenomenon like the synchronization of fireflies here). Now, let's see how we can adopt potential gradient control strategies (i.e., a negative gradient flow) to coordinate the relative heading angles $\theta_i(t) - \theta_j(t)$. Let's consider a quadratic elastic spring potential to the circle $\mathrm{U}_{ij} : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ defined by

$$\mathrm{U}_{ij}\left(\theta_i, \theta_j\right) = a_{ij}\left(1 - \cos(\theta_i - \theta_j)\right),$$

We can drive the affine gradient control law as follows:

$$\dot{\theta}_i = \omega_0 - K \frac{\partial}{\partial \theta_i} \sum_{\{i,j\} \subset E} \mathrm{U}_{ij}\left(\theta_i - \theta_j\right) = \omega_0 - K \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j), \quad i \in \{1, \ldots, n\}.$$

to synchronize the heading angles of the particles for $K > 0$ (gradient descent), respectively, to disperse the heading angles for $K < 0$ (gradient ascent). The controlled phase dynamics above mimic animal flocking behavior. Inspired by these biological phenomena, an area of research is to study these systems in the context of tracking/flocking in swarms of autonomous vehicles.

Simulations:

The animated result for flocking behaviour is here. Codes can be found at link.

4.4. Order Parameter simulations:

In the example, I am considering a network of 100 oscillators with all-to-all connectivity. Here are the plots of order parameters for coupling constants of 0.5, 1, 2 and 3: (code here)

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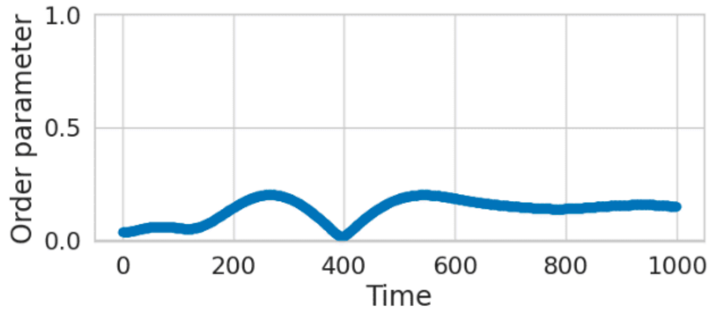
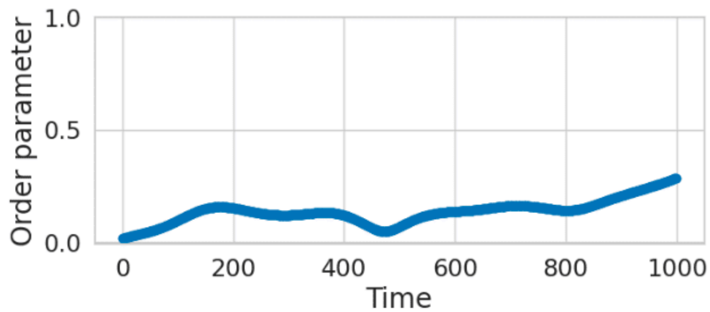
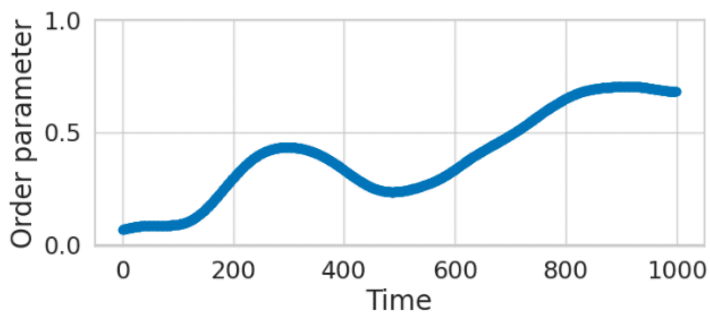


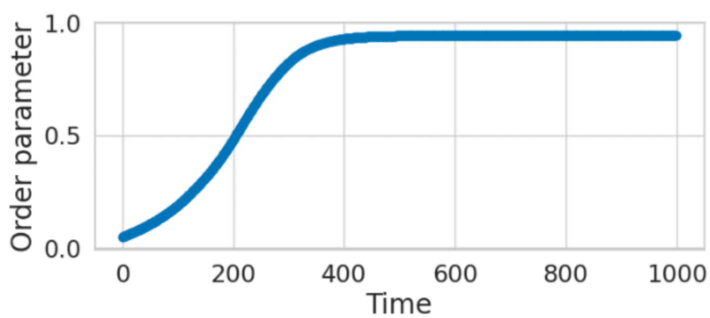
(a) For K = 0.5



(b) For K = 1



(c) For K = 2



(d) For K = 3

Figure 4. Evolution of $r(t)$ with time

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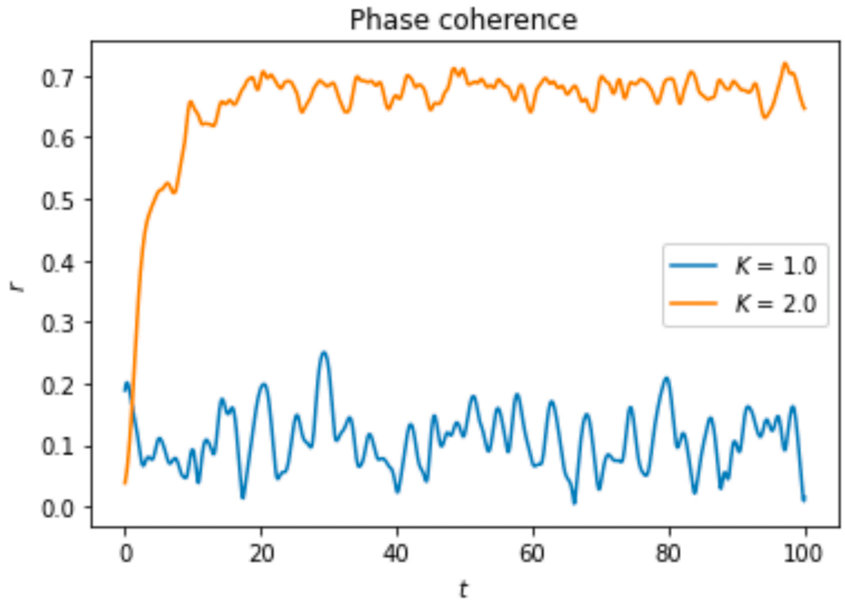

Figure 5. Phase Coherence plot

So, we can see that at 0.5, the order parameter is quite low and also hits zero at some point. For coupling constant of 1, the order parameter is still low but has some increasing behaviour after it hits zero. For 2, there is partial synchronization and the order parameter increases in later time intervals. For the case with K as 3, we have an order parameter that gets close to 1 after some time intervals so we have full synchronization.

In this case, the time series plot illustrates the same where all the trajectories eventually seem to converge.

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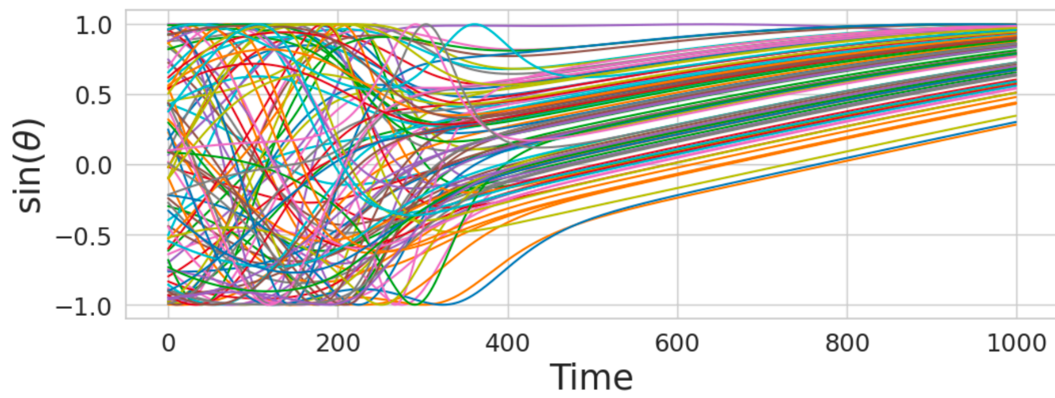

Figure 6. Evolution of trajectories

Now, looking plotting all the oscillators in the complex plane at different times, we get:

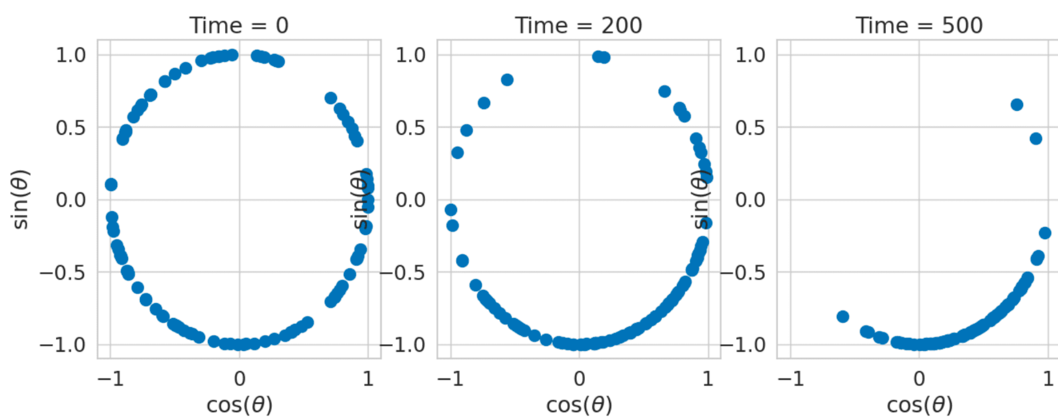

Figure 7. How the oscillators come together with time

4.5. A Slider tool

Another interactive slider tool that I have simulated in MATLAB looks as shown: (inspiration from Cleve Moler's work[4])

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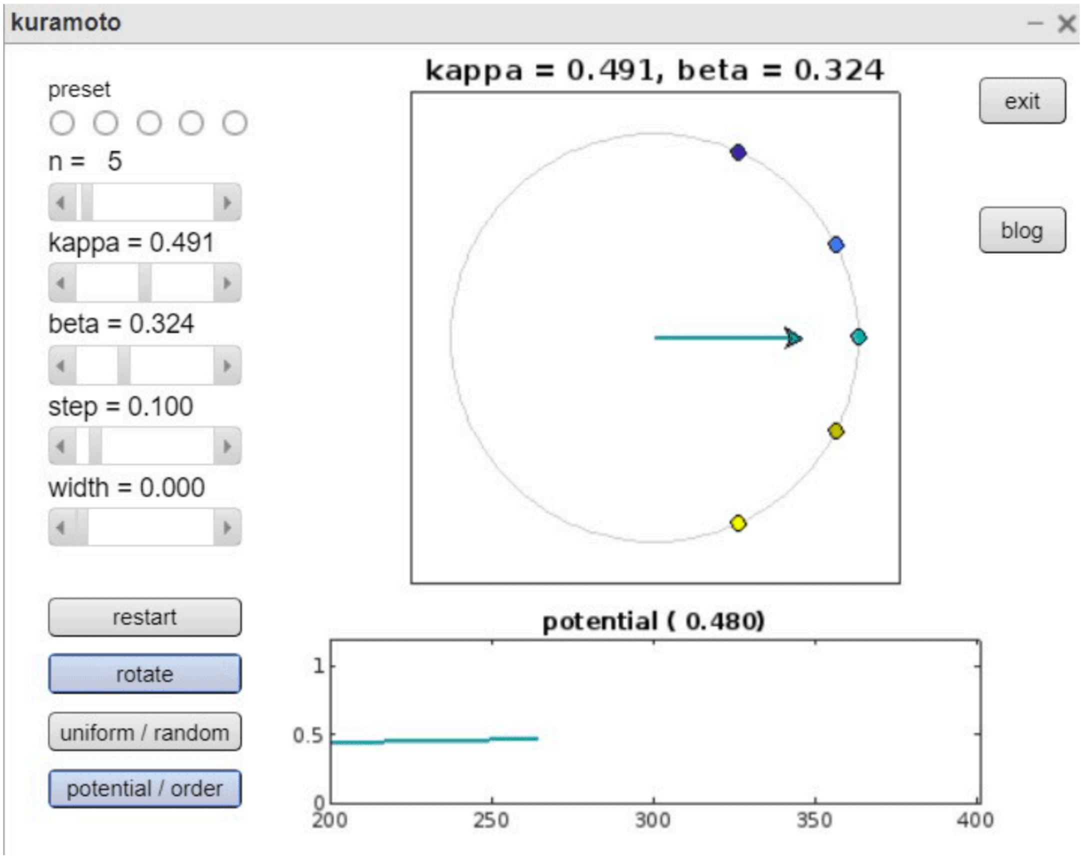

Figure 8. To plug and play

The code for the interactive slider can be found here.

4.6. Manim Animations

In this section, I first started out with simulating the dynamics for a given set of initial phases, and adjacency matrix. Here, n = 5.

4.6.1. Case A: When A is zero

```
x0 = [0, 0.4, 0.8, 1.2, 1.6], w = np.array([1, 2, 3, 4, 5]), A = np.zeros((5, 5))
```

We can clearly see in figure that there is no coupling. We expect the system angles to increase linearly with time at an angular velocity w_i.

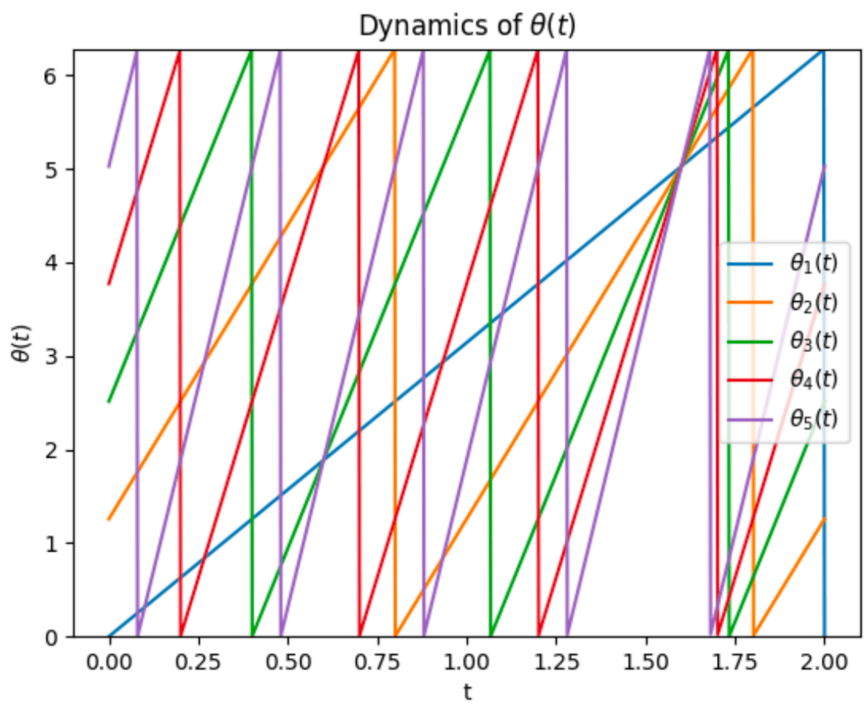

Figure 9. For A = O_{5x5}

4.6.2. Case B: for a Cycle Graph

```
x0 = [0, 2 * np.pi/5 + 1/2, 4 * np.pi/5, 6 * np.pi/5 + 1/10, 8 * np.pi/5],
w = np.array([5, 5, 5, 5, 5])
A = np.array([
    [0, 3, 0, 0, 3],
    [3, 0, 3, 0, 0],
    [0, 3, 0, 3, 0],
    [0, 0, 3, 0, 3],
    [3, 0, 0, 3, 0],])
```

In this case, there is a cyclic coupling structure. Our w values are constant, they are all 5. Now, let us think about the steady state behaviour of the system. Consider an equally spaced cyclic state. Here, $\sum_j \sin(\theta_i - \theta_j) = 0$. If all the oscillators were equally space around the circle, then the corresponding all the sine terms will cancel out, and only the w term will remain. So, the stable state angular velocity is just $w = 5 \forall i$.

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Initially, the system is decoupled. But, in the steady state, the system becomes coupled and all the oscillators are equal spaced around the circle, and oscillate with angular velocity 5.

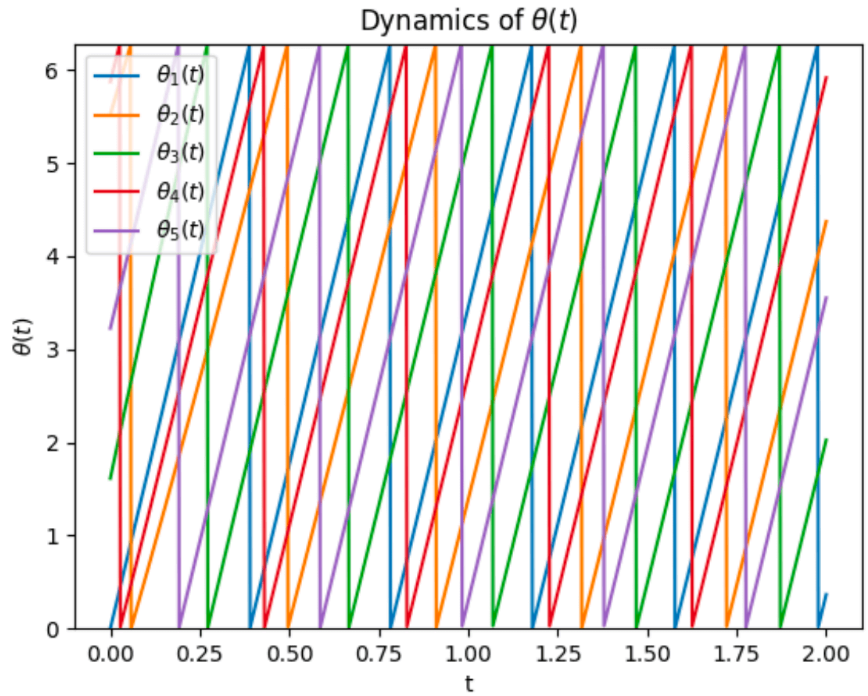

Figure 10. For a cycle graph

Case C: for a Line Graph

```
x0 = [0, 0.4, 0.8, 1.2, 1.6], w = np.array([1, 1, 1, 2.5, 2.5])
A = np.array([
    [0, 10, 0, 0, 0],
    [10, 0, 10, 0, 0],
    [0, 10, 0, 1, 0],
    [0, 0, 1, 0, 10],
    [0, 0, 0, 10, 0],])
```

As the first three oscillators have the same frequency, they get coupled, while the fourth and fifth oscillators also get coupled. This phenomenon of coupling can be observed graphically also.

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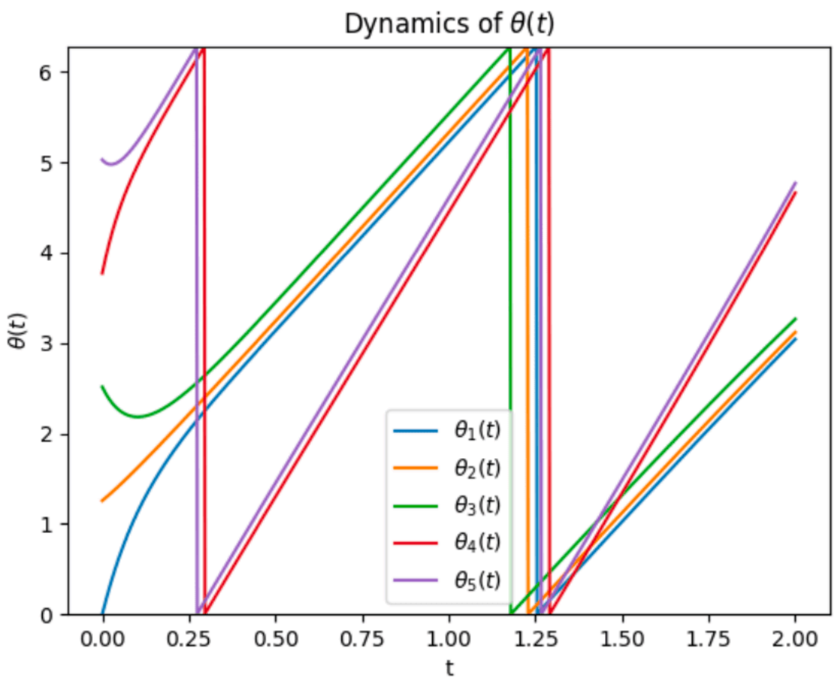

Figure 11. For a line graph

For cases B and C, we also tried animating the dynamics using Manim. All the codes, plots/videos can be found in this folder.

4.7. *Future directions of interest*

- Exploring Synchronization in more complex networks, and applications to smart grids
- Bifurcation analysis for a system of oscillators
- Criterion for phase/frequency synchronization for all Kuramoto oscillator networks (be it identical or non-identical)

5. Supplementary

All the codes can be found in this master folder. Here's the link to the video presentation.

Synchronizing Fireflies: A very interesting simulation demonstrating the synchronization of fireflies that I came across was this.

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6. Conclusions

To conclude, we first start off with an introduction to Kuramoto Oscillators and it's broader applications. Then, we look at a graph theoretic formulation for the same, followed by a detailed discussion of various criterion for stability and synchronization of Kuramoto Oscillators. After that, we broadly analyze and experiment with three unique physical systems that tend to behave like Kuramoto oscillators along with further ablation studies.

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Declarations

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